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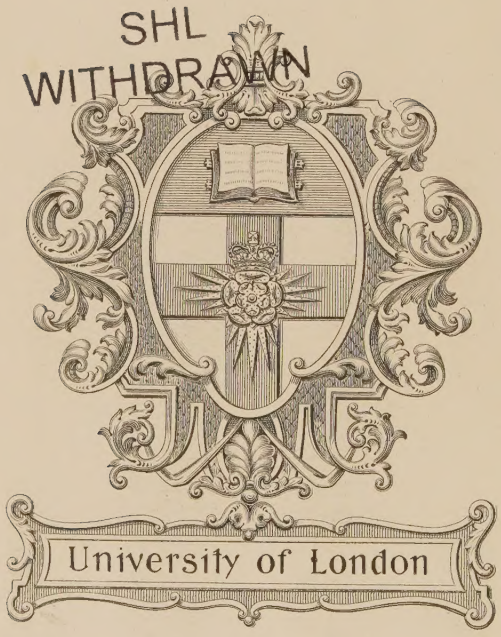

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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

VOL. XXVI.

NOVEMBER, 1894, TO NOVEMBER, 1895.

LONDON:
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1895.

73/40

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HOLBORN, W.C.

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MEMORANDUM

AND

ARTICLES OF ASSOCIATION

OF

THE LONDON MATHEMATICAL SOCIETY.

MEMORANDUM.

1. The name of the Society is "The London Mathematical Society."

2. The Registered Office of the Society shall be situate in England.

3. The objects for which the Society is established are the promotion and extension of mathematical knowledge by :—

(1) Periodical meetings of the Members of the Society, and visitors introduced by them, at the rooms of the Society in London, for the purpose of hearing and discussing communications from the Members of the Society and others on mathematical subjects.

(2) The publication in Transactions or Proceedings of the Society of such of the said communications as in the opinion of the Council of the Society are worthy of such publication.

(3) The distribution of the Transactions or Proceedings of the Society among the Members, and other persons and institutions, both English and Foreign.

(4) The sale to the public of the Transactions or Proceedings of the Society.

(5) The purchase of mathematical books, periodicals, diagrams, models, and instruments to form a library and museum for the use of the Members of the Society.

(6) The award of medals and other marks of distinction, in recognition of contributions to mathematical science.

(7) The making of grants or donations in aid of mathematical investigations, or the publication of mathematical works.

(8) The acquiring of moneys, securities for money, funds, and estates, both real and personal, by gift, subscription, devise, bequest, purchase, or otherwise (but as regards lands and property savouring of realty so far only as the Society may lawfully acquire the same by virtue of the licence of the Board of Trade or otherwise), and the holding, investing, depositing, management, disposal, appropriation, or distribution of the same or any part thereof, and of the proceeds and income arising therefrom, or from the sale or conversion thereof, for all or some of the purposes before mentioned.

(9) The holding or renting rooms or premises for all or some of the purposes before mentioned.

(10) The holding of moneys, securities for money, funds, and estates as in Section 8 mentioned upon special trusts for any of the purposes before mentioned, or other purposes analogous thereto.

(11) Such other lawful means as are or may be incidental or conducive to the objects of the Society.

Provided that, in case the Society shall take or hold any property subject to the jurisdiction of the Charity Commissioners for England and Wales, the Society shall, if required by the Charity Commissioners, vest the same in special Trustees thereof, and provided, that as regards any such property, the Society shall not sell, mortgage, charge, or lease the same without such consent as may be required by law, and, as regards any such property, the Council or Trustees of the Society shall be chargeable for such property as may come into their hands, and shall be answerable and accountable for their own acts, receipts, neglects, and defaults, and for the due administration of such property in the same manner, and to the same extent, as they would, as such Council or Trustees, have been if no incorporation had been effected ;

and the incorporation of the Society shall not diminish or impair any control or authority exerciseable by the Chancery Division or the Charity Commissioners over such Council or Trustees, but they shall, as regards any such property, be subject jointly and separately to such control and authority as if the Society were not incorporated.

4. The income and property of the Society, whencesoever derived, shall be applied solely towards the promotion of the objects of the Society as set forth in this Memorandum of Association, and no portion thereof shall be paid or transferred directly or indirectly by way of dividend, bonus, or otherwise, howsoever, by way of profit to the Members of the Society. Provided that nothing herein shall prevent the payment in good faith of remuneration to any officers or servants of the Society, or to any Member or other person in return for any services actually rendered to the Society or the objects thereof.

5. The 4th paragraph of this Memorandum is a condition on which the licence is granted by the Board of Trade to the Society in pursuance of Section 23 of the "Companies Act, 1867."

6. If any Member of the Society pays or receives any dividend, bonus, or other profit in contravention of the terms of the 4th paragraph of this Memorandum, his liability shall be unlimited.

7. Every Member of the Society undertakes to contribute to the assets of the Society in the event of the same being wound up during the time that he is a Member, or within one year afterwards, for payment of the debts and liabilities of the Society contracted before the time at which he ceases to be a Member, and of the costs, charges, and expenses of winding up the same, and for the adjustment of the rights of the contributories among themselves, such sum as may be required not exceeding one guinea, or in the case of his liability becoming unlimited such other amount as may be required in pursuance of the last preceding paragraph of this Memorandum.

8. True accounts shall be kept of the sums of money received and expended by the Society, and the matters in respect of which such receipt and expenditure takes place, and of the capital and other funds of the Society; and once at least in every year a balance sheet shall be prepared, the accounts shall be examined, and their correctness and that of the balance sheet shall be ascertained by one or more Auditors, either a Member or Members of the Society of

other person or persons to be from time to time appointed at a General Meeting of the Members of the Society, and the balance sheet so audited shall be printed, and a copy be sent to every Member of the Society.

9. If, upon the winding up or dissolution of the Society, there remains, after the satisfaction of all its debts and liabilities, any property whatsoever, the same shall not be paid to or distributed among the Members of the Society, but shall be given or transferred to some other institution or institutions having objects similar to those of the Society, to be determined in General Meeting by the Members of the Society at or before its dissolution, or in default thereof, at the request of the Society or the Liquidators thereof, by such Judge of the High Court of Justice as may have or acquire jurisdiction in the matter.

ARTICLES.

Number of Members.

1. For the purposes of registration the number of Members of the Society is declared not to exceed two hundred and fifty.

2. The Council of the Society hereinafter mentioned may, whenever the objects of the Society require it, increase the number of Members.

Objects.

3. The Society is instituted for the objects expressed in the Memorandum of Association.

Office.

4. The Office of the Society shall be in London.

Constitution.

5. The Society shall consist of Members to be elected or admitted in manner hereinafter provided.

6. The Council of the Society shall consist of sixteen Members of the Society, including the Officers hereinafter mentioned, or of such larger number of Members as shall be prescribed from time to time by By-laws passed at a Special General Meeting of the Society as hereinafter mentioned.

7. The Officers of the Society shall consist of a President, Vice-Presidents, a Treasurer, a Secretary or Secretaries, and such other Officer or Officers (if any) as shall be prescribed from time to time by By-laws passed at a Special General Meeting of the Society.

8. Until the election of a Council and Officers at the First Meeting of the Society in accordance with Article 18, the persons who are at the date of Registration the Council and Officers of the hitherto existing unincorporated body called "The London Mathematical Society" within the meaning of their laws in force at the same date, and have subscribed their names to the Memorandum of Association, shall be the Council and Officers of the Society.

9. A Council and Officers shall be elected at the First Meeting of the Society to be held as provided in Article 18, and subsequently they shall be elected annually by ballot at the Annual General Meeting hereinafter mentioned.

10. Should a vacancy occur among the Council or Officers during the interval between two Annual General Meetings, it shall be filled up by the Council.

11. The business of the Society shall be managed wholly by the Council, subject only to such limitations or directions as shall be expressed in any By-laws or Resolutions passed from time to time by a Special General Meeting of the Society.

Election, Resignation, and Expulsion of Members.

12. Every person who at the date of the Registration of the Society is a Member of the hitherto existing unincorporated body called "The London Mathematical Society" within the meaning of their laws in force at the said date, and who has not before the said date signified his dissent, shall be and become a Member of the Society without election, payment of entrance fee, or further proceedings, and shall have his name entered on the Register of Members of the Society as such Member accordingly. No "Honorary Member" of

the hitherto existing Society shall become a Member of the Society under this rule.

13. Every other candidate for Membership shall be proposed and recommended, according to a form which the Secretaries shall supply, by not less than three Members, of whom one at least must certify his personal knowledge of the candidate; but so much of this rule as relates to personal knowledge of candidates may, by special leave of the Council, be relaxed in the case of candidates not residing in the United Kingdom. The form of proposal shall be read at one of the Ordinary or Annual General Meetings, and the candidate shall be balloted for at the next ensuing Meeting, provided that seven Members shall be present thereat.

14. At the ballot for the election of Members, candidates shall be balloted for collectively; and, if one or more black balls be found in the box, the ballot shall then be taken separately for each candidate; one black ball in three shall exclude.

15. Each candidate thus elected shall be informed thereof by one of the Secretaries, and be supplied with a copy of the Memorandum and Articles of Association and By-laws. He shall pay the contribution which is due from him within six months after the day of his election; otherwise his election shall be void. The Council, however, may, in special cases, extend the period within which such first contribution must be paid.

16. At the first Ordinary or Annual General Meeting at which an elected Member is present after his first contribution has been paid, he shall be admitted according to the following form, after he has signed his name in a book to be kept for the purpose. The President or other Member in the Chair, addressing him by name, shall say, "In the name of the London Mathematical Society, I admit you a Member thereof."

17. Should the Council deem it desirable that any Member should be expelled, they shall call a Special General Meeting to consider the question, and after such Member has had an opportunity at such Meeting of explaining his objection to the Resolution for his expulsion, and provided that twenty Members shall be present at such Meeting, and that, of those who vote, at least two-thirds shall agree by ballot to his expulsion, he shall cease to be a member of the Society.

18. Every person who has been elected a Member of the Society shall continue to be a Member thereof until his decease, or until he has signified in writing to the Society his desire to resign, or until his name has been removed from the list of Members as a defaulter or as having been expelled from the Society. And every person who has ceased to be a Member of the Society, or, in the case of death, his representatives, shall continue liable for all sums due from him to the Society at the time when he ceased to be a Member and shall return all books or other property borrowed by him from the Society, and shall make full compensation for the same if injured or not returned.

SUBSCRIPTIONS OF MEMBERS.

19. An annual subscription of one guinea shall be paid by each Member, except those who have already compounded for their subscriptions under the existing laws of the unincorporated body called by the same name, or shall hereafter compound under the provisions of Article 21. The subscription shall become due in advance on the second Thursday in November in each year. In the case of a newly elected Member the subscription for the current year shall become due on the day of his election, and the next subscription on the second Thursday in November next ensuing.

20. Every newly elected Member shall also pay an entrance fee of one guinea.

21. Any Member may compound for his annual subscriptions by the payment of such sum as shall be prescribed from time to time by By-laws passed at a Special General Meeting of the Society. And every member who now has compounded or shall hereafter compound for his subscription shall not be called upon to pay any additional sum to the Society further or otherwise than such sums as he is liable to contribute under Article 7 of the Memorandum of Association.

22. Should a Member allow his subscription to be one year in arrear, he shall be notified thereof by the Treasurer. Should he, notwithstanding this notification, allow another year to pass without paying up all his arrears, the Treasurer shall lay the matter before the Council, and, unless they shall otherwise order, he shall thereupon cease to be a Member of the Society.

MEETINGS.

23. The first Meeting of the Society shall be held in London within four months of the registration of the Memorandum of

Association. Such Meeting shall have power to elect a Council and Officers, to frame By-laws for the regulation of the affairs of the Society and to pass Resolutions with regard to the affairs of the Society. Provided that no By-law or Resolution which involves such an alteration of or addition to the Articles as can only legally be made by a Special Resolution shall be made, except by a Special Resolution in accordance with Section 51 of the "Companies Act, 1862."

24. An Annual General Meeting shall be held in November in each year.

25. Ordinary Meetings of the Society shall be held on such day or days in each month from November to June, both inclusive, and in such additional months, as the Council shall from time to time determine.

26. The Council may at any time, when it shall seem necessary, call a Special General Meeting of the Society.

27. Ten days' notice shall be sent to every Member resident in the United Kingdom of the time of such Meeting and of the business for which it is summoned, and no business shall be decided upon at such Meeting, except what has been so notified.

28. The Council, upon receiving a written request, addressed to one of the Secretaries, to call a Special General Meeting, shall be bound to accede to such request, provided that it be signed by at least fifteen Members, and that it also state the object for which the Meeting is desired.

29. No motion at such Meeting shall be considered as carried unless fifteen Members be present, and unless of the votes given two-thirds be in favour of it.

30. Except at a Special General Meeting, no question concerning any alteration of the By-laws or the management of the Society shall be entertained.

31. The Chairman may, with the consent of the Meeting, adjourn any business from time to time and from place to place, but no business shall be transacted at any adjourned Meeting other than the business left unfinished at the Meeting at which the adjournment took place.

32. At a Special General Meeting, unless a poll is demanded by at least fifteen Members, a declaration by the Chairman that a resolution

has been carried, or has not been carried, and an entry to that effect in the book of proceedings of the Society, shall be sufficient evidence of the fact without proof of the number or proportion of the votes recorded in favour of or against any such resolution.

33. If a poll is demanded in manner aforesaid, the same shall be taken in such manner as the Chairman directs, and the result of such poll shall be deemed to be the resolution of the Society in Special General Meeting.

Quorum.

34. At an Ordinary Meeting of the Society seven Members shall form a Quorum.

35. At an Annual General Meeting of the Society ten Members shall form a Quorum.

36. At a Special General Meeting of the Society fifteen Members shall form a Quorum.

37. At a Meeting of the Council of the Society five Members shall form a Quorum.

Votes.

38. Every Member who is entitled to a vote at any Meeting of the Society or of the Council shall have one vote and no more, except that in case of an equality of votes the Chairman of the Meeting at which any vote is taken shall be entitled to a casting vote in addition to his vote as a Member.

39. No Member shall be entitled to vote at any Meeting of the Society or of the Council if at the time of such Meeting there shall be any moneys due from him to the Society which have been so due for more than two years.

Notices.

40. Every Member upon his election shall furnish one of the Secretaries with his address, and Members shall give notice to one of the Secretaries of any change of address, and the same shall be entered by one of them in a book to be kept for that purpose. And any notice may be served by the Society upon any Member either personally or by sending it through the post in a prepaid letter to the address furnished by the Member as aforesaid.

41. Any notice if served by post shall be deemed to have been served at the time that the letter containing the same would be delivered at the aforesaid address in the ordinary course of the post, and in proving such service it shall be sufficient to prove that the letter containing the notice was directed to the address furnished by the Member as aforesaid, and put into the post office.

Winding Up.

42. The Society shall be wound up voluntarily whenever a special resolution, as defined by the "Companies Act, 1862," is passed requiring it to be wound up voluntarily.

BY-LAWS

OF

THE LONDON MATHEMATICAL SOCIETY.

I. Of the Council.

1. The Council shall meet at least once in every month during the Session; and any three Members of the Council may, by letter to one of the Secretaries, require an extra Meeting to be called.

2. No President or Vice-President shall hold the same office for more than two years consecutively. He shall, however, be eligible for re-election after the lapse of one year.

3. A month before the Annual General Meeting, the Council shall hold a Meeting, to consider the subject of the election of the new Council and Officers, with a view of recommending a list of names to the Society.

4. In the list so recommended, at least two names of Members not on the Council for the previous year shall appear.

5. Any Member of the Society may send in to the Council a list of names of those Members of the Society whom he considers most fit for election as Members of the Council or as Officers. This list must be sent, duly signed, to one of the Secretaries before noon on the day on which the said Council Meeting shall take place.

6. A list of the names so recommended by the Council, and of all the names suggested by Members of the Society, shall be sent to each

Member simultaneously with the summons to attend the Annual General Meeting. In this list the names of the Nominors or the words, "The Council," as the case may be, shall be attached to each name proposed.

7. Each Member shall have power to add to the list any eligible names which are not already contained in it, and shall make use of it as a balloting list by striking out those names for which he does not vote.

8. Any balloting list which shall contain more names not struck out than the number required by Chapter I., By-law 5, and Article 6 of the Articles of Association, or which shall contain any names of Members who are not eligible for the offices assigned to them, shall be held to be null and void.

9. Members who are unable to attend the General Meeting shall be allowed to vote in the election of the Council and Officers by sending their balloting lists, duly signed, addressed to "The Scrutineers of the London Mathematical Society," at the Registered Office of the Society.

II. Of the Life-composition.

1. Any Member may compound for his annual subscription by the payment of ten guineas in one sum.

III. Of the Honorary Members.

1. The Society may from time to time elect Mathematicians of distinction of foreign countries to be associated with the Society under the designation of "Honorary Members"; but such "Honorary Members" shall not be Members of the Society within the meaning of the provisions of the Companies Acts, and shall not be entered in the Register of Members, or be entitled to vote at any Meeting of the Society, nor shall they be liable to pay any entrance fee, subscription, or other contribution.

2. Every person who at the date of the registration of the Society was an Honorary Member of the Society in its unincorporated form shall become an Honorary Member of the Incorporated Society without further proceedings.

3. Other Honorary Members shall be nominated for election by the Council. No such nomination shall be made unless nine Members at least are present at the Meeting of the Council at which such nomination is made, and of these not less than two-thirds of those who vote shall be in favour thereof.

4. Mathematicians so nominated shall be proposed at the first subsequent Ordinary or Annual General Meeting, and shall be balloted for at the Meeting next following.

5. Honorary Members shall be entitled to be present at all Meetings of the Society.

6. As soon as is convenient after the election of an Honorary Member, a Diploma, signed by the President and one of the Secretaries, and sealed with the seal of the Society, shall be forwarded to him by the Secretaries.

IV. Of the Annual General Meeting.

1. The Annual General Meeting shall be held on the second Thursday in November, at eight o'clock in the evening. The order of procedure at this Meeting shall be determined by the Council.

V. Of the Ordinary Meetings, and Reading of Papers.

1. The Ordinary Meetings of the Society shall commence at eight o'clock in the evening precisely. The order of business thereat shall be arranged by the Council.

2. After the reading of each paper or communication, the Chairman shall invite discussion thereon. When the papers and communications have been disposed of, any Member may, with the permission of the Chairman, propose questions to be discussed either at once or at a subsequent Meeting.

3. Every person who shall desire to bring a paper before the Society shall give notice thereof in writing to one of the Secretaries, stating the title and nature of the paper, whether he wishes to read it himself, and, if so, whether *in extenso*, or in the way of an abstract of its contents.

4. Every paper intended for reading by one of the Secretaries must be sent in at least a week before the day on which it is proposed to be read.

5. In all cases the Council shall decide whether any paper proposed for reading shall or shall not be read.

6. Every Member shall have the right to introduce two visitors at an Ordinary Meeting, on giving their names to the Member in the Chair. But this right extends to no other class of Meetings.

VI. Of the President, Vice-President, and Chairman.

1. The business of the President shall be to preside at all the Meetings of the Society and Council, to regulate the discussions, and to decide questions of order.

2. In the absence of the President one of the Vice-Presidents shall take the Chair, or, if no Vice-President shall be present, the Council shall elect a Chairman; and the Vice-President or other Member in the Chair shall be invested with and shall exercise all the powers and prerogatives of the President.

VII. Of the Treasurer and the Funds of the Society.

1. The Banking Account of the Society shall be kept at such Bank as shall from time to time be determined by the Council, and the Treasurer shall receive for the use of the Society and pay into the said Account all sums of money due or payable to the Society which are not paid in directly to such Account, and for the purposes aforesaid he shall be empowered to endorse all cheques and other negotiable instruments payable to the Society. Subject to the approval of the Council, and under such conditions and restrictions as they may deem expedient, he shall be empowered to pay by cheque on the said Account all sums payable by the Society. He shall also keep particular accounts of all such receipts and payments.

2. All sums of money which there shall not be any present occasion for expending, or otherwise disposing of for the use of the Society, shall be laid out in such securities as shall be approved and directed by the Council. And subject to the provisions of any trust deed relating to such securities, but otherwise at their own absolute discretion, the Council may vary and transpose such securities in, to, or for any others which they shall deem expedient.

VIII. Of the Secretaries.

1. It shall be the duty of the Secretaries to enter in the Minute Book and to read the Minutes of the Proceedings of the Society and Council, to have the custody of all documents of the Society which do not relate to the stocks, shares, funds, and securities belonging to or held in trust for the Society, and to do all such things for the use and service of the Society as appertain to the office of Secretary.

IX. Of the Proceedings of the Society, and Publication of Papers.

1. The Secretaries, under the direction of the Council, shall take charge of the publication of the Proceedings of the Society.

2. These Proceedings shall contain a register of the Mathematical discussions that have taken place, and of the papers that have been read, as well as all business of importance that has been transacted at an Ordinary, Annual, or Special General Meeting of the Society.

3. Any paper read before the Society with a view to publication *in extenso* shall be referred by the Council to two or more Members of the Society, or other persons, who shall report to the Council on its fitness for publication in the Proceedings. After hearing the reports of the referees on the paper, the Council shall decide by ballot whether it shall be printed or not.

4. Every Member whose contribution is not in arrear, and every Honorary Member, shall be entitled to a copy of the Proceedings which shall be published after the date of his election. Copies shall also be sent to such Societies, Public Libraries, or other Institutions, Bodies or Persons, as the Council shall from time to time determine.

5. The author of any paper printed by the Society shall be entitled to twenty-five copies thereof free of all expense; and, on giving timely notice to the Secretaries, he may purchase at cost price as many more as he may desire.

X. Of the Common Seal, and the Custody of Documents.

1. The Common Seal of the Society, together with all deeds and other documents relating to the trust funds of the Society and to the stocks, funds, shares, and securities which are the property of the Society, shall be kept in a box having two separate locks and two

separate keys. One of such keys shall be kept by the President, and the other by the Treasurer, and the box shall be kept in the custody of the Treasurer, and shall be deposited by him at the Bank where the account of the Society is kept, or in such other place as the Council shall from time to time direct.

2. Every deed to which the Common Seal of the Society is to be affixed shall be passed and sealed in Council, and shall be countersigned by the President, or Chairman, and by one other member of the Council.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

VOL. XXVI.

THIRTY-FIRST SESSION, 1894–95

(since the Formation of the Society, January 16th, 1865).

November 8th, 1894.

THE FIRST MEETING OF THE LONDON MATHEMATICAL SOCIETY, as incorporated under the Companies Act, 1867, on October 23rd, 1894, held at 22 Albemarle Street, W.

Mr. A. B. KEMPE, F.R.S., President, in the Chair.

The minutes of the last Meeting of the unincorporated Society, held June 4th, 1894, were read and confirmed.

The President moved, and the Treasurer seconded, a resolution that the By-laws of the Society, which had been framed by the Council, should be adopted.

This resolution was carried unanimously.

The Treasurer read his report. Its reception was moved by Professor M. J. M. Hill, seconded by Professor W. Burnside, and carried unanimously.

The Rev. T. R. Terry, being willing to serve, was appointed Auditor of the report on the motion of the President, seconded by Major MacMahon.

Mr. Tucker stated that the number of members at the date of registration was 216, of whom 101 were compounders.

The Society had to regret the loss, by death, of Mr. William Paice, M.A., of Mr. William Racster, M.A. (*cf.* p. 111, Vol. xxv.), and of one Hon. Foreign Member, Dr. Heinrich Rudolf Hertz (*cf.* p. 93, Vol. xxv.).

The following communications had been made or received:—

A Mechanical Solution of the Problem of Tethering a Horse to the Circumference of a Circular Field, so as to Graze over an nth part of it: Prof. L. J. Rogers.

The Stability of certain Vortex Motions: Mr. A. E. H. Love.

Cyclotomic Quartics: Prof. G. B. Mathews.

On the Application of Elliptic Functions to the Curve of Intersection of Two Quadrics: Mr. J. E. Campbell.

Notes on the Theory of Groups of Finite Order: Prof. W. Burnside.

The Stability of a Deformed Elastic Wire: Mr. A. B. Basset.

The Linear Automorphic Transformation of certain Quantics: Mr. Dallas.

On Bessel's Functions, and relations connecting them with Spherical and Hyper-Spherical Harmonics: Dr. Hobson.

A Theorem of Liouville's: Prof. Mathews.

Note on Non-Euclidean Geometry: Mr. H. F. Baker.

Note on an Identity in Elliptic Functions: Prof. L. J. Rogers.

Note on a Variable Seven-points Circle, analogous to the Brocard Circle of a Plane Triangle: Mr. J. Griffiths.

The Types of Wave-Motion in Canals: Mr. H. M. Macdonald.

On Green's Function for a System of Non-intersecting Spheres: Prof. W. Burnside.

Description of Model of Lord Kelvin's Tetrakaidekahedron: Mr. J. J. Walker.

On a Class of Groups defined by Congruences: Prof. W. Burnside.

Some Properties of the Uninodal Quartic: Mr. W. R. W. Roberts.

Groups of Points on Curves: Mr. F. S. Macaulay.

On a Simple Contrivance for Compounding Elliptic Motions: Mr. G. H. Bryan.

On the Buckling and Wrinkling of Plating Supported on a Framework under the Influence of Oblique Stresses: Mr. G. H. Bryan.

On the Motion of Paired Vortices with a Common Axis: Mr. A. E. H. Love.

On the Existence of a Root of a Rational Integral Equation: Prof. E. B. Elliott.

Pseudo-Elliptic Integrals and their Dynamical Applications: Prof. A. G. Greenhill.

On Regular Difference Terms: Mr. A. B. Kempe.

Theorems concerning Spheres: Mr. S. Roberts.

Second Memoir on the Expansion of certain Infinite Products: Prof. L. J. Rogers.

A Property of the Circumcircle (ii.): Mr. R. Tucker.

A Proof of Wilson's Theorem: Mr. J. Perott.

On the Sextic Resolvent of a Sextic Equation: Prof. W. Burnside.

On the Kinematical Discrimination of the Euclidean and Non-Euclidean Geometries: Mr. A. E. H. Love.

Permutations on a Regular Polygon: Major MacMahon.

The Stability of a Tube: Prof. A. G. Greenhill.

Researches in the Calculus of Variations—

Part V. The Discrimination of Maxima and Minima Values of Integrals with Arbitrary Values of the Limiting Variations.

Part VI. The Theory of Discontinuous or Compounded Solutions: Mr. E. P. Culverwell.

The Solutions of

$$\sinh\left(\lambda\frac{d}{dx}\right)y=f(x),\quad \cosh\left(\lambda\frac{d}{dx}\right)y=f(x),$$

λ a constant: Mr. F. H. Jackson.

A Theorem in Inequalities: Mr. A. R. Johnson.

Some Properties of Two Tucker Circles: Mr. R. Tucker.

Note on Four Special Circles of Inversion of a System of Generalized Brocard Circles of a Plane Triangle: Mr. J. Griffiths.

On the Order of the Eliminant of Two or More Equations: Dr. R. Lachlan.

The same journals had been subscribed for as in the preceding Session. An additional exchange of *Proceedings* had been made for "Les Annales de la Faculté des Sciences de Marseille."

Messrs. Brickmore and Heppel having been appointed Scrutators, the ballot was taken, with the result that the following gentlemen were elected to serve as the Council for the ensuing Session:—Major MacMahon, R.A., F.R.S., President; Prof. M. J. M. Hill, F.R.S., Mr. A. B. Kempe, F.R.S., and Mr. A. E. H. Love, F.R.S., Vice-Presidents; Dr. J. Larmor, F.R.S., Treasurer; Messrs. M. Jenkins and R. Tucker, Hon. Secretaries. Other Members of the Council:—Mr. A. B. Basset, F.R.S., Mr. G. H. Bryan, Lt.-Col. J. R. Campbell, F.G.S., Lt.-Col. A. J. Cunningham, R.E., Prof. E. B. Elliott, F.R.S., Dr. Glaisher, F.R.S., Prof. Greenhill, F.R.S., Dr. Hobson, F.R.S., and Prof. W. H. H. Hudson.

The new President then took the Chair and called upon Mr. Kempe to read his Valedictory Address, the title of which was "Mathematics."

Prof. Elliott moved the following votes of thanks:—To Mr. Kempe for the manner in which he had discharged the duties of President, and to Mr. Kempe and Mr. Basset for the services they had rendered to the Society in connexion with its Incorporation. He concluded his remarks by expressing the hope that Mr. Kempe would allow his address to be printed in the *Proceedings*. These votes, having been seconded by the President, were put to the meeting, and carried unanimously.

The Treasurer moved that "the thanks of the Society be tendered to the Secretaries, Mr. Jenkins and Mr. Tucker, for their most

assiduous and valuable services to the Society, and especially for their services during the past Session, and that the said vote be entered on the minutes of the meeting." The vote having been seconded by Mr. Kempe, and carried, the Secretaries briefly acknowledged the compliment.

The following communications were made:—

The Kinematics of non-Euclidean Geometry: Prof. W. Burnside.

A Generalized Form of the Hypergeometric Series, and the Differential Equation which is satisfied by the Series: Mr. F. H. Jackson.

Third (and concluding) Memoir on certain Infinite Products: Prof. L. J. Rogers.

The following presents were made to the Library:—

"Mathematical Questions—Reprint," Vol. LXI., 8vo; London, 1894.

"Smithsonian Report," 1892; Washington, 1893.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XVIII., St. 9.

"Nyt Tidsskrift for Mathematik," A. Femte Aargang, Nos. 4, 5; and B. Femte Aargang, No. 1; Copenhagen, 1894.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XII., No. 1; Coimbra, 1894.

"Mathematische Annalen," 45e Band, Sonderabdruck, pp. 410–427.

"Physical Society of London—Proceedings," Vol. XIII., Pt. 1; October, 1894.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. I., No. 1; October, 1894.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Klasse, 1894, No. 3; Göttingen.

"Rendiconti del Circolo Matematico di Palermo," Tomo VIII., Fasc. 5; Settembre, Ottobre, 1894.

"Bulletin des Sciences Mathématiques," Tome XVIII., Juillet, Août, Sept.; Paris, 1894.

"Sitzungsberichte der K. Preuss. Akad. der Wissenschaften zu Berlin," 1894, 24–38.

"Acta Mathematica," XVIII., No. 3; Stockholm, 1894.

"Annali di Matematica," Tomo XXII., Fasc. 4; Milano.

"Atti della Reale Accademia dei Lincei," Serie 5, Rendiconti, Sem. 2, Vol. III., Fasc. 6; Roma, 1894.

"Journal für die reine und angewandte Mathematik," Bd. CXIV., Heft 1; Berlin, 1894.

"Educational Times," November, 1894.

"Annals of Mathematics," Vol. VIII., No. 6; University of Virginia, September, 1894.

"Indian Engineering," Vol. XVI., Nos. 11–15.

*Mathematics.** *By* A. B. Kempe, M.A., F.R.S.

Read November 8th, 1894.

The aim of the London Mathematical Society, at its foundation, was "the promotion and extension of mathematical knowledge," and the recent change in our legal constitution, which has been rendered necessary by the success attending our labours during nearly thirty years, in no way alters that aim. Our Memorandum of Association determines our work for the future to be that which it has been in the past—"the promotion and extension of mathematical knowledge."

As we have never had any difficulty as yet with regard to the limits of our operations, and have been content to have them fixed for the future by a formal document, filed at Somerset House, I hope I am not indiscreet or unduly inquisitive in selecting the present moment to ask—What is mathematical knowledge?

The expression "mathematical knowledge" is no doubt an elastic one, admitting of a very wide interpretation; and we are unquestionably not precluded from publishing in our *Proceedings* communications on the historical, bibliographical, educational, or philosophical aspects of mathematical science. But, however large a view we may take of it, the knowledge which we are empowered to promote and extend must after all have some connexion with mathematics, and—What is mathematics?

The question is not one to which an immediate and satisfactory answer can be obtained without difficulty. Our dictionaries are of course compelled to attempt some reply, and inform us that "mathematics" is "the science of number and magnitude,"† "the science which treats of the properties and relations of quantities,"‡ or, more briefly, "the science of quantity."§ These definitions are doubtless in accordance with others given in quite modern times by men of scientific position. Thus, to give but two examples, we find Dr. John Hopkinson, in an address to the Institute of Civil Engineers, stating in May of the present year, that "mathematics has to deal with all questions into which measurement of relative magnitude enters, with

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* Address delivered on retiring from the office of President.
† Latham's *Dictionary*, 1872.
‡ The *Imperial Dictionary*, 1882.
§ The *Century Dictionary*, 1891.

all questions of position in space, and of accurate determination of shape";* and Mr. Venn, in his well-known *Symbolic Logic*,† explains that, when speaking of "mathematics," he refers, "broadly speaking, to questions of number, magnitude, shape, and position." But such definitions are not intended to be descriptive of mathematics in general. Dr. Hopkinson was speaking of a special aspect of the science, viz., its application to engineering; while Mr. Venn, to use his own words, "is quite aware that some exponents of the higher branches would not admit the exhaustiveness of [t]his description," and merely urges that it will include all that can be meant by the anti-mathematical logicians he is addressing. The dictionary definitions are, however, intended to be general; and as such they can only be regarded as relics of a bygone age.

Occasional definitions of a more comprehensive character are certainly to be found scattered here and there in mathematical and other writings. Thus our first President, Professor De Morgan, in his Inaugural Address to the Society, says that, "space and time are the only necessary matters of thought," and "these form the subject-matter of the mathematics";‡ while Dr. Benjamin Peirce, in his memoir on "Linear Associative Algebra," § 1,‖ defines mathematics as "the science which draws necessary conclusions." I do not propose to discuss the validity of these or other definitions of our study which could be referred to; for, while believing that a collection of the views of mathematicians, logicians, and philosophers, as to what constitutes mathematical science, would be most interesting and instructive, one may, I think, without presumption, hazard a doubt whether the definitions which such a collection would contain would be regarded by a body of modern mathematicians as furnishing a satisfactory reply to the question—What is mathematics?

That the question is one which does not admit of any direct and simple answer, I, for one, am unable to believe; and I have ventured to hope that, by calling attention to the fact that such an answer appears to be wanting, I may stimulate some to attempt to supply it. Let me occupy the few minutes during which I feel justified in detaining you in indicating some considerations which should, as it seems to me, be borne in mind in framing that answer.

In the first place, I would protest against any assumption that the

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* The James Forrest Lecture. See *Nature*, Vol. L., p. 42.
† Page ix.
‡ *Proceedings*, Vol. I., p. 1.
‖ *American Journal of Mathematics*, Vol. IX., p 45.

formation of a definition of mathematics is a matter of purely theoretical interest, with which the expert mathematician has little or no concern. No doubt mathematicians have got on very well hitherto without having a definition of their science at hand. I do not suppose that the majority would be prepared with any better answer to the question—What is mathematics? than that the subject was a somewhat wide one, and they did not see any particular advantage to be gained in finding a concise description of the work of the mathematician which would be sufficiently comprehensive to include all its innumerable ramifications. But, because mathematicians in the past have not seen the need of a compendious definition of their study, it does not follow that such a definition would be of no service to us now. Whether it would or not must depend upon the nature of the definition. A mere statement that "mathematics is the science of quantity, number, shape, and position," could hardly have furthered the progress of mathematics at any period of its history; but one which would direct attention to the nature of the essential elements of mathematical thought, and to the causes to which variety in mathematical properties must be attributed, might reasonably be expected to open out new fields, and to indicate lines of connexion between mathematical subjects hitherto regarded as fundamentally distinct.

In framing our definition it is essential to have regard to recent mathematical work. A definition which might have passed when mathematics was really supposed to be the study of quantity, number, shape, and position only, would nowadays be simply misleading. A survey of the field of modern mathematical thought would satisfy even the most superficial observer that the mathematician of the present day studies much besides those subjects. A good deal may, no doubt, be done by the use of a judicious terminology to preserve the respect due to ancient boundaries, and things which are not quantities may be made to pass as "imaginary" quantities. But the consideration of such subjects as the relations of statements to each other, or the possible forms of algebras, or the theory of substitutions, to take but three examples, can scarcely, by any stretch of the imagination, be said to be the study of either quantity, number, shape, or position; and a large proportion of modern mathematical work is open to the same observation. These recent developments are, however, recognised to be as truly mathematical as their older companions, and no definition of mathematics would be adequate which did not so regard them.

In his excellent review in the *American Journal of Mathematics*, Vol. IX., of Professor Klein's Lectures on the "Icosahedron," Dr. Cole remarks that "a characteristic feature of modern mathematics is the predominant importance of theories, like that of groups of operations, which deal with discontinuous quantities. The theories," he says, "which deal mainly with continuity have retreated decidedly into the background." While venturing to demur to the use of the word "quantity" in this connexion, unless we are to carry our respect for the definition of mathematics as "the science of quantity" so far as to call every mathematical conception a "quantity," I would call attention to Dr. Cole's observation as indicating that mathematicians are beginning to appreciate the fact, which is obvious enough when it comes to be noticed, that they have before them in every mathematical investigation a number of individual conceptions.

The individual conceptions dealt with in different investigations are, obviously, of every conceivable description. To enumerate such examples as quantities, points, curves, spaces, algebras, equations, letters, arrangements, substitutions, rotations, differentials, numbers, statements, and quaternions, conveys but a faint idea of the variety exhibited. The various individuals under consideration are not, of course, jumbled together in a mere confused heap, but bear relations to each other which cause them to exhibit orderly arrangement, to fall into sets, and to compose those different systems of individuals which present themselves in the study of different branches of mathematics. The fact that the subject-matter of mathematical thought consists of a number of individual conceptions appears to me to be a most material consideration to be borne in mind in framing a definition of mathematics.

Nor must we disregard the light thrown by the modern German mathematicians upon the relations between various branches of mathematics. They have shown us that certain mathematical subjects, apparently absolutely distinct, are really connected by the closest ties, and, in the face of their revelations, it is difficult to conceive that any existing division of mathematics into "subjects" should be allowed to influence its definition.

It will hardly be suggested, I think, that any exception should be made in the case of the great division into pure and mixed mathematics. The distinction between subject-matter of mathematical thought which is evolved out of the inner consciousness, and that which is obtained as the result of experience and observation, how-

ever important it may be in other respects, does not correspond to any difference in mathematical properties.

But for our purposes we must not be content with the conclusion that divisions are arbitrary; we must go on to inquire why they are so. What is it that has been shown to unite two subjects apparently distinct? The short reply is that it has been shown that a correspondence exists between the subject-matters considered in the two cases. To each individual thing, relation, or property, which is of mathematical importance in the one case, there corresponds respectively an individual thing, relation, or property, in the other. The individuals considered in the one case may differ widely in character from those considered in the other, and similar diversity may exist between the two sets of relations and properties; but amid all this diversity there is something about each of the systems which causes it to resemble the other, and enables us to establish a correspondence between the two. This something is known as "form." What is it?

In answering this question, let me ask you to take a somewhat general view of the subject-matter of thought. From whatever point of view we regard this, the most prominent feature is probably the combination of variety and uniformity which it exhibits. We picture to ourselves things of every imaginable description, differing in every possible way; but the representation also includes objects, here few in number, there many, which do not differ from each other in any respect whatever. A still greater variety is displayed by the pluralities of the things pictured, the differences between which depend, not only upon the number and peculiarities of the individual objects of which each consists, but also on those additional characteristics, of unlimited diversity, which accompany every plurality, and may be concisely referred to as "relations." Here, again, though difference is present, its absence is equally marked; and pluralities, however complex, are rarely unique, but are accompanied by other pluralities, some by one, some by more, to which they bear an undistinguishable resemblance.

Now putting aside the contemplation of the special peculiarities and characteristics of the individuals and relations which come under our observation in such infinite variety, let us confine our attention to the study of the results which flow from the mere fact that this and that individual, or this and that plurality, differ, while this and that do not. We shall not, as might at first sight be supposed, thereby put away everything which gives life and interest to the

subject-matter of our thought, and leave nothing but a mere heap of dry bones. *Form* remains. The like and unlike individuals and pluralities which are contained in any greater plurality must be distributed in some way through the whole body of individuals composing that greater plurality, and the way in which this distribution is effected gives to the latter a characteristic "form," which may be the same, or may differ, in two pluralities of the same number of individuals. Let me fix the ideas as to this by some simple examples.

Let us take first the angular points a, b, c, d, of a regular tetrahedron. These differ in no respect from each other, and the same observation holds as regards the six pairs of those points, viz.,

$$ab,\ ac,\ ad,\ bc,\ bd,\ cd;$$

and as regards the four triads, viz.,

$$abc,\ abd,\ acd,\ bcd.$$

Compare this state of things with that which we find in the case of the angular points p, q, r, s, of a square, where p and r lie at the extremities of one diagonal, and q and s at the extremities of the other. Here the number of individuals dealt with is the same as before, viz., four. As before, they differ from each other in no respect. But when we come to the pairs, the case alters, for, though the four pairs

$$pq,\ qr,\ rs,\ sp$$

differ in no respect, and the same thing holds as regards the two pairs

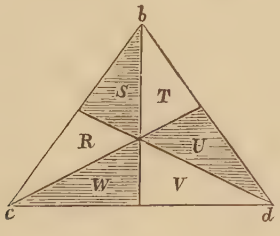
$$pr,\ qs,$$

the first four do differ from the last two. The system composed by the four angular points of a tetrahedron has consequently a different *form*, in the sense in which that word is here used, from that possessed by the system composed by the four angular points of a square.

Again, we might compare the form of the system consisting of the angular points and edges of the tetrahedron with that of the system which consists of the angular points and edges of a regular pentagon. In each case we have to deal with ten individuals; but in the former these divide into two sets, one set containing four individuals (the angular points), and the other six individuals (the edges), the individuals of each set differing in no way from each other, but differing from those in the other set; while in the latter case the division

is into two sets of five individuals, viz., there are five points and five edges. Furthermore, the division into sets of non-differing pairs is quite different in the two cases. In the former, we have five sets of pairs only, consisting respectively of three pairs (of opposite edges), of six pairs (of points), of twelve pairs (of adjacent edges), of twelve pairs (of points and adjacent edges), and of twelve pairs (of points and opposite edges); while, in the latter case, we have seven sets of pairs, consisting respectively of five pairs (of adjacent points), five pairs (of opposite points), five pairs (of adjacent edges), five pairs (of opposite edges), five pairs (of points and opposite edges), ten pairs (of points and adjacent edges), and ten pairs (each consisting of a point and an edge which is neither adjacent to nor opposite it). Without entering upon the consideration of other peculiarities which might be pointed out, it is obvious that the two systems have marked differences of "form."

Let me pass on to give an instance of two systems differing widely in character, but being of the same form. The first of these consists of the four angular points a, b, c, d of our regular tetrahedron, and the twenty-four triangles obtained on the faces of that tetrahedron by joining the angular points to the mid-points of the edges. There will be six of these triangles on each face; one of these faces is shown in the illustration. The twenty-four triangles are of two sorts, there



being twelve of each sort, those of one sort being perversions of those of the other. The three shaded triangles in the illustration are of one sort, the plain ones of the other sort. If, however, we suppose the tetrahedron to admit of being turned inside out, so that the inner sides of the faces can become outer sides, and *vice versâ*, the distinction between the two sorts of triangles will be abolished, and we shall have a system of twenty-eight individuals, four of one sort and twenty-four of another.

The other system is that obtained by taking four independent variables x_1, x_2, x_3, x_4, together with the function $\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4$,

where α, β, γ, δ are all different quantities, and the twenty-three functions obtained from $\alpha x_1+\beta x_2+\gamma x_3+\delta x_4$ by interchanges of the four variables. The form of this system of four variables and twenty-four functions is precisely the same as that of the four angular points and twenty-four triangles of the regular tetrahedron. There is the same distinction of differing and non-differing individuals, pairs, &c., in each case.

This will be more readily appreciated if it be noticed that of the twenty-four triangles there is one, say T, and only one, which is such that

no side of it passes through a,
the longest „ „ „ „ b,
the shortest „ „ „ „ c,
the remaining „ „ „ „ d.

We may therefore represent T by the symbol $(abcd)$, and then the other triangles will be represented by the other like symbols obtained from $(abcd)$ by interchanges of the letters a, b, c, d. Thus we shall have

U represented by $(adcb)$,
V „ „ $(adbc)$,
W „ „ $(acbd)$,
R „ „ $(acdb)$,
and S „ „ $(abdc)$.

The first of the two systems will accordingly be represented by the four letters a, b, c, d, and the twenty-four symbols $(abcd)$, $(bdca)$, &c.

Similarly, we may represent the function $\alpha x_1+\beta x_2+\gamma x_3+\delta x_4$ by the symbol $[x_1x_2x_3x_4]$, and then the other functions will be represented by the other like symbols obtained from $[x_1x_2x_3x_4]$ by interchanges of the letters x_1, x_2, x_3, x_4.

The second of the two systems will accordingly be represented by the four letters x_1, x_2, x_3, x_4, and the twenty-four symbols $[x_1x_2x_3x_4]$, $[x_2x_4x_3x_1]$, &c.

When we observe that, since the four variables x_1, x_2, x_3, x_4 are independent, no distinction is made between them, or between their pairs or triads, so that they compose a system of the same form as the four angular points of the tetrahedron, the correspondence between the two complete systems of twenty-eight individuals becomes pretty obvious.

These illustrations will probably be sufficient to convey a general idea of the nature of "form," and to impress upon you that it owes its existence to the mere presence and absence of difference. It will, I think, be apparent that, simple as are the elements out of which form is constructed, it admits of infinite variety. The possible forms of a system of a given number of individuals are, of course, limited in number, but with the increase of the number of individuals the possibilities of variety mount up with great rapidity.

"Form" thus supplies a limitless field for mathematical investigation; and what I desire next to emphasise is that its study is co-extensive with mathematics itself. If once the broad facts are grasped that the subject-matter of every mathematical investigation consists of a number of individual conceptions; that it must in consequence have a definite "form," in the sense in which that word is here employed; and that, if two subject-matters have the same "form," an exact correspondence may be established between their mathematical properties, however widely they may differ in other respects; then the conviction must inevitably force itself upon us that in considering the mathematical properties of any subject-matter we are merely studying its "form"; and that its other characteristics, except as the means of putting that form in evidence, are, mathematically speaking, wholly irrelevant.

And this conviction will certainly not be weakened by a consideration of the nature of that universal language of symbols, which has been said with truth by Professor Sylvester to be but mathematics itself under another name, viz., algebra.* In the algebraical representation of mathematical facts, we merely strip off all the mathematically irrelevant clothing in which those facts present themselves to us for examination, and reclothe the underlying essential element —form—in a dress more convenient for the purpose of its investigation, viz., one composed of algebraical symbols and formulæ. Thus in the representation of the four angular points of a regular tetrahedron by four letters a, b, c, d, which are not stated to have any special relations to each other, we substitute for a system of four individuals in one dress (that of points), a system of four other individuals (letters) which is of precisely the same "form"; for we make no distinction between a and b and c and d, or between the pairs or triads of those letters. In representing one of the twenty-

* *Messenger of Mathematics*, Vol. IX., p. 83. It need hardly be said that the word "algebra" as here used is to be taken in its widest modern sense.

four triangles by (*abcd*), where the order of the letters is regarded as material, we substitute for a dress of one kind, that of a triangle, another, composed of letters and brackets, which, while representing a single individual, indicates that that individual makes with the four individuals *a*, *b*, *c*, *d* four differing pairs. The representation of another of the individuals by (*bdca*) shows that the pairs which *it* makes with *b*, *d*, *c*, *a*, are respectively of the same sort as those which (*abcd*) makes with *a*, *b*, *c*, *d*.

The form of the system composed by the twenty-four symbols, such as (*abcd*), is readily gathered. Thus the pair (*abcd*), (*badc*) is seen to be undistinguished from the pair (*dcba*), (*cdab*), when we observe that (*badc*) is derived from (*abcd*) by the same interchange of letters as that by which (*cdab*) is derived from (*dcba*).

It would be interesting to analyse different modes of algebraical representation, to point out that they all merely represent "forms," and to indicate the particular species of form represented in different cases, and their peculiar characteristics. That such an analysis would show that algebraical symbols and formulæ in all cases do merely represent "form," there cannot, I think, be any doubt. But this and many other interesting matters with regard to "form," its possibilities and treatment, I cannot touch upon now.* I have, however, I think, said enough to establish at least a *primâ facie* case in favour of the view that the study of mathematical properties is the study of "form," in the sense here understood. If there be any doubt as to the correctness of this view, it will probably be owing to the idea that some particular mathematical property which the doubter has in view does not appear to be accounted for by the consideration of "form" alone. The answer to any such doubt would, I believe, be that there is a failure to appreciate what are the individual things which are being dealt with, and that the system of individuals under consideration is really a larger one than is supposed. There are conceptions involved which have been overlooked, and, when these are taken into account, it will be seen that it is the form of the larger system which is being studied.

If, then, the mathematician in his study of a subject-matter is engaged upon the investigation of its "form," and if "form" is due to the presence and absence of differences between the various indi-

* See "A Memoir on the Theory of Mathematical Form," *Phil. Trans.*, 1886, p. 1, and a Note thereon in the *Proc. Roy. Soc.*, Vol. XLII., p. 193. Also *Proc. Lond. Math. Soc.*, Vol. XXI., p. 147; and *Nature*, Vol. XLIII. (1890), p. 156, "On the Subject-Matter of Exact Thought."

viduals and pluralities of which he conceives that subject-matter to consist, we seem to be within measurable distance of a reply to our question—What is mathematics?

It is notoriously dangerous to attempt to formulate definitions; but, on the other hand, it must not be forgotten that a bad definition may provoke a good one. I would therefore venture provisionally to suggest that:—

Mathematics is the science by which we investigate those characteristics of any subject-matter of thought which are due to the conception that it consists of a number of differing and non-differing individuals and pluralities.

If the result of this attempt be only to elicit conclusive proof that mathematics is something else, and an indication of what it really is, my main object in this brief address will have been attained.

Third Memoir on the Expansion of certain Infinite Products. By L. J. Rogers. Received October 31st, 1894. Read November 8th, 1894.

Abbreviations used in the following Memoir.

(1) $(x) \equiv (1-x)(1-xq)(1-xq^2)\ldots$.

(2) $P(x) \equiv 1/(xe^{\theta i})(xe^{-\theta i}) \equiv 1 + \dfrac{A_1(\theta)}{q_1}x + \dfrac{A_2(\theta)}{q_2!}x^2 + \ldots$, where

$$q_r \equiv 1-q^r, \quad \text{and} \quad q_r! \equiv (1-q)(1-q^2)\ldots(1-q^r).$$

(3) $(-xqe^{\theta i})(-xqe^{-\theta i}) \equiv 1 + \dfrac{B_1(\theta)}{q_1}x + \dfrac{B_2(\theta)}{q_2!}x^2 + \ldots$.

(4) $P(x\lambda)/P(x) \equiv 1 + \dfrac{L_1(\theta)}{q_1}x + \ldots$.

(5) $\lambda_r \equiv 1-\lambda q^{r-1}$, $\mu_r \equiv 1-\mu q^{r-1}$; $\lambda_r! \equiv \lambda_1\lambda_2\ldots\lambda_r$, $\mu_r! \equiv \mu_1\mu_2\ldots\mu_r$, and

$$\lambda_r^2! \equiv (1-\lambda^2)(1-\lambda^2 q)\ldots(1-\lambda^2 q^{r-1}), \text{ \&c.}$$

(6) $H_r(a, b, \ldots / \alpha, \beta, \ldots) \equiv$ coefficient of x^r in $(ax)(bx)\ldots/(\alpha x)(\beta x)\ldots$.

(7) $\phi\{a, b, c, q, x\} \equiv 1 + \dfrac{a_1 b_1}{q_1 c_1}x + \dfrac{a_2! \; b_2!}{q_2! \; c_2!}x^2 + \ldots$.

(8) $f_r \equiv 1 - 2q^{r-\frac{1}{2}}\cos 2\phi + q^{2r-1}$

1. In investigating the properties of the function $A_r(\theta)$, we have hitherto been concerned in establishing relations connecting the coefficients in a supposed identity of the form

$$a_0+a_1A_1(\theta)+a_2A_2(\theta)+\ldots = b_0+2b_1\cos\theta+2b_2\cos 2\theta+\ldots \quad \ldots(1),$$

and applying such relations in a manner so as to obtain various identities involving a single quantity q.

It is easily seen that the method of application (§ 7 of the Second Memoir) implies an identity

$$(q)\left\{1+\frac{qA_2(\theta)}{q_2!}+\frac{q^4A_4(\theta)}{q_4!}+\ldots\right\} = 1+2q\cos 2\theta+2q^4\cos 4\theta+\ldots$$
$$\ldots\ldots\ldots\ldots\ldots(2),$$

and that all the deductions from the subsequent "examples" could have been derived from such an identity.

It is the object of the present memoir to prove identities of type (1) to which we may apply relations of the type

$$a_0+m_1a_1+m_2a_2+\ldots = b_0+n_1b_1+n_2b_2+\ldots,$$

always remembering that we may separate into independent identities those terms which contain even suffixes from those which contain odd.

For this purpose we shall investigate some of the properties of a generalized function similar to $A_r(\theta)$, obtained by expanding the quotient $P(\lambda x)/P(x)$ in powers of x.

Suppose this expansion denoted by

$$1+\frac{L_1(\theta)}{q_1}x+\frac{L_2(\theta)}{q_2!}x^2+\ldots,$$

or by

$$1+\Sigma\frac{L_r}{q_r!}x^r,$$

when the omission of θ involves no ambiguity.

Now, since

$$\frac{P(\lambda x)}{P(x)}(1-2x\cos\theta+x^2) = \frac{P(\lambda xq)}{P(xq)}(1-2\lambda x\cos\theta+\lambda^2x^2),$$

it is easy to see that

$$L_r-2\cos\theta\,.\,L_{r-1}(1-\lambda q^{r-1})+L_{r-2}(1-q^{r-1})(1-\lambda^2q^{r-2}) = 0\ldots(3).$$

We may, moreover, notice a few obvious properties of L_r.

Since, by definition,

$$\lambda_1 + \frac{\lambda_2 L_1}{q_1!}x + \frac{\lambda_3 L_2}{q_2!}x^2 + \ldots = \frac{P(x\lambda)}{P(x)} - \lambda\frac{P(xq\lambda)}{P(x)}$$

$$= (1-\lambda)(1-x^2\lambda)\frac{P(xq\lambda)}{P(x)} \quad \ldots\ldots\ldots(4),$$

we may express L_r in terms of similar coefficients in which λq replaces λ.

The following values of L_r for special values of λ are also worthy of notice.

If $\lambda = 0$, L_r becomes A_r,

$\lambda = q$, $\dfrac{P(\lambda x)}{P(x)}$ becomes $\dfrac{1}{1-2x\cos\theta+x^2}$,

so that $$L_r = \frac{\sin(r+1)\theta}{\sin\theta}q_r!;$$

if $\lambda = 1$, $\dfrac{\lambda_{r+1}L_r}{\lambda_1}$ becomes $2\cos r\theta \,.\, q_r!$, by (4),

$\lambda = \infty$, $\dfrac{L_r}{(-\lambda)^r}$ becomes $\dfrac{B_r}{q^r}$.

Moreover, if $\lambda = q^n$, and q is made equal to unity, then $\dfrac{L_r}{q_r!}$ is the coefficient of x^r in the expansion of $(1-2x\cos\theta+x^2)^{-n}$.

It is easy to obtain the values of L_r by multiplying together the series for $\dfrac{(\lambda x e^{\theta i})}{(x e^{\theta i})}$ and $\dfrac{(\lambda x e^{-\theta i})}{(x e^{-\theta i})}$, and collecting the coefficients of $\dfrac{x_r}{q_r!}$.

It will be seen that L_r can be expressed linearly in terms of $\cos r\theta$, $\cos(r-2)\theta$, &c., the last being $\cos\theta$, or $\cos 0$, according as θ is odd or even.

2. Suppose now that

$$P(\mu x)/P(x) \equiv 1 + \Sigma\frac{M_r}{q_r!}x^r,$$

so that M_r is the same function of μ as L_r is of λ. Then it is clear that M_r may be linearly expressed in terms of L_r, L_{r-2}, ...

Thus, since

$$L_1 = \lambda_1 \,.\, 2\cos\theta, \quad \text{and} \quad L_2 = \lambda_1\lambda_2 \,.\, 2\cos 2\theta + \frac{q_2}{q_1}\lambda_1^2,$$

by direct calculation,

$$\frac{L_1}{q_1} = \frac{M_1}{q_1}\frac{\lambda_1}{\mu_1},$$

$$\frac{L_2}{q_2!} = \frac{M_2}{q_2!}\frac{\lambda_2!}{\mu_2!} + \frac{(\mu-\lambda)\lambda_1}{\mu_1 q_1}.$$

We may derive the general formula for L_r by induction.

Assume

$$\frac{L_r}{q_r!} = \frac{M_r \mu_{r+1}}{q_r!}\frac{\lambda_r!}{\mu_{r+1}!} + \frac{M_{r-2}\mu_{r-1}}{q_{r-2}!}\frac{(\mu-\lambda)\lambda_{r-1}!}{q_1!\,\mu_r!}$$

$$+ \frac{M_{r-4}\mu_{r-3}}{q_{r-4}!}\frac{(\mu-\lambda)(\mu-\lambda q)\lambda_{r-2}!}{q_2!\,\mu_{r-1}!} + \ldots \quad \ldots(1),$$

the general term being

$$\frac{M_{r-2s}\mu_{r-2s+1}}{q_{r-2s}}\frac{(\mu-\lambda)(\mu-\lambda q)\ldots(\mu-\lambda q^{s-1})}{q_s!\,\mu_{r-s+1}!}\lambda_{r-s}!.$$

Using a similar formula for L_{r-1}, and combining it with that above in the expression

$$2\cos\theta\,.\,L_r\lambda_{r+1} - L_{r-1}q_{r+1}(1-\lambda^2 q^{r-1}),$$

which, by § 2 (1), is equal to L_{r+1}, we shall obtain a series of M's and M's multiplied by $2\cos\theta$. However, $2\cos\theta\,.\,M_{r-1}$ may also be transformed by § 2 (1), and we shall finally reduce L_{r+1} to a series of M_{r+1}.

The coefficient of M_{r-2s+1} in $\frac{L_{r+1}}{q_r!}$ is then found to consist in general of three terms, these three reducing to two when

$$r-2s+1 = 0.$$

These will be found, after some reductions and dividing by q_{r+1}, to reduce to the same function of $r+1$ as the above coefficient of M_{r-2s} is of r, thus establishing the identity.

The formula (1) is of great generality, and will be found to be of special importance when the particular values 0, q, 1 are given to λ or μ, since we then obtain expressions for $A_r(\theta)$, $\frac{\sin(r+1)\theta}{\sin\theta}$, $2\cos r\theta$, $L_r(\theta)$, each one in terms of functions of any one other kind.

As a general case, let us expand $P(\lambda x)/P(x)$ in terms of ascending orders of M's. Replacing the L's in $1+\Sigma\frac{L_r}{q_r!}x^r$ by their equivalent series, we get

$$\varphi\left\{\frac{\lambda}{\mu},\ \lambda,\ \mu q,\ q,\ \mu x^2\right\}+\frac{M_1}{q_1}\frac{\lambda_1!}{\mu_1!}x\varphi\left\{\frac{\lambda}{\mu},\ \lambda q,\ \mu q^2,\ q,\ \mu x^2\right\}$$
$$+\frac{M_2}{q_2!}\frac{\lambda_2!}{\mu_2!}x^2\varphi\left\{\frac{\lambda}{\mu},\ \lambda q^2,\ \mu q^3,\ q,\ \mu x^2\right\}+\ldots.$$

If $$\lambda^2x^2=\mu q,$$

the coefficient series may all be expressed as products by Heine's formula

$$\varphi\{a,\ b,\ abx,\ q,\ x\}=\frac{(ax)(bx)}{(abx)(x)},$$

so that, finally,

$$\frac{P(\sqrt{\mu q})}{P(x)}=\frac{(x\sqrt{\mu q})(\mu x\sqrt{\mu q})}{(\mu)(\mu x^2)}\left\{\mu_1+\frac{M_1\mu_2}{q_1!}\frac{x-\sqrt{\mu q}}{1-x\mu\sqrt{\mu q}}\right.$$
$$\left.+\frac{M_2\mu_3}{q_2!}\frac{(x-\sqrt{\mu q})(x-\sqrt{\mu q^3})}{(1-x\mu\sqrt{\mu q})(1-x\mu\sqrt{\mu q^3})}+\ldots\right\}.$$

When $\mu=1$, we get a formula established by Heine

$$\frac{P(q^{\frac{1}{2}})}{P(x)}=\frac{(xq^{\frac{1}{2}})^2}{(q)(x^2)}\left\{1+\frac{x-q^{\frac{1}{2}}}{1-xq^{\frac{1}{2}}}2\cos\theta+\frac{(x-q^{\frac{1}{2}})(x-q^{\frac{3}{2}})}{(1-xq^{\frac{1}{2}})(1-xq^{\frac{3}{2}})}2\cos 2\theta+\ldots\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Again, if $$\lambda x^2=-q,$$

all the Heinean series have product forms, and we get an M-expansion for $P(-q/x)/P(x)$, which, being independent of M, gives us equivalent series in A, $\frac{\sin(r+1)\theta}{\sin\theta}$, and $2\cos r\theta$. We have then an A-series which is equal to a cosine series, and consequently we have a means of employing the formulæ in the second memoir.

We shall, however, make use of a more general formula connecting an A-series with a cosine series, which we now proceed to establish.

3. By putting $\lambda=0$ in § 2, we get an expression for A_r in terms of M. Replacing these in the series

$$1+\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{q_1q_2}A_2+\frac{(u-q^{\frac{1}{2}})(u-q^{\frac{3}{2}})(v-q^{\frac{1}{2}})(v-q^{\frac{3}{2}})}{q_4!}A_4+\ldots\quad\ldots(1),$$

and collecting the coefficients of the several M's, we have

$$\phi\left\{\frac{q^{\frac{1}{2}}}{u},\ \frac{q^{\frac{1}{2}}}{v},\ \mu q,\ q,\ \mu uv\right\}+\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{q_2!\,\mu_3!}\phi\left\{\frac{q^{\frac{3}{2}}}{u},\ \frac{q^{\frac{3}{2}}}{v},\ \mu q^3,\ q,\ \mu uv\right\}M_2$$

$$+\frac{(u-q^{\frac{1}{2}})(u-q^{\frac{3}{2}})(v-q^{\frac{1}{2}})(v-q^{\frac{3}{2}})}{q_4!\,\mu_5!}\phi\left\{\frac{q^{\frac{5}{2}}}{u},\ \frac{q^{\frac{5}{2}}}{v},\ \mu q^5,\ q,\ \mu uv\right\}M_4+\ldots.$$

All these Heinean series are of the form

$$\phi\{a,\ b,\ abx,\ q,\ x\},$$

which $$=(ax)(bx)/(abx)(x),$$

so that (1) finally becomes

$$\frac{(q^{\frac{1}{2}}\mu u)(q^{\frac{1}{2}}\mu v)}{(\mu)(\mu uv)}\left\{\mu_1+\frac{\mu_3 M_2}{q_2!}\,\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-\mu uq^{\frac{1}{2}})(1-\mu vq^{\frac{1}{2}})}\right.$$

$$\left.+\frac{\mu_5 M_4}{q_4!}\,\frac{(u-q^{\frac{1}{2}})(u-q^{\frac{3}{2}})(v-q^{\frac{1}{2}})(v-q^{\frac{3}{2}})}{(1-\mu uq^{\frac{1}{2}})(1-\mu uq^{\frac{3}{2}})(1-\mu vq^{\frac{1}{2}})(1-\mu vq^{\frac{3}{2}})}+\ldots\right\}\ldots(2).$$

If we put $\mu=1$, this latter expression becomes

$$\frac{(q^{\frac{1}{2}}u)(q^{\frac{1}{2}}v)}{(q)(uv)}\left\{1+2\cos 2\theta\,\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-uq^{\frac{1}{2}})(1-vq^{\frac{1}{2}})}+\ldots\right\}\quad\ldots\ldots\ldots(3).$$

Equating the series (1) and (3) together, we have an identity of great generality, to which the formulæ of the second memoir may be applied.

If $$u=v=0,$$

we have the formula § 1 (1), from which the q-identities were obtained. It is also interesting to note that a large number of elliptic functions may be expanded in ascending orders of M, or more particularly A.

If we put $\mu=q$, we have

$$\frac{(q^{\frac{3}{2}}u)(q^{\frac{3}{2}}v)}{(q)(quv)}\,\frac{1}{\sin\theta}\left\{(1-q)\sin\theta+\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-uq^{\frac{1}{2}})(1-vq^{\frac{1}{2}})}(1-q^3)\sin 3\theta+\ldots\right\}$$

$$\ldots\ldots\ldots\ldots(4).$$

Now, if $a_0+a_2A_2(\theta)+\ldots=b_0+2b_2\cos 2\theta+\ldots$

$$=\frac{1}{\sin\theta}(\beta_0\sin\theta+\beta_2\sin 3\theta+\ldots),$$

and any such series as

$$a_0+m_2a_2+m_4a_4+\dots$$

has an equivalent b-series which can be thrown into the form

$$(b_0-b_2)+(b_2-b_4)\,\mu_2+\dots,$$

we obviously get

$$a_0+m_2a_2+m_4a_4+\dots=\beta_0+\mu_2\beta_2+\mu_4\beta_4+\dots \quad \dots\dots\dots(5).$$

From the equality of (1) and (4) we therefore have

$$1+m_2\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{q_1q_2}+m_4\frac{(u-q^{\frac{1}{2}})(u-q^{\frac{3}{2}})(v-q^{\frac{1}{2}})(v-q^{\frac{3}{2}})}{q_4!}+\dots$$

$$=\frac{(q^{\frac{3}{2}}u)(q^{\frac{3}{2}}v)}{(q)(quv)}\left\{q_1+\mu_2\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-uq^{\frac{3}{2}})(1-vq^{\frac{3}{2}})}q_3+\dots\right\} \quad \dots(6).$$

Let $\qquad u=e^{2\phi i} \quad \text{and} \quad v=e^{-2\phi i};$

then, if we write f_r for $1-2q^{r-\frac{1}{2}}\cos 2\phi+q^{2r-1}$, we get

$$1+m_2\frac{f_1}{q_2!}+m_4\frac{f_1f_2}{q_4!}+\dots=\frac{P(q^{\frac{1}{2}},2\phi)}{(q)^2}\left\{\frac{q_1}{f_1}+\mu_2\frac{q_3}{f_3}+\mu_4\frac{q_5}{f_5}+\dots\right\} \quad \dots(7).$$

As specimens of this identity we may quote the following: from p. 322 (12), we have

$$1=\frac{P(q^{\frac{1}{2}},2\phi)}{(q)^2}\left\{\frac{q_1}{f_1}-q\frac{q_3}{f_3}+q^3\frac{q_5}{f_5}-\dots\right\};$$

as is shown by Jacobi, from p. 325 (9), we have

$$1+\Sigma\frac{f_r!\,q^r}{q_{2r}!}=\frac{P(q^{\frac{1}{2}},2\phi)}{(q)^2}\left\{\frac{q_1}{f_1}-q^4\frac{q_5}{f_5}+q^8\frac{q_7}{f_7}-q^{20}\frac{q_{11}}{f_{11}}+\dots\right\};$$

from p. 338, Ex. 1, we have

$$1+\Sigma\frac{f_r!\,q^{r^2}}{q_{2r}!}=\frac{P(q^{\frac{1}{2}},2\phi)}{(q)^2}\left\{\frac{q_1}{f_1}-q^2\frac{q_5}{f_5}+q^4\frac{q_7}{f_7}-q^{10}\frac{q_{11}}{f_{11}}+\dots\right\};$$

from p. 342, we have

$$1+\Sigma\frac{f_r!\,q^r}{q_r!}=\frac{P(q^{\frac{1}{2}},2\phi)}{(q)^2}\left\{\frac{q_1}{f_1}-q^3\frac{q_3}{f_3}+q^9\frac{q_5}{f_5}-q^{18}\frac{q_7}{f_7}+\dots\right\}.$$

In a similar way we may establish the relations

$$\frac{A_1(\theta)}{q_1} + \frac{(u-q)(v-q)}{q_3!} A_3(\theta) + \frac{(u-q)(u-q^2)(v-q)(v-q^2)}{q_5!} A_5(\theta) + \ldots$$

$$= \frac{(uq)(vq)}{(q)(uv)} \left\{ 2\cos\theta + \frac{(u-q)(v-q)}{(1-uq)(1-vq)} 2\cos 3\theta \right.$$

$$\left. + \frac{(u-q)(u-q^2)(v-q)(v-q^2)}{(1-uq)(1-uq^2)(1-vq)(1-vq^2)} 2\cos 5\theta + \ldots \right\} \quad \ldots(8),$$

and deduce many identities from the examples given in the second memoir which connect series of a's and b's with odd suffixes.

4. It will be necessary, in order to establish further properties of $A(\theta)$, to expand the quotient $\phi\{a, b, c, q, x\} / \left(\frac{abx}{c}\right)$ in powers of x, which may be thrown into a very simple form.

If $$\phi\{a, b, c, q, x\} \equiv 1 + a_1 x + a_2 x^2 + \ldots,$$

then we know that

$$(1-q^r)(1-cq^{r-1})\, a_n = (1-aq^{r-1})(1-bq^{r-1})\, a_{n-1},$$

whence, denoting the Heinean series by $\phi(x)$, we have

$$(1-x)\phi(x) - \left(1 - ax - bx + \frac{c}{q}\right)\phi(xq) + \left(\frac{c}{q} - abx\right)\phi(xq^2) = 0.$$

Let $$\phi(x) / \left(\frac{abx}{c}\right) \equiv F(x) \equiv 1 + \beta_1 x + \beta_2 x^2 + \ldots,$$

so that

$$(1-x)\left(1 - \frac{abx}{c}\right) F(x) - \left(1 - ax - bx + \frac{c}{q}\right) F(xq) + \frac{cF}{q}(xq^2) = 0,$$

and by equating the coefficient of x^r to zero

$$(1-q^r)(1-cq^{r-1})\,\beta_r - \left(1 - aq^{r-1} - bq^{r-1} + \frac{ab}{c}\right)\beta_{r-1} + \frac{ab}{c}\beta_{r-2} = 0.$$

Let $$\beta_r = \gamma_r / (1-c)(1-cq) \ldots (1-cq^{r-1}).$$

Then $$(1-q^r)\,\gamma_r - \left(1 - aq^{r-1} - bq^{r-1} + \frac{ab}{c}\right)\gamma_{r-1} + \frac{ab}{c}(1-cq^{r-2})\,\gamma_{r-2} = 0,$$

so that $1+\gamma_1 x+\gamma_2 x^2+\ldots$

is easily seen to $=\dfrac{(ax)(bx)}{(x)\left(\dfrac{abx}{c}\right)}.$

Hence

$$\phi\left\{a, b, c, q, x\right\}/\left(\frac{abx}{c}\right)=1+\Sigma H_r\left(a, b/1, \frac{ab}{c}\right)\frac{x^r}{(1-cq)^{r-1}!}.$$

5. We have seen by § 3 in the first memoir, Vol. XXIV., p. 341, that

$1+\Sigma H_r(x, \lambda e^{\theta i}, \lambda e^{-\theta i})\, A_r(\theta)$

$$=\frac{(\lambda^2)}{(\lambda)^2 P(\lambda, 2\theta)}\,\frac{1}{(xe^{\theta i})}\,\phi\left\{\lambda e^{2\theta i}, \lambda, \lambda^2, q, xe^{-\theta i}\right. \text{ [which, by § 4]}$$

$$=\frac{(\lambda^2)}{(\lambda)^2 P(\lambda, 2\theta)}\left\{1+\Sigma\frac{L_r(\theta)}{q_r!\,(1-\lambda q^{r-1})!}x^r\right\} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

But $1+\Sigma H_r(x, \lambda e^{\theta i}, \lambda e^{-\theta i})\, y^r = 1 \div (xy)\, P(\lambda y),$

by the definition of H,

$$=\left(1+\frac{xy}{1-q}+\frac{x^2y^2}{q_1 q_2}+\ldots\right)\left\{1+\frac{A_1(\theta)}{1-q}\lambda y+\ldots\right\}.$$

Substituting for $H_r(x, \lambda e^{\theta}, \lambda e^{-\theta i})$, from this identity the left side of (1) becomes

$$1+\frac{A_1(\theta)^2}{1-q}\lambda+\frac{A_2(\theta)^2}{q_1 q_2}\lambda^2+\ldots$$
$$+\frac{x}{1-q}\left\{A_1(\theta)+\frac{A_2(\theta)\,A_1(\theta)}{1-q}\lambda+\frac{A_3(\theta)\,A_2(\theta)}{q_2!}\lambda^2+\ldots\right\}$$
$$+\ldots \quad \ldots\ldots\ldots\ldots\ldots(2),$$

the coefficient of $\dfrac{x^r}{q_r!}$ being

$$A_r(\theta)+\frac{A_{r+1}(\theta)\,A_1(\theta)}{1-q}\lambda+\frac{A_{r+2}(\theta)\,A_2(\theta)}{q_2!}\lambda^2+\ldots \quad \ldots(3).$$

This series may be expanded according to ascending orders of A's by means of Vol. XXVI., p. 343, which states that the coefficient of $A_{r+2n}(\theta)$ in $\dfrac{A_{r+s}(\theta)\,A_s(\theta)}{q_s!}$, where $n \not< 0$, is

$$q_{r+s}!/q_{s-n}!\,q_{r+n}!\,q_n!.$$

Hence the coefficient of $A_{r+2n}(\theta)$ in the whole series (3) is

$$\Sigma \frac{\lambda^s q_{r+s}!}{q_{s-n}!\, q_{r+n}!\, q_n!}.$$

Taking r and n as constant and giving s all integral values from n upwards, this becomes

$$\frac{1}{q_n!\, q_{r+n}!}\left\{\lambda^n q_{n+r}! + \lambda^{n+1}\frac{q_{n+r+1}!}{q_1} + \lambda^{n+2}\frac{q_{n+r+2}!}{q_2!} + \dots\right\}$$
$$= \lambda^n/q_n!\,(1-\lambda)(1-\lambda q)\dots(1-\lambda q^{n+r}).$$

Hence the series (3) becomes (with the notation at the beginning of this memoir)

$$\frac{A_r}{\lambda_{r+1}!} + \frac{A_{r+2}}{q_1\lambda_{r+2}!}\lambda + \frac{A_{r+4}}{q_2!\,\lambda_{r+3}!}\lambda^2 + \dots,$$

which, by (1), $$= \frac{(\lambda^2)}{(\lambda)^2 P(\lambda, 2\theta)} \frac{L_r(\theta)}{(1-\lambda^2 q^{r-1})!} \quad \dots\dots\dots\dots\dots\dots (3).$$

We may notice, as a particular case, that

$$\frac{(\lambda^2)}{(\lambda)^2 P(\lambda, 2\theta)}$$
$$= \frac{1}{1-\lambda} + \frac{A_2(\theta)\lambda}{(1-q)(1-\lambda)(1-\lambda q)} + \frac{A_4(\theta)\lambda^2}{q_2!\,(1-\lambda)(1-\lambda q)(1-\lambda q^2)} + \dots.$$

6. These results will now enable us to find an L-expansion of the series

$$1 + \frac{A_1(\theta)\,A_1(\phi)}{1-q}x + \frac{A_2(\theta)\,A_2(\phi)}{q_2!}x^2 + \dots \quad \dots\dots\dots\dots (1).$$

For, by § 2 (1), putting $\lambda = 0$, and writing λ for μ, we have

$$\frac{A_r(\theta)}{q_r!} = \frac{L_r\lambda_{r+1}}{q_r!}\frac{1}{\lambda_{r+1}!} + \frac{L_{r-2}\lambda_{r-1}}{q_{r-2}!}\frac{\lambda}{q_1\lambda_r!} + \frac{L_{r-4}L_{r-3}}{q_{r-4}!}\frac{\lambda^2}{q_2!\,\lambda_{r-1}!} + \dots.$$

Substituting in (1) and rearranging according to ascending orders of $L(\theta)$'s, we have

$$\lambda_1\left\{\frac{1}{\lambda_1} + \frac{A_2(\phi)}{q_1\lambda_2!}\lambda x^2 + \frac{A_4(\phi)}{q_2!\,\lambda_3!}\lambda^2 x^4 + \dots\right\}$$
$$+ \frac{\lambda_2 L_1(\theta)}{q_1!}x\left\{\frac{A_1(\phi)}{\lambda_2!} + \frac{A_3(\theta)}{q_1\lambda_3!}\lambda x^2 + \frac{A_5(\theta)}{q_2!\,\lambda_4!}\lambda^2 x^4 + \dots\right\}$$
$$+ \dots,$$

the coefficient of $\frac{\lambda_r L_r(\theta)}{q_r!} x^r$ being

$$\frac{A_r(\phi)}{\lambda_{r+1}!} + \frac{A_{r+1}(\phi)}{q_1 \lambda_{r+2}!} \lambda x^2 + \frac{A_{r+4}(\phi)}{q_2! \, \lambda_{r+3}!} \lambda^2 x^4 + \ldots \quad \ldots\ldots\ldots\ldots (2).$$

Now, if x were equal to unity, this series would be equal to that on the left side of § 5 (3), but we should obtain a nugatory result, since in this case (1) would be divergent.

We see, however, that (2) will be equal to

$$\frac{(\lambda^2)}{(\lambda)^2 P(\lambda, 2\varphi)} \frac{L_r(\phi)}{(1-\lambda^2 q^{r-1})!} + \text{a multiple of } (x^2-1),$$

so that we may say that

$$1 + \Sigma \frac{A_r(\theta) \, A_r(\phi)}{q_r!} x^r = \text{some multiple of } (x^2-1)$$

$$+ \frac{(\lambda^2)}{(\lambda)^2 P(\lambda, 2\varphi)} \left\{ \lambda_1 + \frac{\lambda_2 L_1(\theta) \, L_1(\phi)}{q_1 (1-\lambda^2)} x + \frac{\lambda_3 L_2(\theta) \, L_2(\phi)}{q_2! \, (1-\lambda^2)(1-\lambda^2 q)} x^2 + \ldots \right\}$$

$$\ldots\ldots\ldots\ldots (3).$$

Since (1) is independent of λ, we get a new form for the right side of (3) by putting $\lambda = q$, which side then becomes, by § 1,

$$\frac{1}{(q) \sin\phi \, . \, P(q, 2\varphi)} \frac{1}{\sin\theta} (\sin\theta \sin\phi + x \sin 2\theta \sin 2\phi + x^2 \sin 3\theta \sin 3\phi + \ldots)$$

$$+ \text{some multiple of } (x^2-1).$$

Although the latter terms are not determined, it is easy to see that, if we apply such examples from the second memoir as were used in § 3 to the relation just obtained, we shall obtain results which are convergent even when $x = 1$, and into which these unknown terms will not enter.

We may virtually say then, for the purpose of utilizing those results of the second memoir which are applicable as in § 3 to a given relation

$$a_0 + a_2 A_2(\theta) + \ldots = \frac{1}{\sin\theta} (\beta_0 \sin\theta + \beta_2 \sin 3\theta + \ldots),$$

or $$a_1 A_1(\theta) + a_3 A_3(\theta) + \ldots = \frac{1}{\sin\theta} (\beta_1 \sin 2\theta + \beta_3 \sin 4\theta + \ldots),$$

that we have such a relation in the equation

$$1+\frac{A_1(\theta)\,A_1(\phi)}{q_1}+\frac{A_2(\theta)\,A_2(\phi)}{q_2!}+\ldots$$

$$=\frac{1}{(q)\sin\phi\,.\,P(q,\,2\phi)}\,\frac{1}{\sin\theta}\,(\sin\theta\sin\phi+\sin 2\theta\sin 2\phi+\ldots)$$

$$=\frac{1}{\sin\theta}\,\frac{\sin\theta\sin\phi+\sin 2\theta\sin 2\phi+\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

since $$2q^{\frac{1}{4}}(q)\sin\phi\,.\,P(q,\,2\phi)=\vartheta_1(\phi,\,q^{\frac{1}{2}}).$$

From p. 338, Ex. 1, we have

$$1+\frac{qA_3(\phi)}{q_2!}+\frac{q^4A_4(\phi)}{q_4!}+\ldots$$

$$=\frac{\sin\phi-q^2\sin 5\phi+q^4\sin 7\phi-q^{10}\sin 11\phi+\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots}\quad\ldots(5).$$

But we have already seen in § 3 that this A-series

$$=\frac{1}{(q)}\,\vartheta_3(\varphi).$$

Hence $$2q^{\frac{1}{4}}(\sin\varphi-q^2\sin 5\phi+q^4\sin 7\phi-q^{10}\sin 11\phi+\ldots)$$

$$=\frac{1}{(q)}\,\vartheta_3(\phi)\,\vartheta_1(\phi,\,q^{\frac{1}{2}}).$$

From the alternate terms in the same identity, we have

$$\frac{A_1(\phi)}{q_1}+\frac{q^2A_3(\phi)}{q_3!}+\frac{q^6A_5(\phi)}{q_5!}+\ldots=\frac{\sin 2\phi-q\sin 4\phi+q^5\sin 6\phi-\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots}.$$

But, by putting $u=v=0$ in § 3 (8), this left side is seen to be

$$\frac{1}{(q)}\,\{2\cos\phi+2q^2\cos 3\phi+2q^6\cos 5\phi+\ldots\};$$

therefore

$$2q^{\frac{3}{4}}(\sin 2\phi-q\sin 4\phi+q^5\sin 6\phi-\ldots)=\frac{1}{(q)}\,\vartheta_2(\phi)\,\vartheta_1(\phi,\,q^{\frac{1}{2}}).$$

From p. 337, Ex. 4,

$$1+\frac{qA_2(\varphi)}{q_2}+\frac{q^2A_4(\varphi)}{q_2q_4}+\ldots=\frac{\sin\phi-q^2\sin 3\phi+q^6\sin 5\phi-\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots}.$$

From p. 342,

$$1+\frac{qA_2(\phi)}{q_1}+\frac{q^2A_4(\phi)}{q_2!}+\ldots=\frac{\sin\phi-q^3\sin 3\phi+q^9\sin 5\phi-\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots},$$

and

$$1+\frac{q(1-q^{\frac{1}{2}})A_2(\phi)}{q_2!}+\frac{q^2(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})}{q_4!}A_4(\phi)+\ldots$$

$$=\frac{\sin\phi-q^{\frac{3}{2}}\sin 3\phi+q^{\frac{9}{2}}\sin 5\phi-\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots}.$$

From p. 339, Ex. 2,

$$1-\frac{q^{\frac{3}{2}}(1-q^{\frac{1}{2}})}{q_2!}A_2(\phi)+\frac{q^2(1-q^{\frac{1}{2}})(1-q^{\frac{3}{2}})}{q_4!}A_4(\phi)-\ldots$$

$$=\frac{\sin\phi-q^{\frac{1}{2}}\sin 3\phi+q^{\frac{5}{2}}\sin 5\phi-\ldots}{\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots}.$$

7. The formula § 2 (2) yields, on putting $x=q^{n+\frac{1}{2}}$, and changing θ into 2ϕ,

$$f_1f_3\ldots f_{2n-1}=\frac{q_{2n}!}{q_n!\,q_n!}\left\{1-\frac{q_n}{q_{n+1}}2q^{\frac{1}{2}}\cos 2\phi+\frac{q_nq_{n-1}}{q_{n+1}q_{n+2}}2q^2\cos 4\phi-\ldots\right\}.$$

But, since

$$A_{2n}(\theta)=\frac{q_{2n}!}{q_n!\,q_n!}\left\{1+\frac{q_n}{q_{n+1}}2\cos 2\theta+\frac{q_nq_{n-1}}{q_{n+1}q_{n+2}}2\cos 4\theta+\ldots\right\},$$

we see that the above product of the n f's is obtained from $A_{2n}(\theta)$, by writing $2q^{\frac{1}{2}r^2}(-1)^r\cos 2r\phi$ instead of $2\cos 2r\theta$. Hence, if

$$a_0+a_2A_2(\theta)+\ldots,=b_0+2b_2\cos 2\theta+\ldots,$$

we have $a_0+a_2f_1+a_4f_1f_3+\ldots=b_0-2b_2q^{\frac{1}{2}}\cos 2\phi+2b_4q^2\cos 4\phi-\ldots$

.....(1),

which is a formula of the type investigated in the second memoir, and from which many there obtained may be easily derived by giving ϕ special values.

For instance, if $\cos 2\phi=0$, we have Ex. 4 on p. 341;

if $\phi=0$, „ Ex. 3 on p. 339;

if $e^{2\phi i}=q^{\frac{1}{2}}$, „ (12) on p. 322.

Moreover, from §§ 3 and 6 in the present memoir, we have from formulæ in the second memoir altered as in § 3 to the form

$$a_0+m_2a_2+\ldots=\beta_0+\mu_2\beta_2+\mu_4\beta_4+\ldots,$$

obtained relations of the type

$$1+\frac{m_2f_1}{q_2!}+\frac{m_4f_1f_3}{q_4!}+\ldots=\frac{P(q^{\frac{1}{2}},\phi)}{(q)^2}\left\{\frac{q_1}{f_1}+\mu_2\frac{q_3}{f_3}+\ldots\right\}$$

and $$1+\frac{m_2A_2(\varphi)}{q_2!}+\frac{m_4A_4(\phi)}{q_4!}+\ldots=\frac{\sin\phi+\mu_2\sin 3\phi+\ldots}{\sin\phi-q\sin 3\varphi+q^3\sin 5\varphi-\ldots}.$$

Hence, by (1), we see that, if

$$(\sin\phi-q\sin 3\phi+q^3\sin 5\phi-\ldots)(1+2c_2\cos 2\phi+2c_4\cos 4\varphi+\ldots)$$
$$=\sin\phi+\mu_2\sin 3\phi+\mu_4\sin 5\phi+\ldots,$$

then the coefficients c and μ are so related that we also have

$$1-2c_2q^{\frac{1}{2}}\cos 2\phi+2c_4q^2\cos 4\phi-\ldots$$
$$=\frac{(q)^2}{P(q^{\frac{1}{2}},2\varphi)}\left\{\frac{q_1}{f_1}+\mu_2\frac{q_3}{f_3}+\mu_4\frac{q_5}{f_5}+\ldots\right\}.$$

We may note, moreover, that, by § 3 (3), and by (1) in the present section,

$$1+\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})(1-2q^{\frac{1}{2}}\cos 2\phi+q)}{q_2!}+\ldots$$
$$=\frac{(q^{\frac{1}{2}}u)(q^{\frac{1}{2}}v)}{(q)(uv)}\left\{1-2q^{\frac{1}{2}}\cos 2\phi\frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-uq^{\frac{1}{2}})(1-vq^{\frac{1}{2}})}+\ldots\right\}.$$

If $u=0$, $v=1$, we have

$$1-\frac{q^{\frac{1}{2}}(1-q^{\frac{1}{2}})f_1}{q_2!}+\frac{q^2(1-q^1)f_1f_3}{q_4!}-\ldots$$
$$=\frac{(q^{\frac{3}{2}})}{(q)}(1-2q\cos 2\phi+2q^4\cos 4\phi-\ldots).$$

In addition to the above formulæ, it is worth while mentioning that we may obtain a very similar set of identities by considering terms which contain odd suffixes in a, A, m, b, &c., but it is scarcely necessary to enter fully into these relations.

8. If $l_0+l_1L_1(\theta)+l_2L_2(\theta)+\ldots=m_0+m_1M_1(\theta)+m_2M_2(\theta)+\ldots,$

we can easily obtain m_0 in terms of the l's, by substituting for the

L's by § 2, and equating coefficients of the various M's. Thus

$$\frac{m_0}{\mu_1} = \frac{l_0}{\mu_1} + q_2! \frac{(\mu-\lambda)\,\lambda_1}{\mu_2!\; q_1} l_2 + q_4! \frac{(\mu-\lambda)(\mu-\lambda q)\,\lambda_2!}{\mu_3!\; q_2!} l_4 + \ldots .$$

Applying this to § 3 (2), which is, of course, equal to the same function of λ's and L's, we get

$$\frac{(q^{\frac{1}{2}}\mu u)(q^{\frac{1}{2}}\mu v)(\lambda)(\lambda uv)}{(q^{\frac{1}{2}}\lambda u)(q^{\frac{1}{2}}\lambda v)(\mu)(\mu uv)} = \frac{\lambda_1}{\mu_1} + \frac{(u-q^{\frac{1}{2}})(v-q^{\frac{1}{2}})}{(1-uq^{\frac{1}{2}})(v-q^{\frac{1}{2}})} \frac{(\mu-\lambda)\,\lambda_1}{\mu_2!\; q_1} \lambda_3$$

$$+ \frac{(u-q^{\frac{1}{2}})(u-q^{\frac{3}{2}})(v-q^{\frac{1}{2}})(v-q^{\frac{3}{2}})}{(1-uq^{\frac{1}{2}})(1-uq^{\frac{3}{2}})(1-vq^{\frac{1}{2}})(1-vq^{\frac{3}{2}})} \frac{(\mu-\lambda)(\mu-\lambda q)\,\lambda_2!}{\mu_3!\; q_2!} \lambda_5 + \ldots,$$

while, from § 6 (3), we get

$$\frac{(\mu^2)(\lambda)^2\, P(\lambda, 2\phi)}{(\mu)^2\,(\lambda^2)\, P(\mu, 2\phi)} = \frac{\lambda_1}{\mu_1} + \frac{\lambda_3 L_2(\phi)}{(1-\lambda^2)(1-\lambda^2 q)} \frac{(\mu-\lambda)\,\lambda_1!}{\mu_2!\; q_1}$$

$$+ \frac{\lambda_5 L_4(\phi)}{(1-\lambda^2 q^3)!} \frac{(\mu-\lambda)(\mu-\lambda q)\,\lambda_2!}{\mu_3!\; q_2!} + \ldots .$$

9. In conclusion, it is interesting to show that there is a formula giving an expression for the product of $L_r(\theta)$ and $L_s(\theta)$, corresponding to and including that given in Vol. XXIV., p. 343, for the product of $A_r(\theta)\, A_s(\theta)$. This formula is

$$\frac{L_s}{q_s!} \frac{L_r}{q_r!} = \frac{\lambda_{r+s+1} L_{r+s}}{\lambda^2_{r+s}!} \frac{\lambda_r!\; \lambda^2_{r+s}!}{q_r!\; \lambda_{r+s+1}!} + \frac{\lambda_{r+s-1} L_{r+s-2}}{\lambda^2_{r+s-2}!} \frac{\lambda_1!\; \lambda_{s-1}!}{q_1!\; q_{s-1}!} \frac{\lambda_{r-1}!\; \lambda^2_{r+s-1}!}{q_{r-1}!\; \lambda_{r+s}!} + \ldots,$$

the general term being

$$\frac{\lambda_{r+s-2t+1} L_{r+s-2t}}{\lambda^2_{r+s-2t}!} \frac{\lambda_{s-t}!\; \lambda_r!\; \lambda_{r-t}!\; \lambda^2_{r+s-t}!}{q_{s-t}!\; q_t!\; q_{r-t}!\; \lambda_{r+s-t+1}!} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $t = 0, 1, 2, \ldots s$, and $s \not> r$.

This may be proved by induction, though the process is somewhat tedious. The formula is easily seen to be true when $s = 1$, reducing in fact merely to the relation (3) in § 1.

Assuming (1) above, we may deduce the corresponding formula for $L_{s+1} L_r$ by multiplying each side of (1) by L_1, and replacing $L_1 L_s$ on the left side, and $L_1 L_{r+s-2t}$ on the right, by terms containing single L's. In this way the products $L_{s+1} L_r$ and $L_{s-1} L_r$ remain on the left, and, when the latter has been replaced by the formula

similar to (1), we get after considerable reduction a new similar formula for $L_{s+1}L_r$.

In this way, from the known formula for L_0L_r and L_1L_r, we may obtain all the required formulæ up to L_r^2, and thus obtain the general result. This will, of course, give A-product and B-product formulæ by putting $\lambda = 0$ and $\lambda = \infty$. The purely trigonometrical formulæ got from $\lambda = 1$ or q require no special notice. The most interesting application of the formulæ, however, is to zonal harmonics, since it gives a general formula for expressing the product of two zonal harmonics as a linear function of the same.

Let $\lambda = q^l$, and afterwards put $q = 1$. Then

$$\frac{P(\lambda x)}{P(x)} = (1-2x\cos\theta+x^2)^{-l},$$

where primarily n is a positive integer,

$$\frac{\lambda_r!}{q_r!} = \frac{l^{(r)}}{r!},$$

where $$l^{(r)} \equiv l(l+1)\ldots(l+r-1),$$

and $$\frac{\lambda_r^2!}{q_r!} = \frac{(2l)^{(r)}}{r!}.$$

By putting $\mu = q^m$, $P(\mu x)/P(x)$ becomes $(1-2x\cos\theta+x^2)^{-n}$, where m is a positive integer, and the formula § 2 (1) connects the coefficients of x^r in $(1-2x\cos\theta+x^2)^{-l}$ with those of $x^r, x^{r-2}, \ldots$ in $(1-2x\cos\theta+x^2)^{-m}$. Just as in the binomial theorem, however, it is not necessary to restrict the values of l and m to positive integers, and we may therefore suppose the resulting formula to hold good for all real values of l and m.

In the case where $m = \frac{1}{2}$, the latter expansion is

$$1+P_1(\theta)\,x+P_2(\theta)\,x^2+\ldots,$$

where $P_r(\theta)$ is the zonal harmonic of order r, so that we have hereby a means of expanding $(1-2x\cos\theta+x^2)^{-l}$ in zonal harmonics. Since, when $q=1$, the Heinean series $\varphi\{a, b, c, q, x\}$ becomes the hypergeometric series $F\{\alpha, \beta, \gamma, x\}$, by writing $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$, and evaluating, we see that the coefficient of $P_r(\theta)$ in the expansion of $(1-2x\cos\theta+x^2)^{-l}$ is

$$\frac{l(l+1)\ldots(l+r-1)}{1\cdot3\cdot5\ldots(2r-1)}\,2^r F\left\{l-\tfrac{1}{2},\ l+r,\ \frac{2r+3}{2},\ x^2\right\}.$$

10. The formula § 9 (1), will also afford a means of expanding the product $P(\lambda y)\,P(\lambda z)\,/P(y)\,P(z)$ according to ascending orders of L's.

We have, in fact, to find the coefficient of $L_n(\theta)$ in the product of

$$\left(1+\frac{L_1}{q_1}y+\ldots\right)\left(1+\frac{L_1}{q_1}z+\ldots\right),$$

after reducing any term $\dfrac{L_r L_s}{q_r!\,q_s!}y^r z^s$ by § 9 (1).

Now, $L_n(\theta)$ can only be derived from such products as

$$y^m z^m \frac{L_{m+n}L_m}{q_{m+n}!\,q_m!}y^n,\quad y^m z^m \frac{L_{m+n-1}L_{m+1}}{q_{m+n-1}!\,q_{m+1}}y^{n-1}z,\ \ldots,$$

and in general $$y^m z^m \frac{L_{m+n+p}L_{m+p}}{q_{m+n-p}!\,q_{m+p}!}y^{n-p}z^p,$$

where $$p = 0,\ 1,\ \ldots\ n\,;$$

it occurring as the $(p+1)^{\text{th}}$ term from the end in the expansion of this general expression by § 9 (1).

We will first keep m constant while p runs through all its values.

The coefficient of L_n in $\dfrac{L_{m+n-p}L_{m+p}}{q_{m+n-p}!\,q_{m+p}!}$ is got by writing

$$s = m+n-p,$$

$$r = m+p,$$

$$r+s-2t = n,$$

so that $$r+s-t = m+n,$$

$$t = m\,;$$

while the coefficient takes the form

$$\frac{\lambda_{n+1}}{\lambda_n^2!}\,\frac{\lambda_{n-p}!\,\lambda_m!\,\lambda_p!\,\lambda_{m+n}^2!}{q_{n-p}!\,q_m!\,q_p!\,\lambda_{m+n+1}!}.$$

Hence, giving p its $(n+1)$ values, we see that this group of terms containing L_n may be written

$$y^m z^m L_n(\theta)\,\frac{\lambda_{n+1}\lambda_m!\,\lambda_{m+n}^2!}{\lambda_n^2!\,\lambda_{m+n+1}!\,q_m!}\left\{\frac{\lambda_n!}{q_n!}y^n+\frac{\lambda_{n-1}!\,\lambda_1}{q_{n-1}!\,q_1}y^{n-1}z+\ldots\right\}.$$

The bracketed series may be simplified by writing

$$y = xe^{\phi i}, \quad z = xe^{-\phi i},$$

in which case it evidently becomes

$$x^n \frac{L_n(\phi)}{q_n!}.$$

We now have to give m all values from 0 to ∞, and we easily see that the whole term containing $L_n(\theta)$ is

$$x^n \frac{L_n(\theta)\, L_n(\phi)}{q_n!} \sum_{\infty}^{m=0} \left\{ \frac{\lambda_{n+1}\, \lambda_m!\, \lambda_{m+n}^2!}{\lambda_n^2!\, \lambda_{m+n+1}!\, q_m!} x^{2m} \right\}$$

$$= x^n \frac{L_n(\theta)\, L_n(\phi)}{q_n!\, \lambda_n!}\, \phi\{\lambda, \lambda^2 q^n, \lambda q^{n+1}, q, x^2\}.$$

Thus $$P(\lambda x e^{\phi i})\, P(\lambda x e^{-\phi i}) / P(x e^{\phi i})\, P(x e^{-\phi i})$$

$$= \phi\{\lambda, \lambda^2, \lambda q, q, x^2\} + x \frac{L_1(\theta)\, L_1(\phi)}{q_1 \lambda_1} \phi\{\lambda, \lambda^2 q, \lambda q^2, q, x^2\}$$

$$+ x^2 \frac{L_2(\theta)\, L_2(\phi)}{q_2!\, \lambda_2!} \phi\{\lambda, \lambda^2 q^2, \lambda q^3, q, x^2\}$$

$$+ \ldots .$$

By the methods of the last section, we get the following zonal harmonic expansion :—

$$\{1 - 2x\cos(\theta - \phi) + x^2\}^{-\frac{1}{2}} \{1 - 2x\cos(\theta + \phi) + x^2\}^{-\frac{1}{2}}$$

$$= F\left\{\tfrac{1}{2}, 1, \tfrac{3}{2}, x^2\right\} + x P_1(\theta)\, P_1(\phi)\, \tfrac{2}{1} F\left\{\tfrac{1}{2}, 2, \tfrac{5}{2}, x^2\right\}$$

$$+ x^2 P_2(\theta)\, P_2(\phi)\, \frac{2.4}{1.3} F\left\{\tfrac{1}{2}, 3, \tfrac{7}{2}, x^2\right\}$$

$$+ \ldots .$$

On the Kinematics of non-Euclidean Space. *By* Prof. W. Burnside.

Received November 7th, 1894. Read November 8th, 1894.

I.

In a note in Vol. xix of the *Messenger of Mathematics*, "On the Resultant of Two Finite Displacements of a Rigid Body," I have shown that a geometrical construction there given is applicable to non-Euclidean space. The construction, or rather its proof, is materially simplified in a note with a similar title in Vol. xxiii of the same journal, but the phraseology used in this second note is wholly that of ordinary space. It is not, I believe, generally known how simply the kinematics of non-Euclidean space may be treated by the methods of ordinary synthetic geometry; and it is my object in the first part of the present paper, by reproducing in a quite general form the construction above referred to, and by applying it to the deduction of certain kinematical theorems, to bring this out clearly.

The elementary geometry of hyperbolic space has been treated in detail by Herr J. Frischauf (*Elemente der Absoluten Geometrie*, Leipzig, 1876), while the leading theorems in the elementary geometry of elliptic space have been given by Mr. S. Newcomb (*Crelle's Journal*, Vol. lxxxiii). Reference may also be made to an address by Prof. G. Chrystal to the Royal Society of Edinburgh (*Proc. R.S.E.*, 1879–80), in which the leading results of Frischauf and Newcomb are partly summarized and partly treated independently. The more elementary of the results obtained by these authors have been assumed as known in the present paper. The distinction between the single and double elliptic spaces,* that is, between spaces in which two straight lines in a plane always meet in one or in two points respectively, is of course taken account of where necessary; but most of the results hold equally well for either. There is, however, a fundamental kinematical distinction between the two cases, which may be stated here, as it is given by Prof. Chrystal in the address above referred to. The distinction is that,

* In his recent writings Prof. Klein uses the terms "elliptic" and "spherical" space for what are here called "single" and "double" elliptic spaces.

while a translation along a complete straight line in single elliptic space is equivalent to a rotation through two right angles round the line, in double elliptic space it is equivalent to no displacement at all.

In what follows, one of two finite straight lines AB, $A'B'$ is continually spoken of as equal to, greater than, or less than the other. It is to be observed that this does not involve the assumption of any such analytical system of measurement as may be used in ordinary space. (It is, in fact, one of the objects of this paper to show how the appropriate metrical systems of elliptic and hyperbolic space may be deduced from purely synthetical considerations.) The test is one of congruency; namely, the point A may be made to coincide with the point A', and the line AB to lie along the line $A'B'$, and it will then be obvious on inspection whether AB is equal to, greater than, or less than $A'B'$. In the same way, it is clear that a test of congruency can be applied to determine whether two intersecting lines AB, AC are or are not at right angles.

The following definitions are introduced to avoid any possible ambiguity.

A *motion* is a displacement of the points of space such that all congruent figures remain congruent. The word "displacement" without qualification is, however, generally here used for "motion" as just defined.

A *rotation* is a motion in which all the points of one straight line are undisplaced. A rotation through two right angles is for shortness called a *half-turn*.

A *translation* is a motion in which a straight line and the two parts into which it divides, or appears to divide, any plane passing through it are respectively displaced into themselves.

Lemma I.—At least one straight line can in general be drawn to meet any two given lines at right angles; and in hyperbolic space there is never more than one such straight line.

If the two straight lines intersect, the line through their point of intersection perpendicular to their plane is a line meeting them both at right angles. Suppose now that the space is hyperbolic, and that AB, AC are the two straight lines. Then, if BC were a straight line meeting AB and AC at right angles, and if A is a finite point, ABC would be a rectilinear triangle, the sum of whose angles is greater than two right angles; while, if A is a point at infinity, then ABC would be a rectilinear triangle whose area is not infinitesimal, while the sum of its angles is equal to two right angles. Neither of these

results is possible (*cf.* Chrystal, *loc. cit.*), and therefore no such line as BC exists. It is to be noticed that, if AB, AC meet at infinity, the line meeting them both at right angles cannot actually be drawn.

Suppose next that the two straight lines AB, CD do not intersect. From every point P of AB draw lines Pp in every possible direction perpendicular to AB, and such that each of them can be brought to congruence with a given finite straight line.

The locus of the extremities of these lines will be called an equidistant surface of AB, and Pp, ... will be called its radii. The equidistant surface remains congruent with itself for all translations along and rotations round its axis AB, and it must therefore be a continuous surface with a definite tangent plane at every point, while the radius through any point is perpendicular to the tangent plane at it. If now the radius to the equidistant round AB is sufficiently small, then, since, by supposition, AB and CD are non-intersecting lines, CD must lie entirely outside (*i.e.*, on the opposite side to AB) of the equidistant. Hence, when the radius is taken larger and larger, there will be some definite finite value of the radius for which CD first meets the equidistant. If the point in which CD first meets an equidistant is a finite point, it necessarily touches it at this point, and the radius to the equidistant through the point is a straight line meeting AB and CD at right angles. If, now, a second equidistant touch CD, then CD is necessarily a line returning into itself, and the space must therefore be elliptic. Hence, again, in this case, not more than one line can be drawn in hyperbolic space to meet two given lines at right angles.

Suppose, secondly, if possible, that CD first meets an equidistant at infinity, so that the space is necessarily hyperbolic. If AB, CD are not in the same plane, this is clearly impossible. For, if from points taken further and further along CD perpendiculars be let fall on the plane ABC, these perpendiculars increase without limit, and, *a fortiori*, the same must be true of the perpendiculars let fall on AB. If, on the other hand, AB, CD are in the same plane, and if through any point C of CD the lines CD', CD'' be drawn to meet AB at infinity, then CD must make with one or the other of these lines an angle less than any finite angle, or, in other words, CD must coincide with either CD' or CD''. Hence this second possibility reduces to the previously considered case in which the two lines meet at infinity.

Lemma II.—If AB be any straight line, and Aa, Bb two straight

lines in the same plane, both of which are at right angles to AB, then successive half-turns round Aa and Bb are equivalent to a translation $2AB$ along AB. Let P be any point in BA produced, and take P' and Q in this line, so that PA and AP', and also $P'B$ and BQ, can be respectively brought to congruence. Then $2AB$ is congruent with PQ. Now the half-turn round Aa brings P to P', and the half-turn round Bb brings P' to Q, so that the two half-turns displace every point of AB through a distance congruent with $2AB$ along AB, while the two halves of the plane $aABb$ on either side of AB are displaced each into itself. The resultant displacement is therefore a translation $2AB$ along AB.

Lemma III.—If Oa, Ob are any two intersecting lines, and if cOc' is perpendicular to both of them, successive half-turns round Oa and Ob are equivalent to a rotation $2aOb$ round cOc'.

Let OP be any line through O in the plane aOb, and take OP', OQ such that the angles POa, aOP' and the angles $P'Ob$, bOQ are respectively equal. The half-turn round Oa changes OP to OP', and Oc to Oc', and the half-turn round Ob changes OP' to OQ and Oc' to Oc. The successive half-turns therefore keep Oc undisplaced and change OP into OQ, and it is evident that the angle POQ is equal to $2aOb$.

Lemma IV.—If Aa, Bb are any two lines, and AB is a line meeting them both at right angles, successive half-turns round Aa and Bb are equivalent to a translation $2AB$ along AB, and a rotation round AB through twice the angle between the planes aAB and ABb.

Through B draw Bb' in the plane aAB perpendicular to AB. Then successive half-turns round Aa and Bb are equivalent to successive half-turns round Aa and Bb' followed by successive half-turns round Bb' and Bb.

The first displacement is, by Lemma II., the same as a translation $2AB$ along AB, and the second, by Lemma III., is the same as a rotation through $2b'Bb$, *i.e.*, through twice the angle between the planes aAB and ABb, round AB.

It is obvious that the resultant of these last two displacements is independent of their order.

These lemmas lead to a very simple construction for the resultant of any given finite displacements. Suppose first that two displacements each consisting of a translation along and a rotation round a given line are to be compounded; and let $a'Aa$, $b'Bb$ be the axes of

the given displacements. Take AB* a line meeting both axes at right angles, and in $a'Aa$ take a' such that half the translation along $a'Aa$ will bring a' to A; then through a' draw $a'\alpha$ perpendicular to $a'Aa$, such that half the rotation round $a'Aa$ brings the plane $\alpha a'A$ into the position $a'AB$. The first displacement is then equivalent to half-turns round $a'\alpha$ and AB, by Lemma IV.; and in the same way $b\beta$ may be constructed meeting $b'Bb$ at right angles, such that the second displacement is equivalent to successive half-turns round AB and $b\beta$. The resultant displacement is therefore equivalent to successive half-turns about $a'\alpha$ and $b\beta$; and, if $\alpha\beta$ be a line meeting these two at right angles, the resultant displacement is the same as a translation $2\alpha\beta$ along $\alpha\beta$ and a rotation round $\alpha\beta$ through twice the angle between the planes $a'\alpha\beta$ and $\alpha\beta b$. Any number of successive displacements may now be compounded in this way, and the axis, translation, and rotation of the resultant screw-motion so determined.

As was pointed out in the introduction, these constructions hold equally well for non-Euclidean as for Euclidean space; but the nature of the displacement arising by compounding two given displacements depends obviously upon the geometrical relations of the lines denoted by aa', $a'A$, AB, Bb, $b\beta$, above.

When the two displacements are translations and the space Euclidean, the resultant displacement is again a translation. Suppose now that the space is hyperbolic, that is, that the two points at infinity on every straight line are real and distinct, and that the axes of the two translations do not lie in the same plane. If the resulting displacement were a translation, it would be necessary that $a'\alpha$ and $b\beta$ should lie in the same plane, but it may be easily shown that this is impossible.

Thus, if the planes through A and B perpendicular to AB be spoken of for a moment as the planes P and Q, $a'\alpha$, and therefore every plane passing through it, is at right angles to the plane P, while $b\beta$ is at right angles to the plane Q. Now, if $a'\alpha$, $b\beta$ lie in a plane, then $b\beta$, being the line of intersection of the planes $a'\alpha b\beta$ and $ABb\beta$, both of which are perpendicular to the plane P, is itself perpendicular to the plane P; and there are therefore two common perpendiculars to the lines P and Q. But this is in contradiction to the fact that in hyperbolic space one line only can be drawn to meet two given lines at right angles.

* The exceptional case in which the points A and B lie at infinity is dealt with at the beginning of Section III.

Hence in hyperbolic space the resultant of two translations along axes that do not lie in the same plane is never a translation.

If the axes of the two translations lie in a plane and meet, their resultant is equivalent to two half-turns about axes perpendicular to the plane, and is thus always a translation whose axis is in the same plane as the given axes.

If the axes lie in a plane and do not meet, the resultant displacement is equivalent to two half-turns about axes lying in the plane, and will thus be a rotation or a translation according as the axes of these half-turns do or do not meet.

In elliptic space, in which all straight lines are of finite length, and every two straight lines in a plane meet, the distinction between a translation and a rotation is lost, for the following reason. The lines drawn in a plane, perpendicular to a given line, all meet in either one common or two common points, according as the space is single or double elliptic space; and the locus of these points when the perpendiculars are drawn in all the different planes through the line is a second line, every point of which is at the same distance from the first line. The relation between the two lines* is reciprocal, and it is immediately evident from the above that a rotation about one of them is equivalent to a translation along the other. If, now, in elliptic space, the two translations to be compounded are along axes not lying in one plane, the lines $a'a$ and $b\beta$ will both meet AB.

Hence $a'a$ and $b\beta$ will only lie in a plane if $ABa'b$ is a plane; and this is contrary to the supposition that the axes of the two translations are non-intersecting lines. Hence the resultant of two translations along non-intersecting axes in elliptic space is never a translation (or rotation). If, on the other hand, the axes lie in one plane, the resultant displacement can be represented indifferently as a translation along some line in that plane or a rotation round the conjugate line.

II.

In a general displacement in Euclidean or hyperbolic space one line only remains unchanged, while in elliptic space two (conjugate) lines remain fixed. This statement, which is true of the general displacement, is therefore also true of the general infinitesimal displacement and of the set of displacements which result from repeating an infinitesimal displacement any (finite or infinite)

* Two such lines will be called conjugate lines.

number of times. There are, however, in Euclidean space certain infinitesimal displacements, namely translations, which keep unchanged each of a doubly-infinite set of straight lines; and the question therefore arises whether there are, in non-Euclidean space, any sets of displacements, arising from the repetition of an infinitesimal displacement, for which more than one (or two) lines remain unchanged.

It is known from considerations of analysis that in hyperbolic space there is no such set of displacements; but that in elliptic space, when a line is given, there are two distinct sets of displacements, each of which keeps a distinct doubly-infinite system of straight lines, of which the given line forms one, unchanged. The latter result is proved by Clifford in his paper on "Biquaternions" (*Proc. Lond. Math. Soc.*, Vol. IV., p. 390); and reference may also be made to a memoir by Sir R. Ball, "On the Theory of the Content" (*Trans. R.I.A.*, 1889).

The preceding lemmas and construction may be applied to obtain and amplify these results by elementary geometrical considerations, which are in part at least distinct from Clifford's.

If round the axis of the displacement one of its equidistant surfaces be described, no line which cuts this surface can remain unaltered by an infinitesimal displacement. For, if P, Q be the points where the line meets the surface, then, since both the surface and the line are changed into themselves by the displacement, the points P, Q must be either unaltered or interchanged; and both these suppositions are clearly impossible. If then a line remains unaltered by a displacement, it must lie on one of the equidistant surfaces of the axis of the displacement. Now in hyperbolic space the tangent plane at any point of an equidistant surface must lie wholly outside it, since the common perpendicular to two lines is also necessarily their only shortest distance. The equidistant is therefore, in this case, not a ruled surface, and no such displacement as that considered is possible.

That, in elliptic space, the equidistant is a ruled surface, may be seen directly as follows.

Let A and B be any two points on a line and its conjugate respectively, and take points A_1, A_2, ... on the line, and B_1, B_2, ... on the conjugate such that the finite lines AA_1, A_1A_2, ..., BB_1, B_1B_2, ... are all equal. Join AB, A_1B_1, A_2B_2 by lines which will be all of equal length, and all at right angles both to AA_1A_2 ... and BB_1B_2 .... Finally, take C, C_1, C_2, ... on AB, A_1B_1, ... such that AC, A_1C_1, ...

are all equal. Then C, C_1, C_2, ... all lie on an equidistant of AA_1, which is evidently at the same time an equidistant of BB_1. Join CC_1, C_1C_2, C_2C_3, ... by straight lines.

If, now, the figure be rotated through two right angles about $A_1C_1B_1$, the points A, B are brought into the positions A_2, B_2, while the lines AA_1 and BB_1 are changed into themselves. The points C and C_2 are therefore interchanged, and hence CC_1C_2 is a straight line. If the points A be kept fixed, and the points B, retaining their relative positions, be displaced continuously along the conjugate line, a complete set of generators of one system of the equidistant is obtained; and the other set will be obtained by taking the points B, B_1, ... in the opposite direction along the conjugate line.

A displacement AA_1 along the line AA_1 and a displacement BB_1 along the conjugate now clearly displace CC_1C_2 along itself, whatever the length AC may be, and wherever B is taken on the conjugate line. Hence a translation along a line and an equal translation along its conjugate leave undisplaced all the generators of one system of all the equidistants of the two lines. If the second translation is reversed in direction, the doubly-infinite set of generators of the other system are undisplaced. To these two displacements, or rather to the velocity-systems connected with them, Clifford has given the names right- and left-vectors. The same words may be used here to denote the corresponding finite displacements, while the two sets of lines which remain undisplaced by a right- or a left-vector may be called, with Clifford, a set of right- or left-parallels. The above reasoning shows that any two of a set of parallels, either right or left, are everywhere at the same distance apart. Moreover, if in the above construction CC_1C_2 is a right-parallel of AA_1A_2, then the lines ACB, $A_1C_1B_1$, ... are left-parallels, and conversely. A right-vector is therefore equivalent to successive half-turns about two left-parallels.

Suppose, now, with the previous notation, that the two displacements, of which $a'Aa$, $b'Bb$ are axes, are both right-vectors. Then $a'a$ and AB are left-parallels, as also are AB and $b\beta$. The resultant displacement, consisting of successive half-turns about $a'a$ and $b\beta$, which are left-parallels, is therefore a right-vector. Right-vectors, therefore, form a group of displacements, in the sense that the resultant of any two right-vectors is again a right-vector; and the same is, of course, true of left-vectors. The groups of displacements thus formed are not, however, like the group of translations in Euclidean space, composed of permutable operations; viz., the re-

sultant of two right- (or left-) vectors depends upon the order in which they are performed.

Finally, it may be shown that a right-vector and a left-vector are always permutable. Thus, let $a'Aa$ be the axis of a right-vector, and $b'Bb$ that of a left-vector, AB being a common perpendicular to these two lines. Take $a'\alpha'$ and $a\alpha$ perpendicular to $a'Aa$ and such that the right-vector is equivalent to successive half-turns round $a'\alpha'$ and AB, and also to successive half-turns round AB and $a\alpha$; and construct $b'\beta'$ and $b\beta$ similarly for the left-vector. Then $a'\alpha'$ and $a\alpha$ are opposite generators of the same system of an equidistant of AB, so that any common perpendicular to them meets AB (necessarily at right angles). So also $b'\beta'$ and $b\beta$ are opposite generators of the other system on another equidistant of AB. The five lines $a'\alpha'$, $b'\beta'$, AB, $b\beta$, $a\alpha$ therefore have a common perpendicular $\alpha'\beta'O\beta\alpha$, and from the construction of the equidistants it follows that $\alpha'\beta$ is equal to $\beta'\alpha$, and the angle between the planes $a'\alpha'\beta$ and $\alpha'\beta b$ is equal to that between $b'\beta'\alpha$ and $\beta'\alpha a$. Hence successive half-turns round $a'\alpha'$ and $b\beta$ are equivalent to successive half-turns round $b'\beta'$ and $a\alpha$; or, in other words, the displacement resulting from the right-vector followed by the left-vector is identical with that resulting from the left-vector followed by the right-vector.

Any displacement in elliptic space is the resultant of a right-vector and a left-vector. For it has been seen that any displacement is equivalent to a rotation Θ round some line, and a rotation Θ' round its conjugate, and, since these two displacements are permutable, they are equivalent to rotations $\frac{1}{2}(\Theta+\Theta')$ round the line and its conjugate, followed by rotations $\frac{1}{2}(\Theta-\Theta')$ round the line and $-\frac{1}{2}(\Theta-\Theta')$ round its conjugate, that is, to a right-vector $\frac{1}{2}(\Theta+\Theta')$ and a left-vector $\frac{1}{2}(\Theta-\Theta')$ with the line for their common axis.*

Now, it has been seen that right-vectors form a group of motions in the sense that the resultant of any two right-vectors is again a right-vector, and that the same is true of left-vectors, while every displacement of the one group is permutable with every displacement of the other. Hence, to determine completely the nature of the general group of motions in elliptic space, it is only necessary to consider the laws according to which right- and left-vectors separately combine.

Through any point of space one, and only one, of a set of right-parallels will pass. Hence, when two right-vectors are given whose

* *Cf.* Clifford on "Biquaternions" (*Proc. Lond. Math. Soc.*, Vol. IV., p. 390).

resultant is required, intersecting lines OA and OB may be taken as their axes; these being the two lines drawn through any chosen point O which belong respectively to the two sets of right-parallels that are displaced into themselves by the two right-vectors.

Through OA draw a plane AOC such that the right-vector whose axis is OA displaces it to AOB; and through OB draw a plane COB such that the plane AOB is displaced into it by the second right-vector. The angles between the pairs of planes AOC, AOB and BOA, BOC will then measure the amplitudes of the two right-vectors. Now every plane contains one, and only one, of a set of right- (or left-) parallels, and therefore the plane AOC must contain a line which is transformed into itself by the resultant displacement; but the plane AOC is changed into BOC by the resultant displacement, and therefore OC must be that axis of the resultant right-vector which passes through O. The amplitude of the resultant is the angle between the planes AOC and BOC, since it displaces one of these planes into the other. The axis and amplitude of the resultant of any two right- (or left-) vectors is thus completely determined. The result may be stated as follows:—

Right-vectors combine according to the same law as finite rotations round a point, the amplitudes of the rotations being twice those of the corresponding right-vectors. It is also clear that exactly the same statement holds concerning left-vectors.

The group of right-vectors (or left-vectors) is therefore isomorphous with the group of rotations round a point; and the structure of the general group* of real motions in elliptic space is thus deduced from

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* It has been suggested by one of the referees, to both of whom I owe my best thanks for the trouble they have taken with this paper, that since there is at present no English treatise on the subject of continuous groups, it would be advisable to give such definitions and explanations of some of the terms used in the present paper as will suffice to make their meaning definite to the reader.

I have attempted in the following note to carry out this suggestion; purposely abstaining from any reference to the analytical form in which Herr Sophus Lie, to whom the theory of continuous groups owes its origin, has presented it.

A set of operations $1, S, S', S'', \ldots,$

which contains every possible combination of the individual operations, taken either directly or inversely, is said to form a group. When the individual operations depend upon a finite number, n, of quantities, each of which is capable of continuous variation through a range which is not infinitely small, the group is spoken of as a "finite continuous group." Since each of the n quantities on which the determination of a particular operation depends is capable of an infinite number of values, the group contains in a quite definite sense ∞^n different operations. To denote such a group Lie uses the phrase "*n-gliedrige continuierliche Gruppe.*"

Such a group necessarily contains infinitesimal operations, *i.e.*, operations which produce an infinitesimal change in any possible operand. If S and T are two infinitesimal operations, the difference of the changes produced by ST and TS in any

purely synthetical considerations. It is, in fact, now seen to arise from the combination of two permutable and isomorphous groups of known type. The structure of this group renders it very simple to enumerate all types of sub-group contained in it; that is, all those sets of motions in elliptic space which have the group property, and at the same time do not include all possible motions. Thus any sub-group must be formed by the combination of the group of right-vectors or one of its sub-groups with a sub-group of the group of left-vectors, or *vice versâ*; and the combination may either be general or may be such that, an isomorphous correspondence having been established between the two sub-groups, corresponding operations are combined together.

Now the group of rotations round a point contains no real sub-group with a doubly-infinite number of operations; its only sub-groups, in fact, being rotations round a fixed axis, which form a singly-infinite set.

Hence the general group of motions in elliptic space which con-

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operand is necessarily infinitesimal in comparison with the changes produced by S or T; so that, when the word "infinitesimal" is used in its ordinary sense, two infinitesimal operations are necessarily permutable; but this, of course, does not involve that the corresponding finite operations, which result from repeating the infinitesimal operations an infinite number of times, are permutable.

The group contains n independent infinitesimal operations, in the sense that every infinitesimal operation of the group can be obtained by a finite combination of them.

Every non-infinitesimal operation of the group can be generated by an infinite number of repetitions of an infinitesimal operation, suitably chosen, and thus the group is completely defined by a set of n independent infinitesimal operations.

On the other hand, n arbitrarily given infinitesimal operations will not in general generate a finite continuous group of ∞^n operations, but an infinite continuous group, *i.e.*, one whose individual operations depend on an infinite number of continuously varying quantities.

The simply-infinite set of operations obtained by repeating an infinitesimal operation (and its inverse) form a group, which is contained within the original group. It is a group whose individual operations are determined by a single continuously varying parameter; and is spoken of by Lie as an "*ein-gliedrige Untergruppe*" of the original group.

In addition to such simply-infinite sub-groups, the original group will in general contain other sub-groups.

Thus, it may happen that $1, \Sigma, \Sigma', \Sigma'', \ldots,$

a set of operations contained in the original group, possess among themselves the group-property defined in the first paragraph. If the number of operations in this set is finite, the sub-group formed by their totality is necessarily discontinuous; if the number is infinite, the sub-group may be either discontinuous or continuous. The latter will be the case, when the individual operations of the sub-group are determined by a number r (necessarily less than n) of continuously varying quantities. The sub-group may then be spoken of as a continuous sub-group of ∞^r operations. Lie's phrase is "*r-gliedrige Untergruppe.*" Such a sub-group again necessarily contains a set of r independent infinitesimal operations, from

tains ∞^6 operations has no sub-group containing ∞^5 operations, while the only two types of sub-groups which contain ∞^4 operations are those arising from the combination of the group of right- (or left-) vectors with those left- (or right-) vectors which keep a given set of left- (or right-) parallels unchanged.

These two groups are analogous to, but not isomorphous with, that group of motions in Euclidean space which consists of all possible screw-motions about a set of parallel axes. Each of the two types contains ∞^2 conjugate sub-groups.

Of sub-groups containing ∞^3 operations, there are three types. Two of these are the groups of right-vectors and left-vectors which are self-conjugate in the main group. The third is the group of

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which it can be generated, and the n infinitesimal operations of the original group can always be chosen so that these r occur among them. It is not, however, generally the case that any r of the n independent infinitesimal operations will generate a sub-group of ∞^r operations; they will, when $r > 1$, generally generate the original group itself.

If now T is any operation of the group, the operations S and $T^{-1}ST$ are called conjugate [(Lie) *gleichberechtigte*] operations, when they are not identical, and T is said to transform S into $T^{-1}ST$.

Similarly, $1,\ \Sigma,\ \Sigma',\ \ldots$

and $1,\ T^{-1}\Sigma T,\ T^{-1}\Sigma' T,\ \ldots$

are called conjugate sub-groups when they are not identical with each other. If these two sub-groups are identical, whatever operation of the original group T may be, the sub-group $1,\ \Sigma,\ \Sigma',\ \ldots$

is called a self-conjugate sub-group. Lie uses the phrase "*invariante Untergruppe*" to denote a sub-group with this property, while Klein writes "*ausgezeichnete Untergruppe.*" Lie uses the word "*ausgezeichnete*" only in connexion with "*ein-gliedrige Untergruppe*"; an "*ausgezeichnete ein-gliedrige Untergruppe*" being, in his phraseology, a simply-infinite continuous sub-group, each of whose operations is permutable with all the operations of the group.

If a continuous sub-group I, containing ∞^r operations, is contained as a self-conjugate sub-group within another more extensive continuous sub-group H, containing ∞^s $(s > r)$ operations, but is not self-conjugate within any continuous sub-group of the original group G that is more extensive than H, then I is transformed into itself by all the ∞^s operations of H.

When I is transformed by all the ∞^n operations of G, there result ∞^{n-s} sub-groups, all of them conjugate to I; and then I is said to form one of a set of ∞^{n-s} conjugate sub-groups [(Lie) *gleichberechtigte Untergruppen*] within G.

When a one-to-one correspondence can be established between the individual operations of two continuous groups, each of which contains ∞^n operations, in such a way that to the product of any two operations of one group in a certain order corresponds the product of the two homologous operations of the other group in the same order, the two groups are said to be holohedrically isomorphous [(Lie) *holoedrisch isomorph*]. Abstractly considered, *i.e.*, when the laws of combination of the individual operations only are taken into account, and not the nature of the operations themselves or of the operand, two holohedrically isomorphous groups are identical. Where the word "isomorphous" is used in the present paper without qualification it is to be regarded as an abbreviation for "holohedrically isomorphous."

rotations round a given point (or the general group of motions in a plane), which is isomorphous with the preceding, but, unlike them, forms one of ∞^3 conjugate sub-groups. It may be regarded as arising from an isomorphous correspondence between the groups of right- and left-vectors established as follows. Through a given point one of every set of right-parallels and one of every set of left-parallels will pass. If, then, a right-vector and a left-vector correspond when their amplitudes are equal and their axes which pass through the given point are identical, the resultant of two right-vectors will correspond to the resultant of the corresponding two left-vectors.

When therefore the group of right-vectors is combined with the group of left-vectors by multiplying together corresponding operations in the two groups, the new group is isomorphous with either of the groups from which it is formed, while it keeps the given point fixed.

Of sub-groups containing ∞^2 operations there is one type, namely, the group of motions which consist of arbitrary rotations round any pair of conjugate lines, and this type contains ∞^4 conjugate sub-groups. It would appear at first sight that the sub-group arising from combining those right-vectors which keep an arbitrarily chosen set of right-parallels unchanged with a similar group of left-vectors would give rise to a new type; but it is an immediate deduction from the constructions in the earlier part of this paper that any set of right-parallels and any set of left-parallels have just two lines in common, these lines being conjugate.

Of sub-groups containing ∞^1 operations there are three types. Of these those right- (or left-) vectors which keep a given set of right- (or left-) parallels unchanged form two types each containing ∞^2 conjugate sub-groups, while the third type consists of screw-motions of given pitch round a given line, and contains ∞^4 conjugate sub-groups.

All discontinuous groups of motions of finite order in elliptic space, corresponding to which there are divisions of the whole of space into a finite number of congruent portions, may be derived in a precisely similar manner from the known finite discontinuous groups of rotations about a point, *i.e.*, from the groups of the regular solids. Owing to the greater number of types of group involved, there is a very much greater variety of such discontinuous groups than of the continuous groups that have just been considered. They need not here be enumerated, as they have been in effect completely classified, though from a rather different point of view, by M. Goursat, in a memoir with the title, "Sur les substitutions orthogonales et les

divisions régulières de l'espace" (*Ann. de l'Ecole Norm. Sup.*, 3me série, tome vi.). Except for the simplest of such discontinuous groups, it is a matter of considerable difficulty to realize the nature of the corresponding division of space into congruent parts; and in the simplest case of all, that namely in which the group consists of a rotation through two right angles round a line and identity, the solution for simple elliptic space is by no means obvious. Before dealing with this particular case, take the case of a cyclical group generated by a right-vector, n repetitions of which lead to identity.

If n planes be drawn through any axis of the right-vector, each of which makes angles $\frac{\pi}{n}$ with the planes on either side of it, the whole of space is divided into n congruent figures which may be called biangles, the space between any two adjacent planes being easily seen to be continuous with the vertically opposite space between them. The right-vector, consisting of a rotation $\frac{\pi}{n}$ round the line, and a translation through $\frac{1}{n}$-th of its length, transforms any one of these biangles successively into each of the others, and n repetitions of it, being equivalent to a rotation π round the line, and a translation through its whole length, which is the same as another rotation π, brings back the original biangle to coincidence with itself, point for point.

If now n is odd, and the generating operation a rotation $\frac{2\pi}{n}$ round the line, the same construction will give n congruent spaces which are transformed into each other by the operations of the cyclical group, though the correspondence of points is not the same as in the former case. If, however, n is even, the n congruent biangles are not transformed into each other, but the original biangle is transformed only into $\frac{n}{2}$ of the biangles, and into each by two operations in two different ways; a different division of space is therefore necessary in this case. When n is 2, it might appear sufficient at first sight to draw a single plane through the line; but in simple elliptic space the two sides of a plane are continuous with each other, so that this would not effect a division of space into two parts.

The requisite division of space into two congruent parts may, however, be obtained as follows. Let A and B be two points taken one on each of two conjugate lines a and b, and bisect the straight segment AB in C. When A and B take all possible positions on a and b respec-

tively, the locus of C is an equidistant of each of these lines, whose "radius" relative to each line is the same, namely, one quarter of the complete straight line. It is easy to verify that on this particular equidistant the generators are at right angles; and, since it is impossible to draw a line from a point on a to a point on b which does not cut the equidistant once, it must divide the whole of space into two parts. Consider now the motion which consists of a rotation through two right angles about one of the generators of this equidistant. Every such line as ACB, used in the construction just given, which meets the generator is brought into the position BCA, so that the rotation interchanges the two conjugate lines a and b. It must, therefore, since only one such equidistant can be drawn with two given conjugate lines, bring the equidistant again into congruence with itself, while the two parts into which space is divided by the equidistant are interchanged. The two parts into which space is divided by this equidistant are therefore congruent with each other, and can be interchanged by a rotation through two right angles about any one of the generators of the equidistant. The construction for the division of space into $2n$ congruent portions, any one of which can be brought to coincidence with each of the others by successive rotations through $\frac{2\pi}{2n}$ round a line, is now almost obvious. With the given line as a generator, such an equidistant as is under consideration is described, and from it $n-1$ more are formed by rotating it through angles $\frac{\pi}{n}$, $\frac{2\pi}{n}$, ... $\frac{(n-1)\pi}{n}$ round the given line. The n equidistants so formed divide space into $2n$ parts with the required properties.

III.

Returning now to the motions of hyperbolic space, it is to be noticed that the construction that has been given for the resultant of any two displacements fails in one case to lead to a definite result; viz., when the axes of the two displacements meet at infinity. This difficulty may be obviated by introducing between the two displacements whose resultant is required two arbitrarily chosen equal and opposite displacements; and combining, to begin with, the first given displacement with the first of the two thus introduced, and the second of these with the second given displacement. The axes of the two displacements thus obtained will not, unless the introduced displacements are specially chosen, meet at infinity, and with them the

original construction may be carried out. The axis of the resultant displacement will necessarily pass through the same point at infinity as the axes of the two given displacements; for this point, being undisplaced by each of the given displacements, is undisplaced by their resultant, and must therefore be one of the two points at infinity on the axis of their resultant. If the two given displacements are translations, the resultant is, as has already been seen, since the axes intersect, also a translation, and in this case a simple construction may be given for the axis and magnitude of the resulting translation. For this purpose I first recall the construction in the case when the axes intersect in a finite point. If AOA', BOB' are the intersecting axes, and if AO, OB' be in direction and magnitude half the translations, then AB' is the axis of the resultant translation, and $2AB'$ its magnitude. Now, let AI, CI be the axes of the two given translations meeting in I at infinity. Then, if the translations are equal in magnitude and opposite in sense as regards I along the two lines, the construction just given shows that the axis of the resultant translation can have no finite point upon it, and therefore in this case it is useless to attempt to construct this axis. In any other case, the axis is a finite line passing through I, and therefore having a second point at infinity on it, say J. Draw a line AC meeting the two given lines, and not passing through the point J; and take B such that AB is half the translation along AI. Join BC and produce it to B', so that BC is equal to CB', and then join B' to D on CI, where CD is half the translation along CI. Then the translation along AI is equivalent to translations $2AC$ and $2CB$ along AC and CB successively, and the translation along CD is equivalent to translations $2CB'$ and $2B'D$ along CB' and $B'D$ successively.

Hence the two given translations are equivalent to $2AC$ and $2B'D$ along these lines, and from the construction it is impossible for these lines to meet at infinity; for, if they did, the axis of the resultant translation would pass through the point in which they met, while neither of the points I and J at infinity on this axis lies on AC. Hence these two equivalent translations can be compounded in the ordinary way.

Returning now to the case in which the two translations are equal in magnitude and opposite in sense, the resultant motion might be characterized as a translation whose axis is at infinity. This is not intended to imply that in hyperbolic space it is correct to speak of lines at infinity, but the phrase is used to describe shortly a motion in hyperbolic space which has nothing completely analogous

to it in Euclidean or elliptic space. Indeed this motion may be equally well described as a rotation whose axis is at infinity. To verify this statement, and to bring out as clearly as possible the nature of this motion, I give the following construction. Draw that line OI meeting the axes AI, CI of the equal and opposite translations at infinity, with respect to which they are symmetrically situated. From O, let fall perpendiculars OA, OC on AI, CI, and take equal lengths AB, CD on AI, CI, equal in magnitude to half the translations, and measured either both towards or both from I, according as the translation along AI is in the direction AI or IA. From B and D draw BP, DP in the plane of the figure perpendicular to AI, CI, and meeting in P, which necessarily lies on OI. From P draw a perpendicular PQ to OA, and produce it to P' so that QP' is equal to PQ; then join $P'O$, and through P' draw a line $P'K$, such that the angle $KP'O$ is equal to the angle DPB, while equal rotations which bring $P'K$ to $P'O$ and PD to PB are in the same sense. Now the two translations are equivalent to successive half-turns round OA, PB, PD, OC. Successive half-turns round PB, PD are equivalent to a rotation round Pp, perpendicular to the plane of the figure, through twice the angle BPD. The half-turn round OA followed by this rotation is equivalent to an equal rotation in the opposite sense round $P'p'$, perpendicular to the plane of the figure, followed by a half-turn round OA. This equal rotation in the opposite sense round $P'p'$ is equivalent to successive half-turns round $P'K$ and $P'O$; while, since the angle AOC is equal to the angle $P'OP$, successive half-turns round OA, OC are equivalent to successive half-turns round OP', OP. Hence, finally, the two translations are equivalent to successive half-turns round $P'K$ and OP. Now OP passes through I, and the resultant displacement leaves I unchanged, so that $P'K$ must also pass through I. The motion under consideration is therefore equivalent to successive half-turns about two lines in the same plane with the two original axes, and passing through the same point at infinity with them; in other words, it may be regarded as a rotation about an axis at infinity perpendicular to the plane of the figure. By such a motion every point in the plane AIC is displaced along the circle of infinite radius described through it with I as centre. This brings out in a striking manner the fact that in hyperbolic space a circle of infinite radius is not the same as a straight line. If through the lines AI, CI, ... passing through the same point I in the original plane, planes be drawn perpendicular to this plane, the motion in question displaces each such plane into

another of the set; and if in these planes lines $A'I$, $C'I$, ... be drawn, so that each pair of lines such as AI, $A'I$ or CI, $C'I$ is congruent with each other pair, then $A'I$, $C'I$, ... lie in a plane, and are displaced among themselves by the motion. It is also easy to see that any two motions which keep the point I unchanged and displace every finite line passing through I are permutable with each other.

When the two component displacements about axes intersecting at infinity are rotations, the axis of the resultant rotation may be found at once by the same construction as that used when the axes intersect in a finite point.

It is now possible to analyse the general group of real motions in hyperbolic space, so far as concerns the complete enumeration of all types of sub-group contained within it. Owing to the fact, which will be proved immediately, that the group contains no self-conjugate sub-groups, it does not appear possible to present the structure of the group itself, without the consideration of imaginary motions, in a form in any way analogous to that in which the group of motions in elliptic space has been presented.

It has been proved in the earlier part of this paper that no infinitesimal motion in hyperbolic space transforms more than one line into itself. Now any continuous sub-group must contain some infinitesimal displacement, an infinitesimal screw-motion of given pitch, about some line. If then the sub-group is self-conjugate it must contain every conjugate operation within the main group, and therefore must contain a similar infinitesimal screw-motion about every line in space. But from such a set of motions, infinitesimal screw-motions of any pitch whatever can be constructed, and therefore the group in question must coincide with the main group of motions.

Again, a continuous sub-group which does not coincide with the main group, must be such that all of its operations transform either some one point, some one line, or some one plane into itself. For, if not, the group must contain infinitesimal motions displacing every point in three directions which do not all lie in a plane; and from these may be compounded infinitesimal motions displacing every point in all possible directions, and therefore also finite motions which will displace every point to every other point of space. If, then, the group contains an infinitesimal operation whose axis passes through some chosen point, it must contain conjugate operations whose axes pass through every other point of space, and from this property it may easily be seen to coincide with the main group.

Now there are ∞^2 points at infinity, ∞^3 finite points, ∞^3 planes, and ∞^4 lines in space. Hence there can be no sub-group containing ∞^5 operations; for, if there were there would be ∞^1 such conjugate sub-groups, and therefore the point, line, or plane, which is undisplaced by the group, would have only ∞^1 different positions. Also any sub-group containing ∞^4 operations must keep a point at infinity fixed. Now it has been seen that any two displacements whose axes meet at infinity have for their resultant another displacement whose axis passes also through the same point at infinity. Hence the totality of displacements whose axes meet in a point at infinity do actually form a group, and since there are ∞^2 such axes and ∞^2 displacements corresponding to each axis, the group contains ∞^4 operations. There is, then, one type of such sub-group, and the type contains ∞^2 conjugate sub-groups.

In any type of sub-group containing ∞^3 operations, there must be ∞^2 or ∞^3 conjugate sub-groups, and in the former case the sub-group must be self-conjugate within a sub-group containing ∞^4 operations Now the sub-group just considered has been seen to contain two sub-groups with ∞^3 operations, namely, those sub-groups made up of all its translations and of all its rotations respectively; and from their nature these are self-conjugate within the larger sub-group. Hence arise two types of sub-groups containing ∞^3 operations, one consisting entirely of translations, and the other entirely of rotations, each keeping a point at infinity fixed, and each forming one of a set of ∞^2 conjugate sub-groups. The only other possible types of sub-group containing ∞^3 operations must contain ∞^3 conjugate sub-groups, and must therefore keep either a finite point or a plane unchanged. Now the group of rotations round a point does actually consist of ∞^3 operations, as also does the general group of motions in a plane, so that these two types exist and are completely accounted for.

To simplify the discussion of the remaining sub-groups it may be pointed out that of the sub-groups containing ∞^3 operations the last two types are simple and contain no self-conjugate sub-groups, while the first two types contain self-conjugate sub-groups of the same type, or rather identical self-conjugate sub-groups, consisting of those motions, which, as has been seen, may be indifferently regarded as translations or rotations, whose axes lie at infinity. These sub-groups, moreover, are self-conjugate within groups of ∞^4 operations. Thus arises a single type of sub-group containing ∞^2 operations, and consisting of ∞^2 conjugate sub-groups. Every other type of sub-group containing ∞^2 operations must contain within it ∞^4 conjugate sub-

groups. Hence it must either keep a line unchanged, or else a point at infinity, and a plane passing through it. Both of these types actually exist, the first consisting of all possible displacements with a given line for axis, and the latter of translations in a plane along lines passing through the same point at infinity. Lastly, sub-groups containing ∞^1 operations must occur in conjugate sets of ∞^4 at most, and must therefore be contained self-conjugately in the two preceding types.

Now displacements with a given line for axis are all permutable with each other, so that every sub-group is contained in such a group self-conjugately. The first of the two preceding types, therefore, gives rise to an infinite number of types of simply-infinite sub-groups, each consisting of those screw-motions with a given line for axis which have a given pitch, and including as limiting cases simply-infinite sets of translations and rotations respectively. The second of the two preceding types contains a single self-conjugate sub-group, namely, the set of motions which have been spoken of as rotations about an axis at infinity. This forms the one other type of simply-infinite sub-group.

The only discontinuous groups of motion of finite order in hyperbolic space are the known finite groups of rotations round a point; for such a group cannot contain any displacement other than a rotation, as otherwise it could not be of finite order, and for the same reason it cannot contain rotations about non-intersecting axes. On the other hand, of discontinuous groups of motion, whose orders are not finite, there is in hyperbolic space an infinite variety. The truth of this statement may be made clear as follows, by considering certain discontinuous groups of plane motions. If from a point O three equal lines OA, OB, OC are drawn in a plane and making equal angles with each other, and through A, B, C lines are drawn perpendicular respectively to OA, OB, OC, these lines, when OA is infinitesimal, will form an infinitesimal equilateral triangle, whose angles are infinitesimally less than $\frac{\pi}{3}$. As OA is taken greater and greater the angles of the triangle become less and less, and for a certain length of OA each pair of sides will meet at infinity, and the angles of the triangles will be zero. Hence, equilateral triangles can be constructed in hyperbolic space, whose angles are $\frac{\pi}{n}$, where n is any integer greater than 3.

If, now, such a triangle be drawn in a plane, and on each of its sides an equal triangle be constructed, and if this process be continued

indefinitely, the whole plane will be divided into an infinite number of such congruent equilateral triangles without gaps or overlapping, $2n$ triangles being ranged round every angular point. When planes are drawn through the sides of the triangles perpendicular to their plane, the whole of space is divided into what may be described as equilateral prisms, all of which are congruent with each other. Moreover, by rotations $\frac{2\pi}{3}$, π, $\frac{2\pi}{n}$ about perpendiculars to the plane of the triangles through O, the middle point of a side and an angular point respectively, this infinite set of equilateral prisms is brought to congruence with itself. Hence, of necessity, these three rotations generate a discontinuous group of motions.

Another very simple, but interesting, illustration of the division of space into congruent parts, and of the corresponding discontinuous group of motions, arises in connexion with the regular solids. From a point O, perpendicular to a line OI, draw n lines OO_1, OO_2, ... OO_n, equal and equally inclined to each other; and through their extremities draw lines O_1I, O_2I, ... to meet OI at infinity. By taking a section of the prismatic figure so formed at a sufficiently great distance from O, the size of the section can be made as small as desired, and, therefore, the dihedral angles at the edges must be the same as for an infinitesimal figure. Each of these dihedral angles is, therefore, $\frac{n-2}{n}\pi$, which, for $n = 3, 4, 5$, gives $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{3\pi}{5}$. Hence if a regular solid be described with its vertices at infinity, the internal dihedral angle between two adjacent faces will be $\frac{\pi}{3}$ for a tetrahedron, cube, or dodecahedron, $\frac{\pi}{2}$ for an octohedron, and $\frac{3\pi}{5}$ for an icosahedron.

With the exception of the last, these angles are all submultiples of four right angles; and, therefore, in the first four cases, if the original solid is rotated about its edges, through the dihedral angle, the new figures so formed rotated about their edges, and so on indefinitely, the whole of space will be exactly filled, without gaps, with congruent figures. It may be added here, without proof, as the result depends only on certain simple inequalities, that there are only four* other ways

* While these pages are passing through the press, I have become acquainted with a paper by Signor L. Bianchi: "Sulle divisioni regolari dello spazio non euclideo in polyedri regolari" (*Rendiconti, Accademia dei Lincei*, 1893), in which it is stated that there are only two modes of division of hyperbolic space into congruent regular polyhedra. It appears to me that Signor Bianchi has introduced an unnecessary limitation into his discussion; but it is impossible to discuss this point adequately in a footnote, and I shall hope to return to it in a future paper.

of dividing hyperbolic space into equal and congruent regular solids. These are: (i) cubes, there being twenty cubes arranged round each vertex with icosahedral symmetry; (ii) and (iii) dodecahedra, there being either eight or twenty dodecahedra arranged round each vertex with octohedral and icosahedral symmetry respectively; (iv) icosahedra, there being twelve icosahedra arranged round each vertex with dodecahedral symmetry.

There is a marked difference, as regards discontinuous groups of motion, between hyperbolic space, on the one hand, and elliptic and Euclidean space, on the other. It has been seen above that for elliptic space there are only a finite number of types of such groups, and in Euclidean space Herr Schönflies (*Krystallsysteme und Krystallstructur*: Teubner, 1891), among others, has shown that there are just 65 types.

IV.

Returning now again to the motions of elliptic space, it is interesting to point out that it is only necessary to investigate some analytical form of the group of rotations round a point (a problem of group-theory) in order to pass on from the foregoing purely synthetical considerations to the complete metrical system for elliptic space.

The most symmetrical analytical form of the group of rotations round a point is that in which it is regarded as that group of homogeneous projective transformations of three variables q_1, q_2, q_3 which keep the form

$$q_1^2+q_2^2+q_3^2$$

unchanged. Hence, if q_1, q_2, q_3, q_4, q_5, q_6 are six independent variables, the group of motions in elliptic space can be expressed as that group of homogeneous projective transformations of these variables which keep the two forms

$$q_1^2+q_2^2+q_3^2 \quad \text{and} \quad q_4^2+q_5^2+q_6^2$$

unchanged.

If, now, new variables p_1, p_2, p_3, p_4, p_5, p_6 are introduced, such that

$$p_1 = q_1+q_4, \quad p_2 = q_2+q_5, \quad p_3 = q_3+q_6,$$

$$p_4 = q_1-q_4, \quad p_5 = q_2-q_5, \quad p_6 = q_3-q_6,$$

the group, expressed in terms of the p's, is that homogeneous projective group which keeps unchanged

$$p_1p_4+p_2p_5+p_3p_6$$

and

$$p_1^2+p_2^2+p_3^2+p_4^2+p_5^2+p_6^2$$

The p's may therefore be regarded as homogeneous line-coordinates in ordinary space, and when they are so regarded the equation

$$p_1^2+p_2^2+p_3^2+p_4^2+p_5^2+p_6^2=0$$

represents a quadric surface which contains no real points. The group of motions in elliptic space is, therefore, abstractly considered, identical with that group of projective transformations in ordinary space which preserves unchanged a purely imaginary quadric; and this is the starting-point from which the metrical relations of elliptic space are actually derived.

[*Added, December 28th.*

Since the group of real motions in hyperbolic space is a simple group, it is not possible to determine its analytical form by a process precisely analogous to that employed above for the group of elliptic motions. On the other hand, the group having been exhaustively analysed, it becomes a problem of pure group-theory to make this determination. It will be simplified by the following considerations. To every motion of hyperbolic space corresponds a transformation of the points at infinity, and no motion keeps more than two points at infinity unchanged. Hence between the group of motions in hyperbolic space and the group of transformations of the points at infinity, that they involve, there is a one-to-one correspondence; *i.e.*, the groups are, abstractly considered, identical. Now, the points at infinity form a doubly-infinite set, and, therefore, a transformation group of the points at infinity may be represented as a transformation-group of points in an ordinary plane. Again, if IJ, $I'J'$ be any two lines of hyperbolic space, and PK, $P'K'$ any other two lines meeting the former two respectively at right angles, a single motion can be found which will bring IJ, PK into the positions $I'J'$, $P'K'$. Hence the group of transformations of the points at infinity, or the corresponding transformation-group of points in a plane, is such that it contains a single transformation which will bring any three arbitrarily chosen points into any other three arbitrarily chosen positions; or, in the phraseology of group-theory, the transformation-group is triply-transitive. The group of motions of hyperbolic space is, therefore, capable of being represented in the form of a triply-transitive group of ∞^6 transformations of points in a plane. Now, it can be shown that of such groups there is one type, and one only—groups between which a one-to-one correspondence can be established, being, of course, regarded as identical (*cf.* Lie-Scheffers, *Vorlesungen über continuierliche Gruppen*, pp. 355, 356).

The particular form of the group which it is most convenient to consider here is that group of point-transformations which arises from an even number of inversions at all real circles of a plane. It is easy to see that this group contains ∞^6 transformations, and that it contains one, and just one, transformation which will displace any three given points into any other three. Moreover, if the equation to any circle be written in the form

$$\alpha(x^2+y^2+1)+2\beta x+2\gamma y+\delta(x^2+y^2-1)=0,$$

so that the square of its radius is

$$\frac{\alpha^2+\beta^2+\gamma^2-\delta^2}{(\alpha+\delta)^2},$$

the group in question, when expressed in terms of the symbols α, β, γ, δ, is easily found to be that homogeneous projective group which keeps

$$\alpha^2+\beta^2+\gamma^2-\delta^2$$

unchanged, this latter condition corresponding to the fact that, by inversion at real circles, a real circle necessarily remains a real circle.

If now, finally, α, β, γ, δ are regarded as homogeneous point-coordinates in ordinary space, the group of hyperbolic motions is seen to be identical with that projective group of ordinary space which transforms a real quadric with imaginary generators into itself.]

Thursday, December 13th, 1894.

Major MACMAHON, R.A., F.R.S., President, and subsequently A. E. H. LOVE, Esq., F.R.S., Vice-President, in the Chair.

The following gentlemen were elected members of the Society:—William Henry Young, M.A., formerly Fellow of Peterhouse, Cambridge; William Montgomery Coates, M.A., Fellow and Assistant Tutor of Queens' College, Cambridge; Philip Herbert Cowell, B.A., Fellow of Trinity College, Cambridge; Gilbert Harrison John Hurst, B.A., Scholar of King's College, Cambridge; Horace J. Harris, B.A.,

University College, London; Ernest William Brown, M.A., Fellow of Christ's College, Cambridge, and Professor of Mathematics in Haverford College, Pennsylvania.

The Treasurer having read the Auditor's report, the adoption of the Treasurer's report was moved by Mr. Kempe, seconded by Prof. Rogers, and carried unanimously. A vote of thanks to the Auditor for the trouble he had taken was moved by Prof. Hill, seconded by Mr. Walker, and carried.

The following communications were made:—

On Maxwell's Law of Partition of Energy: Mr. G. H. Bryan.

The Spherical Catenary; and The Transformation of Elliptic Functions: Prof. Greenhill.

On certain Definite Theta-Function Integrals: Prof. Rogers.

Groups defined by Congruences (second paper): Prof. W. Burnside.

Vibrations in Condensing Systems: Dr. J. Larmor.

On the Integration of Allégret's Integral: Mr. A. E. Daniels.

On the Complex Number formed by two Quaternary Matrices Dr. G. G. Morrice.

The Chairmen, Messrs. Bryan, Greenhill, Rogers, Larmor, and Walker took part in the discussions on the papers.

The following presents were received:—

"The Imperial University of Japan Calendar," 1893–4.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XVIII., St. 10, 11; Leipzig, 1894.

"Proceedings of the Cambridge Philosophical Society," Vol. VIII., Part 3; 1894.

"Proceedings of the Royal Society," Vol. LVI., Nos. 338–339.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. VIII., No. 3; 1893–4.

"Bulletin de la Société Mathématique de France," Tome XXII., No. 8; Paris.

"Bulletin des Sciences Mathématiques," Tome XVIII., Oct., Nov.; 1894.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXVIII., Livr. 3 and 4; Harlem, 1894.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. I., No. 2.

"Journal of the College of Science, Japan," Vol. VIII., Pt. 1; Tokyo, 1894.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 8–9, Sem. 2^a; Roma, 1894.

"Educational Times," December, 1894.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. VIII., Fasc. 8–10; Napoli, 1894.

"Observations made during 1889 at the United States Naval Observatory," 4to; Washington, 1893.

Balbin, V.—"Tratado de Geometría Analítica," 8vo; Buenos Ayres, 1888. "Tratado de Estereometría Genética," 8vo; Buenos Ayres, 1894. "Método de los Cuadrados Mínimos," 8vo; Buenos Ayres, 1889. "Elementos de Calculo de los Cuaterniones," 8vo; Buenos Ayres, 1887. "Geometría Plana Moderna," 8vo; Buenos Ayres, 1894.

D'Ocagne, M.—"Mémoire sur les Suites Récurrentes," 4to pamphlet.

"Annales de l'Ecole Polytechnique de Delft," Tome VIII., Livr. 1–2; Leide, 1894.

"Annales de la Faculté des Sciences de Toulouse," Tome VIII., Fasc. 4; Paris, 1894.

"Journal für die reine und angewandte Mathematik," Bd. CXIV., Heft 2; Berlin, 1894.

"Transactions of the Royal Irish Academy," Vol. XXX., Parts 13 and 14; Dublin, 1894.

"Indian Engineering," Vol. XVI., Nos. 16–20; Oct. 20th–Nov. 17th.

On a Class of Groups defined by Congruences. (*Second Paper.*)

By W. Burnside. Received December 7th, 1894. Read December 13th, 1894.

1. *Introduction.*

In a paper printed in Vol. XXV of the Society's *Proceedings*, I have discussed the groups defined by a congruence of the form

$$z' \equiv \frac{\alpha z+\beta}{\gamma z+\delta} \pmod{p},$$

where p is prime, and α, β, γ, δ are rational integral functions of the roots of an irreducible congruence of the n^{th} degree to the same prime modulus.

This discussion was greatly facilitated by the fact that the groups defined by a congruence of the same form in which the coefficients are ordinary integers had been already exhaustively analysed.

Now the corresponding group in two non-homogeneous variables, namely, the group defined by the congruences

$$x' \equiv \frac{\alpha x+\beta y+\gamma}{\alpha'' x+\beta'' y+\gamma''}, \quad y' \equiv \frac{\alpha' x+\beta' y+\gamma'}{\alpha'' x+\beta'' y+\gamma''} \pmod{p},$$

has not hitherto been the subject of any similar discussion. If the

determinants of all the substitutions be unity, it is known to be a simple group of order

$$(p^2+p+1)\,p^3(p+1)(p-1)^2 \quad \text{or} \quad \tfrac{1}{3}(p^2+p+1)\,p^3(p+1)(p-1)^2,$$

according as p is congruent to -1 or 1, mod 3; but beyond this nothing is known of the type and number of the cyclical and other sub-groups contained in it.

The present paper is intended, to some extent at least, to fill this gap; and it is an almost necessary preliminary to the discussion, which I hope to undertake later, of the similar groups in which the coefficients are rational integral functions of the roots of an irreducible congruence.

The last paragraph of the paper deals shortly with the two exceptional cases of $p = 2$ and $p = 3$. Passing over these, it is clear that, since the number giving the order of the group in terms of p depends on whether p is of the form $3m+1$ or $3m-1$, these two cases require separate treatment.

The greater part of the paper is occupied with a detailed discussion of the case in which p is of the form $3m-1$. On passing on to the case in which p is of the form $3m+1$, it is found that, though the results are different in form from those of the former case, they are closely analogous to them, while the process of arriving at them is practically the same in the two cases. I have, therefore, not thought it necessary to repeat in detail all the steps of the reasoning in this second case, which would have considerably increased the length of the paper, but have simply pointed out the necessary modifications of the processes employed, and stated the results.

A limitation on the generality of the results, which is not essential, and is more apparent than real, as the subjoined footnote will show, has been introduced, in the assumption that p^2+p+1 in the one case, and $\frac{1}{3}(p^2+p+1)$ in the other, is the product of not more than two prime factors.*

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*

p	p^2+p+1		p	$\frac{1}{3}(p^2+p+1)$	
5	31	= prime	7	19	= prime
11	133	= 7.19	13	61	= prime
17	307	= prime	19	107	= prime
23	553	= 7.79	31	331	= prime
29	871	= 13.67	37	469	= 7.67
41	1723	= prime	43	631	= prime
47	2257	= 37.61	61	1261	= 13.97
53	2863	= 7.409	67	1519	= 7^2.31
59	3581	= prime	73	1801	= prime
71	5113	= prime	79	2107	= 7^2.43
83	6973	= 19.367	97	3169	= prime
89	8011	= prime			

The results obtained may be summarized as follows.

Case 1. $p \equiv -1 \pmod 3$.

The orders of the highest cyclical sub-groups are p^2+p+1, p^2-1, p^2-p, p, and $p-1$, and every substitution of the group occurs in some cyclical sub-group whose order is one of these numbers.

The order and type of the sub-groups within which these cyclical sub-groups are contained self-conjugately is then determined. For each cyclical sub-group of order p^2+p+1, this is a group of order $3(p^2+p+1)$, and it is shown that every sub-group containing substitutions whose orders are equal to or factors of p^2+p+1 must be contained within a sub-group of order $3(p^2+p+1)$.

Finally, every sub-group which contains no substitutions whose order is equal to or a factor of p^2+p+1 is shown to be contained either within one of two sub-groups whose orders are $p^3(p+1)(p-1)^2$ or within a sub-group of order $6(p-1)^2$. The first two of these three general types are both isomorphous with the general linear homogeneous group in two variables, while the third is isomorphous with the permutation-group of three symbols. In this third case, the form of the sub-group is limited to a few easily recognised types, and in the two former the problem of determining all possible types is not essentially distinct from the corresponding problem for the general linear group in two homogeneous variables.

Case 2. $p \equiv 1 \pmod 3$.

The orders of the highest cyclical sub-groups are $\frac{1}{3}(p^2+p+1)$, $\frac{1}{3}(p^2-1)$, $\frac{1}{3}(p^2-p)$, p, $p-1$, and $\frac{1}{3}(p-1)$, and every substitution of the group occurs in some cyclical sub-group whose order is one of these numbers.

The other results in this case are exactly the same as in the former case if the orders of all the sub-groups there mentioned be divided by 3.

In the first case, the non-homogeneous group is holohedrically isomorphous with the homogeneous group given by

$$\left.\begin{aligned} x' &\equiv \alpha x + \beta y + \gamma z \\ y' &\equiv \alpha' x + \beta' y + \gamma' z \\ z' &\equiv \alpha'' x + \beta'' y + \gamma'' z \end{aligned}\right\} \pmod p,$$

and advantage is taken of this to avoid entirely working with the non-homogeneous form. To give completeness to the paper I have ventured to deal at length with the reduction of a homogeneous sub-

stitution in three variables to its canonical form, although this problem has been completely treated for the general case of n variables by M. C. Jordan, in his *Traité des Substitutions*. It would, in fact, be at least as lengthy to quote M. Jordan's general results and apply them to the particular case of $n = 3$, as it is to obtain the results for the particular case *ab initio*. The group which is the subject of investigation is referred to sometimes as the main group, and sometimes as the group G.

2. *On the Representation of G as a Permutation Group.*

Consider the p^3-1 quantities $Ax+By+Cz$ formed by giving A, B, C any integral values from 0 to $p-1$, with the exception of simultaneous zero values. They may be arranged in p^2+p+1 sets of $p-1$ each, according to the following scheme

$$nx, \quad n(y+kx), \quad n(z+ky+k'x),$$

$$n = 1, 2, \ldots p-1; \quad k, k' = 0, 1, 2, \ldots p-1.$$

Now any substitution of the homogeneous group which changes $Ax+By+Cz$ into $A'x+B'y+C'z$ also changes $k(Ax+By+Cz)$ into $k(A'x+B'y+C'z)$. Hence, if one member of any one of the above p^2+p+1 sets is changed by the substitution into a member of a second set, then all the members of the first are changed into the various members of the second set. If, then, each set is regarded as a single entity, and is represented by the symbol $\{Ax+By+Cz\}$, the group is isomorphous with a permutation group of the p^2+p+1 symbols

$$\{x\}, \quad \{y+kx\}, \quad \{z+ky+k'x\},$$
$$k, k' = 0, 1, 2, \ldots p-1.$$

Now from the enumeration of all possible types of substitution given in the succeeding section, it follows that no substitution can keep more than $p+2$ of these symbols unchanged, this maximum number occurring in the case of substitutions of the type

$$x' \equiv \alpha x, \quad y' \equiv \alpha y, \quad z' \equiv \beta z,$$

which leaves unchanged the symbols

$$\{x\}, \quad \{y+kx\}, \quad \{z\}, \quad k = 0, 1, 2, \ldots p-1.$$

Hence the permutation-group of the p^2+p+1 symbols is holohedrically isomorphous, *i.e.*, abstractly considered, identical with the group defined by the congruences.

If now $Ax+By+Cz$, $A_1x+B_1y+C_1z$ are any two linear functions, one of which is not a multiple of the other, and if $A'x+B'y+C'z$, $A_1'x+B_1'y+C_1'z$ are any other pair satisfying the same condition, the coefficients being, as is always supposed, unless otherwise stated, real integers, it is easy to see that six other constants P, Q, R, P', Q', R' may be determined in a variety of ways, so that the congruences

$$A'x'+B'y'+C'z' \equiv Ax+By+Cz,$$

$$A_1'x'+B_1'y'+C_1'z' \equiv A_1x+B_1y+C_1z,$$

$$P'x'+Q'y'+R'z' \equiv Px+Qy+Rz,$$

give, on solution for x', y', z', a substitution of determinant unity. Hence the permutation-group is doubly-transitive, and therefore its order must be $(p^2+p+1)(p^2+p)\,m$, where m is the order of the sub-group obtained by keeping any two symbols unchanged. The type of this sub-group may be obtained at once, for, if $\{y\}$ and $\{z\}$ are the two unchanged symbols, two of the defining congruences of every one of its substitutions must be of the form

$$y' \equiv \beta y, \quad z' \equiv \gamma z.$$

The most general substitution satisfying this condition is

$$x' \equiv \alpha x+\alpha' y+\alpha'' z, \quad y' \equiv \beta y, \quad z' \equiv \gamma z,$$

where $$\alpha\beta\gamma \equiv 1,$$

and conversely the totality of substitutions of this type form a group.

Now the congruence $$\alpha\beta\gamma \equiv 1$$

has $(p-1)^2$ distinct solutions; for to α and β any values from 1 to $p-1$ may be assigned, and then γ is determinate; while α' and α'' may each have any value from 0 to $p-1$.

The number of distinct sets of defining congruences of the above type is therefore $p^2(p-1)^2$. If now the congruence

$$z^3-1 \equiv 0$$

has no real solution except unity, that is $p \equiv -1 \pmod 3$, each set of defining congruences gives a different substitution, and the order of the sub-group is $p^2(p-1)^2$.

If, however, $$z^3-1 \equiv 0$$

has three different real roots, 1, ϵ, ϵ^2, or if $p \equiv 1 \pmod 3$, the three

sets of congruences

$$x' \equiv \alpha x + \alpha' y + \alpha'' z, \qquad y' \equiv \beta y, \qquad z' \equiv \gamma z,$$
$$x' \equiv \epsilon\alpha x + \epsilon\alpha' y + \epsilon\alpha'' z, \qquad y' \equiv \epsilon\beta y, \qquad z' \equiv \epsilon\gamma z,$$
$$x' \equiv \epsilon^2\alpha x + \epsilon^2\alpha' y + \epsilon^2\alpha'' z, \qquad y' \equiv \epsilon^2\beta y, \qquad z' \equiv \epsilon^2\gamma z$$

give the same substitution, and the order of the sub-group is $\frac{1}{3}p^2(p-1)^2$.

Hence, when p is an odd prime greater than 3, the order of the main group is

$$(p^2+p+1)(p^2+p)p^2(p-1)^2 \quad \text{or} \quad \tfrac{1}{3}(p^2+p+1)(p^2+p)p^2(p-1)^2,$$

according as p is of the form $3m-1$ or $3m+1$. When p is 2 or 3, the order of the group is given by the former of these two expressions. These two special cases are, however, exceptional, and will be considered later.

When the characteristic congruence, as defined in the next section, is irreducible, no linear function of x, y, z with real coefficients is altered into a multiple of itself; and when it is the product of a linear factor and an irreducible quadratic factor there is one such function. An inspection of the other types of substitution, which are given explicitly in the next section, shows that in other cases there may be 3, $p+1$ or $p+2$ linear functions which are changed into multiples of themselves. The substitutions of the group, therefore, when expressed as a doubly-transitive permutation group of p^2+p+1 symbols, must either permute all the symbols or must keep 1, 3, $p+1$ or $p+2$ symbols unchanged.

Case I. $p \equiv -1 \pmod 3$.

3. *On the Typical Forms of the Substitutions of G.*

Let
$$\left.\begin{aligned} x' &\equiv ax + by + cz \\ y' &\equiv a'x + b'y + c'z \\ z' &\equiv a''x + b''y + c''z \end{aligned}\right\} \pmod p,$$

be any substitution S, of determinant unity. Then

$$Ax' + By' + Cz'$$
$$= (Aa + Ba' + Ca'')x + (Ab + Bb' + Cb'')y + (Ac + Bc' + Cc'')z.$$

Hence $Ax+By+Cz$ is transformed into a multiple λ, of itself, if

$$\begin{aligned} A(a-\lambda)+ \quad Ba' \quad + \quad Ca'' \quad &\equiv 0, \\ Ab \quad +B(b'-\lambda)+ \quad Cb'' \quad &\equiv 0, \\ Ac \quad + \quad Bc' \quad +C(c''-\lambda) &\equiv 0, \end{aligned}$$

so that λ is given by

$$\begin{vmatrix} a-\lambda, & a', & a'' \\ b, & b'-\lambda, & b'' \\ c, & c', & c''-\lambda \end{vmatrix} \equiv 0.$$

This congruence is known as the characteristic congruence of the substitution, and it is well known that if T is any other substitution of the same form as S, then $T^{-1}ST$ has the same characteristic congruence as S;* which is the same as saying that all conjugate substitutions within the group have the same characteristic congruences. The converse of this theorem is not generally true.

If, however, the characteristic congruence has three unequal roots, whether real or imaginary, then all substitutions which have such a common characteristic congruence are conjugate substitutions. This theorem is of so great importance for what follows that I give a formal proof of it.

Suppose, then, that $\lambda_1, \lambda_2, \lambda_3$ are the three unequal roots of the above congruence. Corresponding to λ_1, the ratios $A : B : C$ are given by

$$A_1 : B_1 : C_1$$

$$:: \lambda_1^2-\lambda_1(b'+c'')+b'c''-b''c' : b\lambda_1+b''c-bc'' : c\lambda_1+bc'-b'c.$$

If, then, $\xi = x,$

$$\eta = -(b'+c'')x+by+cz,$$

$$\zeta = (b'c''-b''c')x+(c'a''-c''a')y+(a'b''-a''b')z,$$

the substitution S may be written in the form

$$\lambda_1^2\xi'+\lambda_1\eta'+\zeta' = \lambda_1(\lambda_1^2\xi+\lambda_1\eta+\zeta),$$

$$\lambda_2^2\xi'+\lambda_2\eta'+\zeta' = \lambda_2(\lambda_2^2\xi+\lambda_2\eta+\zeta),$$

$$\lambda_3^2\xi'+\lambda_3\eta'+\zeta' = \lambda_3(\lambda_3^2\xi+\lambda_3\eta+\zeta),$$

---

* Jordan, *Traité des Substitutions*, p. 98.

while every other substitution with the same characteristic equation can be represented in this form when ξ, η, ζ are replaced by three other independent linear functions of x, y, z with real integral coefficients. In particular, the above form, when ξ, η, ζ are replaced by x, y, z, may be taken as the type of all substitutions whose characteristic congruences have the three unequal roots λ_1, λ_2, λ_3.

If, now, in this form x, y, z be replaced by

$$\begin{aligned} &\alpha x + \beta y + \gamma z, \\ &\alpha' x + \beta' y + \gamma' z, \\ &\alpha'' x + \beta'' y + \gamma'' z, \end{aligned}$$

and a corresponding change be made in the accented symbols, the resulting substitution is that represented by TST^{-1}, where T is the substitution

$$\begin{aligned} x' &\equiv \alpha x + \beta y + \gamma z, \\ y' &\equiv \alpha' x + \beta' y + \gamma' z, \\ z' &\equiv \alpha'' x + \beta'' y + \gamma'' z. \end{aligned}$$

This will not generally be a substitution of determinant unity, so that TST^{-1} is not necessarily conjugate to S within the group considered. It remains to be shown that T can be expressed in the form $T_1 T_2$, where T_1 is a substitution of determinant unity, where T_2 is permutable with S. Writing the substitution S, for a moment, in the abbreviated form

$$X' \equiv \lambda_1 X, \quad Y' \equiv \lambda_2 Y, \quad Z' \equiv \lambda_3 Z,$$

it is evidently permutable with every substitution of the form

$$X' = \kappa_1 X, \quad Y' = \kappa_2 Y, \quad Z' = \kappa_3 Z,$$

and this latter will certainly be a substitution with real coefficients if

$$\kappa_1 = f(\lambda_1), \quad \kappa_2 = f(\lambda_2), \quad \kappa_3 = f(\lambda_3),$$

where $f(\lambda)$ is any rational function of λ with real coefficients. The determinant of this substitution is $f(\lambda_1) f(\lambda_2) f(\lambda_3)$, which may be given any value from 1 to $p-1$, by suitably choosing $f(\lambda)$. Hence, whatever the determinant n of T may be, a substitution of determinant n may be found which is permutable with S; and, since the complete set of substitutions of determinant n arise by combining any one of them with the group of substitutions of determinant unity, it follows that T can be expressed in the required form $T_1 T_2$.

It may be pointed out that the theorem thus proved, and the proof

itself, hold equally well whatever the number of variables involved may be.

The characteristic congruence may be (i) irreducible, (ii) the product of an irreducible quadratic factor and a linear factor, or (iii) the product of three linear factors; and it is clearly only in the last case that it can have equal roots. A typical form of any substitution for which the three roots are all unequal has already been found.

Suppose, now, that the congruence has two equal roots, so that the roots may be taken as α, β, β; these being real numbers. Exactly as before, two independent linear functions of x, y, z may be found (here necessarily with real coefficients) which the substitution multiplies by α and β, so that taking these to replace x and y, the substitution may be written

$$\xi' \equiv \alpha\xi,$$

$$\eta' \equiv \beta\eta,$$

$$z' \equiv \alpha''\xi + \beta''\eta + \beta z.$$

Hence $$z' + P\xi' + Q\eta' \equiv \beta z + (\alpha'' + P\alpha)\,\xi + (\beta'' + Q\beta)\,\eta.$$

If $$\beta'' \equiv 0,$$

and P is chosen so that $$P(\beta - \alpha) \equiv \alpha'',$$

then $$z' + P\xi' + Q\eta' \equiv \beta(z + P\xi + Q\eta);$$

so that, writing $$\zeta = z + P\xi + Q\eta,$$

the substitution takes the form

$$\xi' = \alpha\xi,$$

$$\eta' = \beta\eta,$$

$$\zeta' = \beta\zeta.$$

If, however, $\beta'' \not\equiv 0$, it is impossible to reduce the substitution to this form. In this case, if

$$P(\beta - \alpha) \equiv \alpha'',$$

$$Q \equiv 0,$$

and $$\zeta \equiv z + P\xi,$$

the substitution may be written

$$\xi' \equiv \alpha\xi, \quad \eta' = \beta\eta, \quad \zeta' \equiv \beta\left(\zeta + \frac{\beta''}{\beta}\eta\right),$$

and if, further, η be written for $\frac{\beta''}{\beta}\eta$, the form will be

$$\xi' \equiv \alpha\xi, \quad \eta' \equiv \beta\eta, \quad \zeta' \equiv \beta(\zeta+\eta).$$

Every substitution whose characteristic congruence has two equal roots must come under one of these two types, but it is immediately evident that a substitution of the one type cannot be conjugate to one of the other type. On the other hand, a repetition of the previous reasoning will show that all substitutions of the first of these two types with a common characteristic congruence are conjugate.

If the characteristic congruence has three equal roots, each must be unity. In this case one linear function of x, y, z with real coefficients can be found which is unaltered by the substitution, and, if this be denoted by ξ, the substitution can be expressed in the form

$$\xi' \equiv \xi,$$
$$y' \equiv \alpha'\xi + \beta' y + \gamma' z,$$
$$z' \equiv \alpha''\xi + \beta'' y + \gamma'' z,$$

where $$\begin{vmatrix} \beta'-\lambda & \gamma' \\ \beta'' & \gamma''-\lambda \end{vmatrix} \equiv (1-\lambda)^2.$$

Now, $Py' + Qz' \equiv (P\beta' + Q\beta'')\, y + (P\gamma' + Q\gamma'')\, z + (P\alpha' + Q\alpha'')\, \xi,$

and the congruences $$P \equiv P\beta' + Q\beta'',$$
$$Q \equiv P\gamma' + Q\gamma''$$

are, from the above equation of condition, equivalent to each other. Hence, if P and Q are determined from

$$P(\beta'-1) + Q\beta'' \equiv 0,$$
$$P\alpha' \qquad + Q\alpha'' \equiv 1,$$

then $$Py' + Qz' = Py + Qz + \xi;$$

and, when η is written for $Py + Qz$, the substitution takes the form

$$\xi' = \xi,$$
$$\eta' = \xi + \eta,$$
$$z' = a''\xi + b''\eta + z.$$

Here again

$$Lz' + M\xi' + N\eta' \equiv Lz + (La'' + M + N)\,\xi + (Lb'' + N)\,\eta,$$

and if

$$La''+N\equiv 0,$$
$$Lb'' \equiv 1,$$
$$Lz'+M\xi'+N\eta'\equiv Lz+M\xi+N\eta+\eta;$$

hence, writing

$$\zeta = Lz+M\xi+N\eta,$$

the substitution becomes

$$\xi'\equiv\xi,$$
$$\eta'\equiv\xi+\eta,$$
$$\zeta'\equiv\eta+\zeta.$$

It has been assumed that b'' is different from zero; if, however, b'' were zero, the corresponding typical form would be

$$\xi'\equiv\xi,$$
$$\eta'\equiv\xi+\eta,$$
$$\zeta'\equiv\zeta,$$

so that again, when the characteristic equation has three equal roots, there are two distinct types.

4. *On the Orders of the Substitutions of G, and on their Distribution in Cyclical Sub-Groups.*

When the characteristic congruence

$$\lambda^3-\alpha\lambda^2+\beta\lambda-1\equiv 0$$

of a substitution is irreducible, the roots are, according to Galois' theory, of the form λ, λ^p, λ^{p^2}, where

$$\lambda^{p^2+p+1}-1\equiv 0.$$

Now, if the real form of the substitution is

$$x'\equiv ax+by+cz,$$
$$y'\equiv a'x+b'y+c'z,$$
$$z'\equiv a''x+b''y+c''z,$$

then

$$\alpha\equiv a+b'+c'',$$
$$\beta\equiv b''c'-b'c''+ca''-c''a+a'b-ab',$$

and α and β can evidently, by suitably choosing the substitution, take all possible values. Hence all cubic congruences in which the coefficient of the leading term is unity, while the constant term is

negative unity, must occur among the characteristic congruences. Among those that are irreducible must therefore occur congruences satisfied by a primitive root of

$$\lambda^{p^2+p+1}-1 \equiv 0.$$

If $$X' \equiv \lambda X, \quad Y' \equiv \lambda^p Y, \quad Z' \equiv \lambda^{p^2} Z$$

is a substitution in its typical form corresponding to such a congruence, its order m is the least integer which satisfies

$$\lambda^m \equiv \lambda^{mp} \equiv \lambda^{mp^2},$$

and, since in the case considered $p-1$ has no factor in common with p^2+p+1, this least value of m is p^2+p+1.

Moreover, the roots of any other irreducible characteristic congruence can be clearly expressed in the form λ^r, λ^{rp}, λ^{rp^2}, so that the corresponding substitutions are r^{th} powers of substitutions of orders p^2+p+1. The orders of all substitutions, therefore, whose characteristic congruences are irreducible are either p^2+p+1 or a factor of this number.

When the characteristic congruence is resolvable into a linear factor and an irreducible quadratic factor, so that

$$\lambda^3-\alpha\lambda^2+\beta\lambda-1 \equiv (\lambda-n)(\lambda^2-\alpha'\lambda+\beta'),$$

where n, α', β' are real, the quadratic congruence

$$\lambda^2-\alpha'\lambda+\beta' \equiv 0$$

may be any whatever, since α and β can take all possible values, and among such congruences must occur those satisfied by a primitive root μ of

$$\mu^{p^2-1}-1 \equiv 0.$$

The typical form of the corresponding substitution is

$$X' \equiv \mu X, \quad Y' \equiv \mu^p Y, \quad Z' \equiv \mu^{-(p+1)} Z,$$

and its order, which is the least integer m satisfying

$$\mu^m \equiv \mu^{mp} \equiv \mu^{-m(p+1)},$$

is p^2-1. The roots of every other irreducible quadratic congruence can be expressed in the form μ^r, μ^{rp}, where r is not a multiple of $p+1$; and therefore the order of every substitution whose characteristic congruence has an irreducible quadratic factor is either p^2-1 or some factor of this number which is not at the same time a factor of $p-1$.

It has been seen that all other substitutions can be reduced, without the use of imaginaries, to one of the five following typical forms:—

$$\text{(i)}\quad x' \equiv \alpha x, \quad y' \equiv \beta y, \quad z' \equiv \gamma z, \quad \alpha\beta\gamma \equiv 1;$$

$$\text{(ii)}\quad x' \equiv \alpha x, \quad y' \equiv \beta y, \quad z' \equiv \beta z, \quad \alpha\beta^2 \equiv 1;$$

$$\text{(iii)}\quad x' \equiv \alpha x, \quad y' \equiv \beta y, \quad z' \equiv \beta(z+y), \quad \alpha\beta^2 \equiv 1;$$

$$\text{(iv)}\quad x' \equiv x, \quad y' \equiv y+z, \quad z' \equiv z;$$

$$\text{(v)}\quad x' \equiv x, \quad y' \equiv y+x, \quad z' \equiv z+y.$$

The orders of these types can be determined by inspection. For (i) or (ii) the order is $p-1$, or a factor of $p-1$; for (iii) it is $p(p-1)$, or a factor of this number which itself contains p as a factor; for (iv) and (v) the order is p.

The result of this discussion is to show that the main group contains cyclical sub-groups whose orders are p^2+p+1, p^3-1, p^2-p, $p-1$, p, or factors of these numbers, and that every substitution of the group, except identity, is contained in some such sub-group.

I go on next to discuss the number of cyclical sub-groups of each type, and their distribution into conjugate sets.

Order p^2+p+1. The type of substitution S which will generate a cyclical sub-group of order p^2+p+1 is

$$X' \equiv \lambda X, \quad Y' \equiv \lambda^p Y, \quad Z' \equiv \lambda^{p^2} Z,$$

where

$$X \equiv \lambda^2 x + \lambda y + z,$$

$$Y \equiv \lambda^{2p} x + \lambda^p y + z,$$

$$Z \equiv \lambda^{2p^2} x + \lambda^{p^2} y + z.$$

If a substitution T is permutable with S, it must keep the same three (imaginary) elements unchanged, and must therefore be of the form

$$X' \equiv \kappa_1 X, \quad Y' \equiv \kappa_2 Y, \quad Z' \equiv \kappa_3 Z.$$

If, now, $\kappa_1 \equiv f(\lambda), \quad \kappa_2 \equiv f(\lambda^p), \quad \kappa_3 \equiv f(\lambda^{p^2}),$

this is a real substitution, since when expressed in terms of x, y, z the coefficients are symmetric functions of $\lambda, \lambda^p, \lambda^{p^2}$, and therefore real. But, if the κ's are not of the above form, the coefficients are unsymmetric functions of $\lambda, \lambda^p, \lambda^{p^2}$ are therefore necessarily imaginary.

Now, any rational function of λ with real coefficients is some power of λ_1, a primitive root of

$$\lambda_1^{p^3-1}-1 \equiv 0$$

and if $$\lambda_1^r \equiv f(\lambda),$$

$$\lambda_1^{rp} \equiv [f(\lambda)]^p \equiv f(\lambda^p),$$

and $$\lambda_1^{rp^2} \equiv f(\lambda^{p^2}).$$

The determinant of

$$X' \equiv f(\lambda)\, X, \quad Y' \equiv f(\lambda^p)\, Y, \quad Z' \equiv f(\lambda^{p^2})\, Z$$

is then only unity when

$$\lambda_1^{r(p^2+p+1)}-1 \equiv 0,$$

or when r is a multiple of $p-1$.

But in this case $$f(\lambda) \equiv \lambda_1^{\kappa(p-1)} \equiv \lambda^\kappa,$$

and therefore the only substitutions with which S is permutable are its own powers.

The substitution S, therefore, forms one of a set of $\dfrac{N}{p^2+p+1}$ conjugate substitutions, the symbol N denoting the order of the main group. Now, the only powers of S which have the same multipliers (*i.e.*, the same characteristic congruence) as S are clearly S^p and S^{p^2}, and to each set of three substitutions such as S^r, S^{rp}, S^{rp^2} contained in the cyclical sub-group generated by S, which belong to the same characteristic congruence, there corresponds such a set of $\dfrac{N}{p^2+p+1}$ conjugate substitutions. There are, therefore, in all $\frac{1}{3}(p^2+p)\dfrac{N}{p^2+p+1}$ substitutions whose orders are p^2+p+1 or one of its factors, and these form $\frac{1}{3}\dfrac{N}{p^2+p+1}$ conjugate cyclical sub-groups of order p^2+p+1, each of which must therefore be contained self-conjugately in a sub-group of order $3(p^2+p+1)$.

Order p^2-1. The type of substitution S which will generate a cyclical sub-group of order p^2-1 is

$$X' \equiv \mu X, \quad Y' \equiv \mu^p Y, \quad Z' \equiv \mu^{-(p+1)} Z,$$

where μ is a primitive root of

$$\mu^{p^2-1}-1 \equiv 0.$$

By reasoning almost identical with that used in the previous case, it may be shown that this substitution is permutable only with its own powers, so that S is one of a set of $\frac{N}{p^2-1}$ conjugate substitutions.

The only power of S which has the same multipliers as S is S^p, and therefore this set of conjugate substitutions consists of $\frac{1}{2}\frac{N}{p^2-1}$, no one of which is a power of any other, and their p^{th} powers. These $\frac{1}{2}\frac{N}{p^2-1}$ substitutions generate as many conjugate cyclical sub-groups of order p^2-1, each of which is therefore contained self-conjugately in a sub-group of order $2(p^2-1)$.

That the substitutions contained in these cyclical sub-groups, whose orders are not $p-1$ or a factor of $p-1$, are all different, may be verified by noticing that they form $\frac{1}{2}(p^2-p)$ different sets, each set having the same multipliers; while each set with common multipliers are shown above to contain $\frac{N}{p^2-1}$ conjugate substitutions. The total number of substitutions contained in the main group, then, whose orders are equal to or factors of p^2-1, without being equal to or factors of $p-1$, is $\frac{1}{2}\frac{Np}{p+1}$.

Order p^2-p. The type of substitution which generates a cyclical sub-group of order p^2-p is

$$x' \equiv \alpha(x+y), \quad y' \equiv \alpha y, \quad z' \equiv \alpha^{-2}z,$$

where α is a primitive root, mod. p.

Considered as an operation of the permutation group, this is an operation belonging to the sub-group which keeps the two symbols $\{y\}$ and $\{z\}$ fixed. The general type of such sub-group is

$$x' \equiv \alpha x+\alpha' y+\alpha'' z, \quad y' \equiv \beta y, \quad z' \equiv \gamma z, \quad \alpha\beta\gamma \equiv 1,$$

and, since the permutation group is doubly transitive, there are $\frac{1}{2}(p^2+p+1)(p^2+p)$ such sub-groups all conjugate to each other. I shall then first consider the number of cyclical sub-groups of order p^2-p contained in the sub-group that keeps $\{y\}$ and $\{z\}$ fixed, and

their relation to each other. It will then be easy to extend the results to the totality of such cyclical sub-groups.

The necessary and sufficient conditions that the typical substitution of the sub-group, above written, should be of order $p(p-1)$ are that (i) α should be a primitive root, mod. p; (ii) either $\beta \equiv \alpha$ and $\alpha' \not\equiv 0$, or $\gamma \equiv \alpha$ and $\alpha'' \not\equiv 0$. Taking first $\beta \equiv \alpha$ and $\alpha' \not\equiv 0$, the n^{th} power of the substitution

$$x' \equiv \alpha x + \alpha' y + \alpha'' z, \quad y' \equiv \alpha y, \quad z' \equiv \alpha^{-2} z$$

is $$x' \equiv \alpha^n x + n\alpha' \alpha^{n-1} y + n\alpha'' \alpha^{2-2n} z, \quad y' \equiv \alpha^n y, \quad z' \equiv \alpha^{-2n} z.$$

Hence neither of the substitutions

$$x' \equiv \alpha x + y + Az, \quad y' \equiv \alpha y, \quad z' \equiv \alpha^{-2} z,$$

$$x' \equiv \alpha x + y + Bz, \quad y' \equiv \alpha y, \quad z' \equiv \alpha^{-2} z,$$

can be a power of the other, when A and B are different; and therefore the p substitutions obtained from either of these by writing for A or B all values from 0 to $p-1$ generate p different cyclical sub-groups of order p^2-p. Moreover, every substitution of the sub-group that keeps $\{y\}$ and $\{z\}$ fixed, whose order is a factor of p^2-p without at the same time being p or a factor of $p-1$, and for which $\alpha \equiv \beta$, is contained in one of these cyclical sub-groups. For let

$$x' \equiv \alpha^s x + \alpha' y + \alpha'' z, \quad y' \equiv \alpha^s y, \quad z' \equiv \beta^{-2s} z$$

be such a substitution.

The $\left[s+\kappa(p-1)\right]^{\text{th}}$ power of

$$x' \equiv \alpha x + y + Az, \quad y' \equiv \alpha y, \quad z' \equiv \alpha^{-2} z$$

is $$x' \equiv \alpha^s x + \left[s+\kappa(p-1)\right] \alpha^{s-1} y + \left[s+\kappa(p-1)\right] A\alpha^{2-2s} z,$$

$$y' \equiv \alpha^s y, \quad z' \equiv \alpha^{-2s} z,$$

and κ, A can be chosen in one way so that this is the same as the given substitution.

There are, therefore, within the sub-group which keeps $\{y\}$ and $\{z\}$ fixed, p cyclical sub-groups of order p^2-p for which $\alpha \equiv \beta$, and there are therefore p more for which $\alpha \equiv \gamma$. Moreover, these cyclical sub-groups are all conjugate with the larger sub-group considered.

For it may be verified by actual calculation that the substitution

$$\left(x+\frac{B-A}{\alpha^{-2}-\alpha}z,\ y,\ z\right)^*$$

transforms $(\alpha x+y+Az,\ \alpha y,\ \alpha^{-2}z)$

into $(\alpha x+y+Bz,\ \alpha y,\ \alpha^{-2}z)$;

while $(x,\ -z,\ y)$

transforms $(\alpha x+y,\ \alpha y,\ \beta z)$

into $(\alpha x+z,\ \beta y,\ \alpha z)$.

The sub-group which keeps $\{y\}$ and $\{z\}$ fixed contains, then, $2p$ conjugate cyclical sub-groups of order p^2-p, and the substitutions of these cyclical groups whose orders are not p or factors of $p-1$ are all different.

The $\frac{1}{2}(p^2+p+1)(p^2+p)$ conjugate sub-groups each of which keeps two symbols fixed contain in all $(p^2+p+1)(p^3+p)\,p$ conjugate cyclical sub-groups of order p^2-p. This number is equal to $\frac{N}{p(p-1)^2}$, and therefore each such cyclical sub-group is contained self-conjugately in a group of order $p(p-1)^2$. The type of this group is given by

$$x' \equiv \alpha x+\alpha' y,\quad y' \equiv \beta y,\quad z' \equiv \gamma z,\quad \alpha\beta\gamma \equiv 1.$$

Each of the cyclical sub-groups contains $(p-1)(p-2)$ substitutions, whose orders are neither p nor $p-1$ or one of its factors, and the main group therefore contains $\frac{N(p-2)}{p(p-1)}$ substitutions whose orders are factors of p^2-p, which are different from p and from $p-1$ and its factors.

Order p. There are two types of sub-group of order p, and of these I first consider those of the form

$$(x+z,\ y,\ z).$$

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* Where there is no risk of confusion, the substitution

$$x' \equiv ax+by+cz,\quad y' \equiv a'x+b'y+c'z,\quad z' \equiv a''x+b''y+c''z$$

will in future be written in the abbreviated form

$$(ax+by+cz,\ a'x+b'y+c'z,\ a''x+b''y+c''z).$$

The sub-group which keeps $\{y\}$ and $\{z\}$ fixed contains p^2-1 substitutions of this type, which are given generally by

$$(x+\alpha y+\beta z,\ y,\ z),$$

where α and β take all possible values. Now, the substitution

$$(ax,\ by,\ cz)$$

will transform $(x+\alpha y+\beta z,\ y,\ z)$

into $(x+\alpha' y+\beta' y,\ y,\ z)$,

if $a\alpha \equiv b\alpha',\quad a\beta \equiv c\beta'.$

Since $abc \equiv 1,$

these congruences give $a^3 \equiv \dfrac{\alpha'\beta'}{\alpha\beta},$

which, when α, β, α', β' are finite, always has a real solution, in the case $p \equiv -1 \pmod 3$, which is under consideration. On the other hand,

$$(x,\ z,\ -y)$$

transforms $(x-y,\ y,\ z)$

into $(x+z,\ y,\ z)$,

and therefore the whole set of p^2-1 substitutions are conjugate within the main group. The p^2 substitutions give $p+1$ cyclical sub-groups contained in the sub-group which keeps $\{y\}$ and $\{z\}$ fixed. Every sub-group keeping two symbols fixed similarly contains $p+1$ such cyclical sub-groups; but these are not all distinct, for the cyclical sub-groups occurring in the groups keeping any two of the $p+1$ symbols $\{y\}$, $\{z\}$, $\{y+nz\}$, $n = 1, 2, \ldots p-1$, are evidently all the same. Hence the main group contains $\frac{\frac{1}{2}(p^2+p+1)(p^2+p)}{\frac{1}{2}(p+1)p}(p+1)$ such cyclical sub-groups, which are all conjugate to each other, as also are all their substitutions. This number, expressed as before, is $\dfrac{N}{p^3(p-1)^2}$, so that each such sub-group is contained self-conjugately in a sub-group of order $p^3(p-1)^2$. Thus the cyclical sub-group generated by

$$(x,\ y+z,\ z)$$

is self-conjugate within the group given by

$$(ax+by+cz,\ b'y+c'z,\ c''z),\quad ab'c'' \equiv 1.$$

The total number of substitutions of order p and of this first type contained in the main group is $\frac{N}{p^3(p-1)}$.

The second type of substitution of order p is

$$(x+y,\ y+z,\ z).$$

The n^{th} power of this substitution is

$$\left(x+ny+\tfrac{1}{2}n(n-1)z,\ y+nz,\ z\right),$$

and the conditions that the substitution should be transformed into its n^{th} power by

$$(ax+by+cz,\ a'x+b'y+c'z,\ a''x+b''y+c''z)$$

are easily found to be

$$a' \equiv a'' \equiv b'' \equiv 0, \quad an \equiv b', \quad b'n = c'', \quad c' \equiv bn+\tfrac{1}{2}an(n-1).$$

These give

$$a \equiv \frac{1}{n}, \quad b' \equiv 1, \quad c'' = n, \quad c' \equiv bn+\tfrac{1}{2}(n-1).$$

Hence the sub-group given by all substitutions of the form

$$\left(\frac{1}{n}x+by+cz,\quad y+\left[bn+\tfrac{1}{2}(n-1)\right]z,\quad nz\right)$$

is the sub-group of greatest order which contains the cyclical sub-group generated by

$$(x+y,\ y+z,\ z)$$

self-conjugately. Since b, c may take all possible values, and n all values except zero, the order of this sub-group is $(p-1)p^2$. Hence the cyclical sub-group is one of a conjugate set of $\frac{N}{(p-1)p^2}$ contained in the main group.

By transforming $(x+y, y+z, z)$ with a substitution which keeps $\{z\}$ fixed, it may be seen at once that all possible sub-groups of the type considered may be obtained for which $\{z\}$ is unchanged; and hence the conjugate set of sub-groups just obtained contains all sub-groups of the order p and of the second type.

The substitutions of these groups are necessarily all different, and all conjugate with each other; and the number of such substitutions contained in the main group is $\frac{N}{p^2}$.

Order $p-1$. The cyclical sub-groups of order $p-1$, unlike the sub-groups of other orders, do not form a single conjugate set. If α is any primitive root, mod. p, α^r, α^s, $\alpha^{-(r+s)}$ will be the multipliers of a substitution of order $p-1$, if and only if the greatest common factor of r and s is prime relatively to $p-1$. The cyclical sub-group generated by $(\alpha^r x, \alpha^s y, \alpha^{-(r+s)}z)$ will contain $\phi(p-1)$ substitutions of order $p-1$, where $\phi(n)$ is the symbol used in the theory of numbers for the number of integers less than and prime to n. Two of these substitutions will have the same multipliers if the set of quantities α^{mr}, α^{ms}, $\alpha^{-m(r+s)}$ is identical with the set α^r, α^s, $\alpha^{-r+s)}$ for some value of m different from unity; and it may be at once verified that the only values of r, s, and m for which this can be the case are given by

$$r+s \equiv 0, \quad m \equiv p-2 \pmod{p-1}.$$

Hence in a cyclical substitution arising from a substitution with the multipliers, α, α^{-1}, 1, the sets of multipliers of the substitutions of order $p-1$ are the same in pairs, and the sub-group contains only $\frac{1}{2}\phi(p-1)$ such sets of multipliers; whereas in every cyclical sub-group of order $p-1$ which arises from a substitution with multipliers no one of which is unity the sets of multipliers of the $\phi(p-1)$ substitutions of order $p-1$ are all different.

Now, the number of ways in which two distinct symbols r, s, less than $p-1$, may be chosen so that their highest common factor is prime relatively to $p-1$, excluding simultaneous zero values, is

$$\phi(p-1)\,\psi(p-1),^*$$

where
$$\psi(p-1) = (p-1)\left(1+\frac{1}{q_1}\right)\left(1+\frac{1}{q_2}\right)\ldots,$$

q_1, q_2, ... being the different prime factors of $p-1$.

If r, s, $-(r+s)$, the indices of a set of multipliers of a substitution of order $p-1$, are all different, then

$$r, s; \quad r, -(r+s); \quad s, -(r+s);$$
$$s, r; \quad -(r+s), r; \quad -(r+s), s$$

will appear in the above solution as six distinct ways of choosing r and s, which, however, all lead to the same set of multipliers.

If, on the other hand, r, r, $-2r$ are the multipliers of a substitution of order $p-1$, then

$$r, r; \quad r, -2r; \quad -2r, r$$

---

* *Cf.* Jordan, *Traité des Substitutions*, p. 96.

will appear as three distinct ways of choosing r and s, which again all lead to the same set of multipliers.

In this latter case, r must be prime to $p-1$, and may therefore have $\varphi(p-1)$ values. There are, then, $3\phi(p-1)$ such solutions of the problem of choosing r and s, leading to $\phi(p-1)$ sets of multipliers. Subtracting these $3\phi(p-1)$ solutions from the total number, there remain

$$\phi(p-1)\left[\psi(p-1)-3\right]$$

solutions, leading to $\frac{1}{6}\phi(p-1)\left[\psi(p-1)-3\right]$

further sets of multipliers; and the number of distinct sets of multipliers is therefore in all

$$\tfrac{1}{6}\phi(p-1)\left[\psi(p-1)+3\right].$$

Of these sets of multipliers $\frac{1}{2}\phi(p-1)$ occur in a cyclical sub-group arising from a substitution whose multipliers are α, α^{-1}, 1; while it has been seen that the sets of multipliers of the substitutions of order $p-1$ in any other cyclical sub-group of this order are all distinct. Hence there are $\frac{1}{6}\psi(p-1)$ further types of cyclical sub-group of order $p-1$, each type containing an entirely distinct collection of sets of multipliers of the substitutions of order $p-1$ from all the others. The total number of types of cyclical sub-group of order $p-1$ is therefore $\frac{1}{6}\psi(p-1)+1$.

The cyclical sub-group arising from the substitution

$$(\alpha x,\ \alpha^{-1}y,\ z)$$

is transformed into itself by an operation which transforms the substitution itself into its $(p-2)^{\text{th}}$ power, that is, into

$$(\alpha^{-1}x,\ \alpha y,\ z).$$

The general form of an operation which will effect this transformation is

$$(ay,\ bx,\ cz) \quad abc \equiv -1,$$

and the group that arises by combining together these substitutions in all possible ways, containing all substitutions of the above forms together with those of the form

$$(a'x,\ b'y,\ c'z) \quad a'b'c' \equiv 1,$$

is of order $2(p-1)^2$. Hence this type of cyclical sub-group of order

$p-1$ is self-conjugate in a group of order $2(p-1)^2$, and therefore forms one of a set of $\frac{N}{2(p-1)^2}$ conjugate sub-groups. The remaining types contain no substitutions which can be transformed into powers of themselves, and hence, to find the sub-groups within which they are self-conjugate, it is only necessary to find the substitutions permutable with them. When the multipliers of the generating substitution $(a^r x, a^s y, a^{-(r+s)} z)$, so that $r \pm s \not\equiv 0$, it is seen at once that the only substitutions with which the cyclical sub-group is permutable are those of the form

$$(ax,\ by,\ cz) \quad abc \equiv 1,$$

forming a group of order $(p-1)^2$. Each of the $\frac{1}{6}\psi(p-1)-1$ types, coming under this head, is therefore self-conjugate in a group of order $(p-1)^2$, and each forms one of a set of $\frac{N}{(p-1)^2}$ conjugate sub-groups.

The remaining type of cyclical sub-group arises from a substitution of the form

$$(ax,\ ay,\ a^{-2}z).$$

The conditions that this substitution should be permutable with

$$(ax+by+cz, \quad a'x+b'y+c'z, \quad a''x+b''y+c''z)$$

are

$$c \equiv c' \equiv a'' \equiv b''' \equiv 0,$$

and the order of the sub-group so defined is $p(p+1)(p-1)^2$. This remaining type therefore forms one of a set of $\frac{N}{p(p+1)(p-1)^2}$ conjugate sub-groups.

It would not be easy to determine, from the above enumeration of the sets of conjugate groups of order $p-1$, the total number of substitutions contained in the main group whose orders are equal to or factors of $p-1$, but the number in question may be obtained independently in the following manner.

The sub-group of order $(p-1)^2$ whose type is

$$(ax,\ by,\ cz) \quad abc \equiv 1$$

is self-conjugate within a group of order $6(p-1)^2$ obtained by combining the group itself with all those substitutions which permute $\{x\}$, $\{y\}$, $\{z\}$ among themselves. It forms therefore one of

$\frac{N}{6(p-1)^2}$ conjugate sub-groups. Any one of the $(p-1)^2$ substitutions belonging to the original group which keeps three symbols only fixed appears in that group only; but a substitution of the form (ax, ay, cz) appears in each of the $\frac{1}{2}(p+1)p$ conjugate groups which keeps $\{z\}$ and any pair of the symbols $\{x\}$, $\{y\}$, $\{x+\kappa y\}$, $\kappa = 1, 2, \ldots p-1$ fixed. Now, of the $(p-1)^2-1$ substitutions in the original group, other than identity, $3(p-2)$ keep $p+2$ symbols fixed. Hence the total number of substitutions in the main group whose orders are equal to or factors of $p-1$ is

$$\frac{N}{6(p-1)^2}\left[(p-1)^2-1-3(p-2)+\frac{3(p-2)}{\frac{1}{2}p(p+1)}\right],$$

or

$$\frac{N(p-2)}{6(p-1)^2}\left[p-3+\frac{6}{p(p+1)}\right].$$

As a partial verification of the accuracy of the enumeration that has now been completed of the number of substitutions of each different order that are contained in the main group, it may be observed that the sum of

$\frac{N(p^3+p)}{3(p^2+p+1)}$, the number of substitutions whose orders are equal to or factors of p^2+p+1,

$+\ \frac{Np}{2(p+1)}$, the number whose orders are equal to or factors of p^2-1, without being factors of $p-1$,

$+\ \frac{N(p-2)}{p(p-1)}$, the number whose orders are equal to or factors of p^2-p, while different from p, $p-1$, or its factors,

$+\ \frac{N}{p^3(p-1)}$, the number whose orders are p, and which are of the type $(x+z, y, z)$,

$+\ \frac{N}{p^2}$, the number whose orders are p, and which are of the type $(x+y, y+z, z)$,

$+\ \frac{N(p-2)}{6(p-1)^2}\left[p-3+\frac{6}{p(p+1)}\right]$, the number whose orders are equal to or factors of $p-1$,

$+1$, the identical substitution,

is N, as it should be.

5. *On the Sub-Groups of G which contain Substitutions of Order p.*

Before going on to a general discussion of the various types of sub-group contained in the group of substitutions considered, it will be convenient to begin by obtaining certain results relative to sub-groups whose order is divisible by p, as these will materially shorten certain portions of the subsequent discussion.

Suppose first that a sub-group g of order m contains a substitution of the type

$$(x+y,\ y+z,\ z).$$

If g contains the cyclical sub-group arising from this substitution self-conjugately, m must be equal to or a factor of $(p-1)p^2$. If this is not the case, and if at the same time m is not divisible by p^2, g must contain either $\frac{m}{p}$ or $\frac{m}{(p-1)p}$ conjugate sub-groups of order p. In the latter case, each will be self-conjugate within a sub-group formed by all substitutions of the type (p. 76)

$$\left\{\frac{1}{n}x+by,\ y+\left[bn+\tfrac{1}{2}(n-1)\right]z,\ nz\right\},$$

and no two sub-groups of this type have a common substitution except identity. Hence, in this case, g will contain only $\frac{m}{(p-1)p}$ substitutions other than those contained in the $\frac{m}{(p-1)p}$ sub-groups of order $(p-1)p$; while in the former case g contains only $\frac{m}{p}$ substitutions whose orders are different from p. It follows that in either case g can contain no substitutions whose orders are factors of p^2+p+1 or $p+1$; and therefore that m is a factor of $(p-1)^2p$. But from this it is easily seen that g must contain the sub-group of order p self-conjugately. Hence, when the sub-group of order p is not contained self-conjugately in g, m must be divisible by p^2.

Suppose next that the sub-group contains a substitution of the type

$$(x+y,\ y,\ z).$$

If the cyclical sub-group arising from this substitution is contained self-conjugately in g, then m must be equal to or a factor of $p^3(p-1)^2$. If this is not the case, g contains substitutions conjugate to the given one. Any such substitution has among the $p+1$ symbols unchanged by it at least one in common with those unchanged by the given substitution; for, if

$$\{ax+by+cz\} \quad \text{and} \quad \{a'x+b'y+c'z\}$$

are two of the unchanged symbols, then

$$\{(a'b-ab')\,y+(a'c-ac')\,z\}$$

is unchanged by both cyclical sub-groups.

If now the notation be changed so that $\{z\}$ is a common unchanged symbol for the two groups, while the first is generated by

$$(x+Ay+Bz,\ y,\ z),$$

which involves no loss of generality, three different cases may occur.

Firstly, all $p+1$ unchanged symbols may be the same for the two groups, so that the second is generated by

$$(x+A'y+B'z,\ y,\ z).$$

The two then generate a group of order p^2, given by all substitutions which are of the type

$$(x+\alpha y+\beta z,\ y,\ z)\ ;$$

and this, moreover, interchanges p^2 symbols transitively.

When this is not the case, the second cyclical sub-group must be generated by a substitution of the form

$$z' \equiv z,$$
$$x'+\alpha y' \equiv x+\alpha y,$$
$$x'+\beta y' \equiv x+\beta y+z,$$

or by one of the form

$$z' \equiv z,$$
$$x'+\alpha y' \equiv x+\alpha y,$$
$$x'+\beta y' \equiv x+\beta y+x+\alpha y.$$

In the first of these alternative cases, the second substitution may be written in the form

$$(x+\gamma z,\ y+\delta z,\ z),$$

where $$(\beta-\alpha)\,\gamma \equiv -\alpha, \quad (\beta-\alpha)\,\delta \equiv 1.$$

The two substitutions then generate a sub-group of order p^2 or p^3, according as A is or is not zero.

In the second alternative case, the second substitution is

$$(ax+by,\quad cx+dy,\quad z),$$

where $$a \equiv \frac{\beta-2\alpha}{\beta-\alpha}, \quad b \equiv \frac{-\alpha^2}{\beta-\alpha}, \quad c \equiv \frac{1}{\beta-\alpha}, \quad d \equiv \frac{\beta}{\beta-\alpha},$$

so that $$a+d \equiv 2.$$

The two substitutions $(x+Ay+Bz,\ y,\ z)$

and $(ax+by,\ cx+dy,\ z), \qquad a+d \equiv 2, \quad ad-bc \equiv 1,$

then generate either the general linear group in two homogeneous variables of determinant unity, or a group within which it is contained.

Hence, again, in this case, with a single exception, the order m of the sub-group must be divisible by p^2; while, in the exceptional case, the sub-group g must itself contain, as a sub-group, a group of order $p(p^2-1)$, isomorphous with the general linear group in two homogeneous variables. This latter sub-group, keeping one symbol fixed, interchanges the remainder in two transitive sets of p^2-1 and $p+1$.

Returning now to the first case, and putting on one side those groups which contain a sub-group of order p self-conjugately, it has been seen that the order m of a group g, containing a substitution of the type

$$(x+y,\ y+z,\ z),$$

that is, a substitution of order p that keeps only one symbol fixed, must be divisible by p^2. The sub-group of order p^2 contained in g is of the type that contains

$$(x+y,\ y+z,\ z)$$

self-conjugately; and this is given by all substitutions of the form

$$(x+\alpha y+\beta z,\ y+\alpha z,\ z).$$

The group therefore contains substitutions of the type

$$(x+z,\ y,\ z),$$

and, unless the cyclical sub-group arising from this is contained self-conjugately (which cannot be the case when a factor of p^2+p+1 or an odd factor of $p+1$ divides m), the preceding investigation again applies here.

It follows, therefore, that if a sub-group contains, not self-conjugately, a sub-group of order p which keeps only one symbol fixed, its order must be divisible either by p^3 or by $p^2(p^2-1)$; for either it must contain two distinct types of sub-group of order p^2, or it must contain sub-groups of orders p^2 and $p(p^2-1)$.

Suppose now that the sub-group g contains operations displacing all the symbols. Then, (i) if it contain a sub-group of type

$$(x+\alpha y+\beta z,\ y,\ z)$$

which displaces the symbols in two transitive sets of p^2 and $p+1$, it must contain a sub-group conjugate to this, displacing the symbols

in two other sets. Hence g must be transitive in the p^2+p+1 symbols. Also the conjugate sub-group of order p^2 must have one undisplaced symbol in common with the given sub-group of order p^2, and, if, again changing the notation, this be taken for z, the two sub-groups are of the forms

$$(x+\alpha y+\beta z,\ y,\ z)$$

and

$$z' \equiv z,$$
$$x'+Ay' \equiv x+Ay,$$
$$x'+By' \equiv x+By+\alpha\,(x+Ay)+\beta z.$$

The latter contains the operation

$$(x-z,\ y+z,\ z),$$

and this, taken with the former sub-group, generates a sub-group of order p^3.

Again, (ii) if g contain a sub-group, order p^2, of the type

$$(x+\alpha z,\ y+\beta z,\ z),$$

it will contain a conjugate sub-group with a different undisplaced symbol. Now, the given sub-group may be written in the form

$$ax'+by'+cz' \equiv ax+by+cz+\alpha' z,$$
$$a'x'+b'y'+c'z' \equiv a'x+b'y+c'z+\beta' z,$$
$$z' \equiv z;$$

and, therefore, the conjugate sub-group may be taken without loss of generality in the form

$$(x,\ y+\alpha x,\ z+\beta x).$$

The two conjugate sub-groups therefore contain the two substitutions

$$(x+z,\ y,\ z\ ),$$
$$(\ x,\ y,\ z+x),$$

which, as has been seen, generate a sub-group of order $p\,(p^2-1)$, and also the two substitutions

$$(x+z,\ y,\ z),$$
$$(\ x,\ y+x,\ z),$$

which generate a sub-group, order p^2, of different type from

$$(x+\alpha z,\ y+\beta z,\ z).$$

Hence g, containing two sub-groups of order p^2 of different types, must contain sub-groups of order p^3, and its order must be divisible by $p^3(p^2-1)$. It is also again necessarily transitive in the p^2+p+1 symbols.

Lastly, (iii) if g contain the sub-group of order $p(p^2-1)$ arising from

$$(x+y, \quad y, \quad z)$$

and

$$(x, \quad x+y, \quad z),$$

which displaces the symbols in two transitive sets of p^2-1 and $p+1$, keeping one fixed, it contains a conjugate sub-group, displacing the symbols in two other sets, and it is therefore transitive in all the symbols.

The order of the group is therefore at least $(p^2+p+1)p(p^2-1)$. Now, no operation displacing all the symbols is permutable with an operation of order p, and hence the sub-group g would contain at least $(p^2+p+1)(p+1)$ conjugate sub-groups of order p. But the sub-group arising from

$$(x+y, \quad y, \quad z),$$

$$(x, \quad y+x, \quad z)$$

contains only $p+1$ sub-groups of order p, and each of these is common to $p+1$ of the p^2+p+1 such conjugate sub-groups. Hence each sub-group of g which keeps one symbol fixed must contain further substitutions of order p, beyond those contained in the sub-group of order $p(p^2-1)$ of the above type. Among the substitutions keeping $\{z\}$ fixed, there must therefore be, besides the simultaneous types

$$(x+y, \quad y, \quad z),$$

$$(x, \quad x+y, \quad z),$$

simultaneous types either of the form

$$(x+y, \quad y, \quad z),$$

$$(x+z, \quad y, \quad z),$$

or of the form

$$(x+y, \quad y, \quad z),$$

$$(x, \quad y+z, \quad z).$$

In either case the order of the sub-group must be divisible by p^3; since, as in former cases, there will be two distinct types of sub-group of order p^2.

The final result of this discussion of sub-groups containing operations of order p may be stated as follows:—

If a sub-group contains substitutions displacing all the symbols (*i.e.*, substitutions whose orders divide p^2+p+1), and if it also contain substitutions of order p, the sub-group must be transitive in all the p^2+p+1 symbols, and its order must be divisible by p^3.

In the proof of this result it is first shown that, if the sub-group g contain a cyclical sub-group of order p, not self-conjugate, it must contain sub-groups of one of the three types,

$$\text{(i)}\quad (x+\alpha y+\beta z,\quad y,\quad z),$$

$$\text{(ii)}\quad (x+\alpha z,\quad y+\beta z,\quad z),$$

$$\text{(iii)}\quad \left\{\begin{matrix}(x+y, & y, & z)\\ (x, & x+y, & z)\end{matrix}\right\}.$$

Now, if the substitutions of g do not all keep $\{z\}$ fixed, there must, when the sub-group contained in g is of types (ii) and (iii), be conjugate sub-groups, and then the reasoning already given shows that g must be transitive, and of order divisible by p^3, independently of the additional supposition that it contains substitutions displacing all the symbols.

The same is true when g contains a sub-group of type (i), unless the symbols $\{y\}$, $\{z\}$, $\{y+\kappa z\}$, $\kappa = 1, 2, \ldots p-1$, form a single transitive set of symbols for the group g.

Hence the result may also be stated in the following form:—

If the substitutions of a sub-group g do not all keep some one symbol fixed, and if the order of g is divisible by p, then g must be transitive in the complete set of p^2+p+1 symbols, and its order must be divisible by p^3, unless it interchanges the symbols in two transitive sets of p^2 and $p+1$.

The most general group of this latter type is evidently one of order $p^3(p-1)^2(p+1)$, whose substitutions are of the type

$$(ax+by+cz,\quad b'y+c'z,\quad b''y+c''z),$$

$$a(b'c''-b''c') \equiv 1,$$

which contains as a self-conjugate sub-group

$$(x+\alpha y+\beta z,\quad y,\quad z).$$

6. *On the Transitive Sub-Groups of G.*

I go on now to consider the sub-groups which contain cyclical sub-groups of order p^2+p+1. Such sub-groups are necessarily transitive.

Let g denote one of them, and let $(p^2+p+1)\,m$ be its order. Then, if the cyclical sub-group of order p^2+p+1 is contained self-conjugately in g, it has been seen that m must be 3.

If not, g contains either $(p^2+p)\,m$ or $(p^2+p)\,\frac{m}{3}$ operations displacing all the symbols. In the former case there are only m substitutions left over, and therefore the sub-group of order m keeping one symbol fixed is contained self-conjugately in g, and must consist of the identical substitution only, so that m is 1.

If m is not unity, it must be divisible by 3, and the number of substitutions in g which do not displace all the symbols is

$$(p^2+p+1)\,m-\tfrac{1}{3}p\,(p+1)\,m.$$

Now, with the exception of identity, no substitution is permutable with a substitution of order p^2+p+1, so that each of the remaining operations, except identity, forms one of a conjugate set, whose number is a multiple of p^2+p+1. It follows that

$$\frac{m}{3} \equiv 1 \pmod{p^2+p+1},$$

or
$$m = 3\left[1+\lambda\,(p^2+p+1)\right],$$

where λ is an integer.

Now, m is a factor of $p^3\,(p-1)^2\,(p+1)$, and it has been seen that, if m contains p as a factor, it must contain p^3.

Hence (i), if m is not a multiple of p,

$$3\left[1+\lambda\,(p^2+p+1)\right]$$

is a factor of
$$(p-1)^2\,(p+1),$$

i.e., of
$$3\left[1+\frac{p-2}{3}\,(p^2+p+1)\right];$$

and therefore of
$$3\left(\frac{p-2}{3}-\lambda\right),$$

which is impossible unless this last expression is zero.

In this case, then,
$$\lambda = \frac{p-2}{3},$$

and
$$m = (p-1)^2(p+1).$$

If (ii) m is a multiple of p^3, it follows at once that

$$\lambda = p-1,$$

and
$$m = 3p^3.$$

Hence the only possible orders for groups containing substitutions of order p^2+p+1, not self-conjugately, are

$$(p^2+p+1)(p-1)^2(p+1),$$

and $$(p^2+p+1)\,3p^3.$$

It is immediately obvious that no sub-group of the latter order can exist. For its sub-groups that keep one symbol fixed would be of order $3p^3$, and these would necessarily contain groups of order p^3 as self-conjugate sub-groups. But a group of order p^3 is self-conjugate only within one of order $p^3(p-1)^2$, and 3 is not a factor of $p-1$.

If a transitive sub-group of order $(p^2+p+1)(p-1)^2(p+1)$ exists, its sub-group keeping one symbol fixed is of order $(p-1)^2(p+1)$.

Suppose that this sub-group contains m_1, m_3, m_{p+2} substitutions keeping respectively just 1, 3, and $p+2$ symbols fixed. Now, each substitution keeping r symbols fixed belongs to r different sub-groups keeping one symbol fixed. Hence the total number of different substitutions belonging to the p^2+p+1 conjugate sub-groups which each keep one symbol fixed is

$$1+(p^2+p+1)\left(m_1+\frac{m_3}{3}+\frac{m_{p+2}}{p+2}\right).$$

Neither 3 nor $p+2$ can be a factor of p^2+p+1, and therefore $\frac{m_3}{3}$ and $\frac{m_{p+2}}{p+2}$ must be integers. Writing n_3 and n_{p+2} for them, and n_1 for m_1,

$$(p-1)^2(p+1) = 1+n_1+3n_3+(p+2)\,n_{p+2},$$

and
$$(p^2+p+1)(p-1)^2(p+1)-\tfrac{1}{3}(p^2+p)(p-1)^2(p+1)$$
$$= 1+(p^2+p+1)(n_1+n_3+n_{p+2}),$$

the two sides of the latter equation representing two ways of counting all the substitutions in the sub-groups which do not displace all the symbols. Combining these equations, there results

$$2n_3+(p+1)\,n_{p+2} = \tfrac{1}{3}p(p+1)(p-2),$$

whence
$$(p-1)\,n_3 = (p+1)\left[(p-1)(\tfrac{2}{3}p^2+p)-n_1\right],$$
$$(p-1)\,n_{p+2} = 2n_1-p^2(p-1).$$

Now, it is, on the other hand, easy to show that the sub-group can contain no substitution that keeps $p+2$ symbols fixed.

For any such substitution

$$S, \quad \text{or} \quad (\alpha x,\ \alpha y,\ \beta z),$$

cannot be contained self-conjugately, and a substitution Σ conjugate

to S and keeping $\{z\}$ fixed is necessarily of the form

$$(\alpha x+\gamma z,\quad \alpha y+\gamma' z,\quad \beta z),$$

so that $S\Sigma^{-1}$ would be of order p, contrary to supposition.

Hence $$n_{p+2}=0;$$

and therefore $$n_1=\tfrac{1}{2}p^2(p-1).$$

But, if the greatest cyclical sub-group, whose order is a factor of p^2-1, contained in the sub-group considered, is of order $\frac{p+1}{q_1}\frac{p-1}{q_2}$, where $\frac{p-1}{q_2}$ is the greatest factor of $p-1$ dividing this number, it contains $\left(\frac{p+1}{q_1}-1\right)\frac{p-1}{q_2}$ substitutions that keep only one symbol fixed, and, together with its conjugate sub-groups, it must contain $\epsilon(p-1)^2(p+1-q_1)$ such substitutions, where ϵ is 1 or $\frac{1}{2}$. The total number of such substitutions contained in the sub-group is the sum of a number of such terms as this, and is therefore divisible by $\frac{1}{2}(p-1)^2$. Hence the above found value for n_1 is impossible, and a sub-group of the type supposed does not exist. The only sub-groups, therefore, which contain substitutions of order p^2+p+1 are those already found of order $3(p^2+p+1)$.

Before going on to the intransitive sub-groups, there is one other type of transitive sub-group, the possibility of which it is necessary to consider. There might be a sub-group g, of order $(p^2+p+1)m$, containing no substitutions of order p^2+p+1. Here, and in dealing with the intransitive sub-groups, I make the limitation, already referred to in the introduction, that p^2+p+1 is the product of not more than two prime factors, which will be represented by p_1 and p_2. If, now, g contains no substitutions of order p_1p_2, it must contain $\epsilon p_2 m$ and $\epsilon' p_1 m$ conjugate sub-groups of orders p_1 and p_2 respectively, where ϵ, ϵ' are either 1 or $\frac{1}{3}$. If they are not both $\frac{1}{3}$, there would be a number of substitutions in g, displacing all the symbols, greater than the order of the group, and this is impossible.

Hence, since all the substitutions of these sub-groups are distinct, the group contains $\frac{1}{3}(p_1-1)p_2m+\frac{1}{3}(p_2-1)p_1m$ substitutions displacing all the symbols, leaving over

$$\tfrac{1}{3}(p_1p_2+p_1+p_2)$$

substitutions.

Suppose, now, first that m is not divisible by p; and, if possible, let the sub-group contain a substitution S of the type

$$(-x,\ -y,\ z).$$

If S is transformed into S' by any substitution which keeps $\{z\}$

unchanged, $S'S^{-1}$ would be a substitution of order p. Hence the sub-group either contains no substitutions of this type, or else such a substitution must be permutable with all the substitutions of the sub-group which keep $\{z\}$ unchanged. Now, it is substitutions of the type S which transform substitutions of order p^2-1 into their own p^{th} powers.

Hence, g must contain at least $\frac{2}{3}p_1p_2m$ substitutions that keep one symbol only fixed, or else that sub-group of g which keeps $\{z\}$ unchanged must contain a substitution of type S self-conjugately. On the former supposition the number of substitutions of g which keep either one or no symbols unchanged would exceed the order of g; and this is impossible. Passing to the latter supposition, the general type of substitution which is permutable with S is

$$(ax+by,\ a'x+b'y,\ c''z), \qquad (ab'-a'b)\,c'' \equiv 1,$$

and that sub-group of g which keeps $\{z\}$ unchanged must be contained within this group. Now, this group is identical with the general linear group in the homogeneous variables, and therefore any sub-group of it which contains distinct cyclical sub-groups whose orders are factors of $p+1$ must also contain substitutions of order p. Hence that sub-group of g which keeps one symbol unchanged must contain a substitution of order 3 self-conjugately. It will, therefore, be a sub-group of dihedral type, and m will be of the form

$$2\,\frac{p+1}{q_1}\,\frac{p-1}{q_2}.$$

Of the substitutions of this sub-group exclusive of identity, $\left(\frac{p+1}{q_1}-1\right)\frac{p-1}{q_2}$ keep one symbol unchanged, and $\left(\frac{p+1}{q_1}+1\right)\frac{p-1}{q_2}-1$ keep $p+1$ symbols unchanged.

Hence the p_1p_2 conjugate sub-groups contain

$$1+p_1p_2\left(\frac{p+1}{q_1}-1\right)\frac{p-1}{q_2}+\frac{p_1p_2}{p+1}\left(\left[\frac{p+1}{q_1}+1\right]\frac{p-1}{q_2}-1\right)$$

distinct substitutions.

Now, if $\frac{p+1}{q_1}$ is greater than 3, this quantity is greater than

$$\tfrac{1}{3}(p_1p_2+p_1+p_2)\,m;$$

and, if $\frac{p+1}{q_1}$ is equal to 3, $\frac{1}{p+1}\left(\left[\frac{p+1}{q_1}+1\right]\frac{p-1}{q_2}-1\right)$ cannot be an integer, and therefore in any case the second supposition is inadmissible.

If, now, p is a factor of m, then $3p^3$ must be a factor of m, and the sub-group of g that keeps one symbol unchanged cannot contain the group of order p^3 self-conjugately. It must therefore contain at least $p+1$ conjugate sub-groups of order p^3. But this is the number that is contained in the most general sub-group that keeps one symbol unchanged, and it is easy to see that any sub-group containing these $p+1$ groups of order p^3 is at least as extensive as this most general sub-group. The sub-group g would therefore, in this case, coincide with the main group.

Hence, finally, no transitive sub-group of the type in question can exist.

7. *On the Intransitive Sub-Groups of G.*

Among the intransitive sub-groups contained in the main group there are two classes the discussion of which is practically involved in the known results obtained by former writers in connexion with the general homogeneous integral group in two variables. These are (i) the sub-groups contained in the sub-group of order

$$p^3(p-1)^2(p+1)$$

which keeps one symbol fixed, and (ii) the sub-group contained in the group of the same order which interchanges the symbols in two transitive sets of p^3 and $p+1$.

It has already been seen incidentally that there is an intransitive sub-group of order $6(p-1)^2$, namely, (iii) the group of type

$$\left\{\begin{matrix} (\ y, & z, & x) \\ (-y, & x, & z) \\ (\ ax, & by, & cz) \end{matrix}\right\}, \qquad abc \equiv 1,$$

which either leaves the three symbols $\{x\}$, $\{y\}$, $\{z\}$ unchanged or interchanges them among themselves.

I shall first show that any intransitive sub-group not belonging to the first two classes is necessarily either contained in a sub-group of the type just given, or is a sub-group of the transitive group of order $3(p^2+p+1)$.

Suppose that N is the order of such a sub-group g, and n the order of the highest sub-group contained at once in g and in one of class (iii). Then g must contain $\frac{N}{n}$ conjugate sub-groups of order n.

Now, two such conjugate sub-groups can only have substitutions

in common if the symbols which they interchange are of the forms

$$\{x\}, \quad \{y\}, \quad \{z\}$$

and

$$\{x+Ay\}, \quad \{x+By\}, \quad \{z\};$$

so that the sub-groups contain conjugate substitutions

$$(\alpha x, \quad \beta y, \quad \gamma z)$$

and $\quad x'+Ay' \equiv \alpha(x+Ay), \quad x'+By' \equiv \beta(x+By), \quad z' \equiv \gamma z.$

Moreover, the multipliers α and β cannot be equal for all the substitutions of the two sub-groups, as in that case the sub-groups would not be distinct. But, if α and β are unequal, it is easy to verify that the two substitutions just written will generate the sub-group formed by all substitutions of the type

$$(ax+by, \quad a'x+b'y, \quad \gamma z), \qquad (ab'-a'b)\,\gamma \equiv 1,$$

and the order of this sub-group is equal to or a multiple of $p(p^2-1)$. Now, by supposition, the substitutions of g do not all keep $\{z\}$ unchanged. Hence (p. 86, bottom) the group, if not transitive in all the p^2+p+1 symbols, must interchange the symbols in two transitive sets of p^2 and $p+1$. But, by supposition, the latter is not the case, and therefore, finally, the $\frac{N}{n}$ conjugate sub-groups of order n contained in g have no common substitutions except identity. The $N\left(1-\frac{1}{n}\right)$ distinct substitutions thus accounted for must contain all the substitutions of g whose orders are equal to or factors of $p-1$, as otherwise there would be a second set of $N\left(1-\frac{1}{n'}\right)$ substitutions, which with the previous set would give a number greater than the order of the group. The remaining substitutions of the sub-group, if any, must either displace all the symbols or must keep one symbol unchanged; and in the latter case their orders must divide p^2-1, since the group can contain no substitutions of order p. If there are substitutions displacing all the symbols, their number must be $\epsilon N\left(1-\frac{1}{p_1}\right)$, where p_1 is a factor of p^3+p+1, and ϵ is either 1 or $\frac{1}{3}$. If there are substitutions which keep one symbol unchanged, and if $\frac{p+1}{q_1}\,\frac{p-1}{q_2}$ is the highest order of any such substitution, there will be a set of $\eta N\left(1-\frac{q_1}{p+1}\right)$ substitutions, conjugate to this substitution,

and to those of its powers which keep only one symbol unchanged, where η is 1 or $\frac{1}{2}$; and, if this does not exhaust all the substitutions of the group, there must be further sets similar to the last. Hence, finally,

$$N = 1 + N\left(1 - \frac{1}{n}\right) + \epsilon N\left(1 - \frac{1}{p_1}\right) + \Sigma\eta N\left(1 - \frac{q_1}{p+1}\right),$$

where ϵ is 0, 1, or $\frac{1}{3}$, and each η is 0, 1, or $\frac{1}{2}$. This equation cannot clearly be satisfied if ϵ is unity. If $\epsilon = \frac{1}{3}$, there cannot be more than one term under the sign of summation, and, if there is such a term, η must be $\frac{1}{2}$, so that either

$$N = 1 + N\left(1 - \frac{1}{n}\right) + \tfrac{1}{3}N\left(1 - \frac{1}{p_1}\right) + \tfrac{1}{2}N\left(1 - \frac{q_1}{p+1}\right)$$

or

$$N = 1 + N\left(1 - \frac{1}{n}\right) + \tfrac{1}{3}N\left(1 - \frac{1}{p_1}\right).$$

The least possible values of $1 - \frac{1}{n}$, $1 - \frac{1}{p_1}$, and $1 - \frac{q_1}{p+1}$ are $\frac{1}{2}$, $\frac{6}{7}$, and $\frac{2}{3}$, and therefore the first equation is impossible. The second equation gives

$$\tfrac{1}{3} + \frac{1}{N} = \frac{1}{3p_1} + \frac{1}{n},$$

and can only be satisfied by

$$N = 3p_1, \quad n = 3.$$

Corresponding to this solution there are the intransitive sub-groups of the sub-groups of order $3(p^2+p+1)$.

Finally, if $\epsilon = 0$, so that

$$N = 1 + N\left(1 - \frac{1}{n}\right) + \Sigma\eta N\left(1 - \frac{q_1}{p+1}\right),$$

there can be only one term again under the sign of summation $\left(\text{the least value of } 1 - \frac{q_1}{p+1} \text{ is } \frac{2}{3}\right)$, and η must be $\frac{1}{2}$. Hence

$$N = 1 + N\left(1 - \frac{1}{n}\right) + \tfrac{1}{2}N\left(1 - \frac{q_1}{p+1}\right)$$

or

$$\tfrac{1}{2} + \frac{1}{N} = \frac{1}{n} + \frac{q_1}{2(p+1)},$$

and the only solution of this equation is

$$n = 2, \quad N = 2\frac{p+1}{q_1},$$

The corresponding type of sub-group belongs to the first class, all of its substitutions keeping one symbol unchanged.

The only other possibility is represented by the equation

$$N = 1 + N\left(1 - \frac{1}{n}\right),$$

so that

$$n = N,$$

and the sub-group g is contained within the above considered sub-group of order $6(p-1)^2$.

The intransitive sub-groups of the sub-group of order $3(p^2+p+1)$, which exist when p^2+p+1 is not a prime, are of simple type and need not be explicitly dealt with, so that it is now only necessary to consider the various sub-groups of the three general types of intransitive sub-groups specified at the beginning of this section. The first two types, though obviously not conjugate to each other within the main group, are holohedrically isomorphous, and therefore, when the various types of sub-group contained in the one have been investigated, those contained in the other may be immediately written down. This isomorphism may be established in the following manner:—

Type (ii) contains the group of order p^2,

$$(x + ay + bz,\ y,\ z),$$

self-conjugately, and is generated by combining this group with the group generated by

$$(x,\ y+z,\ z),$$
$$(x,\ y,\ y+z),$$
$$(\alpha x,\ \alpha^r y,\ \alpha^{-(r+1)} z),$$

α a primitive root mod. p.

If these three substitutions are denoted by A, B, C, and if the substitution

$$(x + ay + bz,\ y,\ z)$$

is denoted by $S_{a,b}$ a simple calculation gives

$$AS_{a,b}A^{-1} = S_{a,a+b},$$
$$BS_{a,b}B^{-1} = S_{a+b,b},$$
$$CS_{a,b}C^{-1} = S_{a\alpha^{r-1},\ b\alpha^{-r-2}}.$$

Type (i) contains the sub-group of which $S'_{a,b}$ or

$$(x+az,\ y+bz,\ z)$$

is the type self-conjugately; and is generated by combining this group with the group arising from

$$A' \text{ or } (x-y,\ y,\ z),$$
$$B' \text{ or } (x,\ y-x,\ z),$$
$$C' \text{ or } (\alpha^{-r}x,\ \alpha^{r+1}y,\ \alpha^{-1}z);$$

moreover,

$$A'S'_{a,b}A'^{-1} = S'_{a,a+b},$$
$$B'S'_{a,b}B'^{-1} = S'_{a+b,a},$$
$$C'S'_{a,b}C'^{-1} = S'_{a\alpha^{r-1},\ b\alpha^{-r-2}}.$$

Now, except as regards the symbols in terms of which they are written, A', B', C' are identical with the inverses of A, B, C; and it is well known that, among the various ways in which a group can be isomorphously connected with itself, that in which two inverse operations are taken as corresponding operations always occurs. Hence an isomorphous correspondence is established between the two types by taking A, B, C, $S_{a,b}$ as corresponding substitutions to A', B', C' and $S'_{a,b}$. It is, therefore, only necessary to deal in detail with one of the two types, and the first will be chosen, as lending itself rather more readily to calculation. This may, for shortness, be referred to as the sub-group H.

The order of the greatest possible sub-group of H which contains no substitutions whose orders are factors of $p+1$ is $p^3(p-1)^2$. Such a sub-group, if it exists, must, by Sylow's theorem, contain either a single self-conjugate sub-group of order p^3, or $(p-1)^2$ conjugate sub-groups of this order, since $(p-1)^2$ contains no factor of the form $\kappa p+1$ except itself and unity. Now, every group of order p^3 is of the type

$$(x+\alpha y+\beta z,\ y+\gamma z,\ z),$$

and is obviously self-conjugate within the group of type

$$(ax+by+cz,\ b'y+c'z,\ c''z),$$

whose order is $p^3(p-1)^2$. Hence H only contains $p+1$ conjugate sub-groups of order p^3, and therefore the supposition that a sub-group of H of order $p^3(p-1)^2$ contains $(p-1)^2$ conjugate sub-groups of order p^3 is impossible.

Hence any sub-group of H which contains no substitutions whose

orders are $p+1$ or one of its factors must be contained as a sub-group within a group of the type

$$(ax+by+cz,\ b'x+c'z,\ c''z).$$

Consider now sub-groups of H which contain substitutions whose orders are factors of $p+1$. If such a sub-group contains the operation

$$(x+z,\ y,\ z),$$

it must contain a conjugate substitution in which $\{y\}$ is not an unchanged symbol, and it must therefore contain the whole of the self-conjugate sub-group

$$(x+\alpha z,\ y+\beta z,\ z).$$

Every sub-group of H containing substitutions whose orders are factors of $p+1$ must therefore either contain this self-conjugate sub-group of order p^2, or, containing none of the operations of this sub-group, it must be a sub-group of one of the p^2 sub-groups of H whose type is

$$\left\{\begin{array}{lll}(x+y, & y, & z)\\ (x, & x+y, & z)\\ (\alpha^r x, & \alpha^{-r-1}y, & \alpha z)\end{array}\right\}.$$

Moreover, if it contains the sub-group of order p^2, it must be merihedrically isomorphous with some sub-group of the group of type just written, and, therefore, must be generated by some sub-group of the group just written, combined with the group of order p^2 given by

$$(x+\alpha z,\ y+\beta z,\ z).$$

Again, every sub-group of the group given by

$$\left\{\begin{array}{lll}(x+y, & y, & z)\\ (x, & x+y, & z)\\ (\alpha^r x, & \alpha^{-r-1}y, & \alpha z)\end{array}\right\}$$

contains as a self-conjugate sub-group those substitutions which multiply z by unity; and any such sub-group is a sub-group of the group

$$\left\{\begin{array}{lll}(x+y, & y, & z)\\ (x, & x+y, & z)\end{array}\right\},$$

of which all possible types of sub-group are known.

Hence, by starting from the known sub-groups of this last sub-group, all the sub-groups of H which contain substitutions that keep only one symbol fixed may be constructed. They will consist (i) of these sub-groups themselves, (ii) of those obtained by combining them with substitutions of the form

$$(\alpha x,\ \beta y,\ \gamma z),$$

and (iii) of those obtained by combining the sub-groups of type (ii) with the group

$$(x+az,\ y+bz,\ z).$$

To every type of sub-group of H thus obtained will correspond an isomorphous sub-group of

$$(x+by+cz,\quad b'y+c'z,\quad b''y+c''z),$$

which may or may not be conjugate within the main group to the sub-group of H with which it corresponds.

The other intransitive sub-groups that require consideration are, as has been seen, the sub-groups of a group I, of type

$$\left\{\begin{matrix}(\ y,\ & z,\ & x\ )\\ (-y,\ & x,\ & z\ )\\ (\ ax,\ & by,\ & cz)\end{matrix}\right\},$$

and they must contain substitutions of order 3, since the sub-group

$$\left\{\begin{matrix}(-y,\ & x,\ & z\ )\\ (\ ax,\ & by,\ & cz)\end{matrix}\right\}$$

is contained within H.

Now the sub-group $\qquad (ax,\ \ by,\ \ cz),\quad abc \equiv 1,$

is contained self-conjugately within I, and is generated by the two permutable operations of order $p-1$,

$$(\alpha^{-1}x,\ \ \alpha y,\ \ z)\quad \text{and}\quad (\alpha^{-1}x,\ \ y,\ \ \alpha z).$$

Every possible sub-group of this Abelian group may now be written down, and combined either with

$$(\ y,\ \ z,\ \ x),$$
$$(-y,\ \ x,\ \ z),$$

or with the former of these two substitutions alone.

The sub-groups thus obtained will evidently not be all distinct from I; but in this way all possible sub-groups of I are obtained.

The actual enumeration of all possible types of intransitive sub-group would be excessively laborious, and it is doubtful whether it would serve any useful purpose; but the preceding analysis supplies the means for determining directly whether a sub-group of any given type actually exists or not.

8. *Case II.* $p \equiv 1 \pmod 3$.

I now go on to the case in which $p \equiv 1 \pmod 3$, in which the congruence

$$z^3 \equiv 1 \pmod p$$

will have three different real roots. These will be denoted by $1, \theta, \theta^2$.

The homogeneous group of determinant unity

$$(ax+by+cz, \quad a'x+b'y+c'z, \quad a''x+b''y+c''z)$$

is no longer holohedrically isomorphous to the non-homogeneous group

$$x' \equiv \frac{ax+by+c}{a''x+b''y+c''}, \quad y' \equiv \frac{a'x+b'y+c'}{a''x+b''y+z''},$$

for the three different homogeneous substitutions

$$\left[\theta^r(ax+by+cz), \quad \theta^r(a'x+b'y+c'z), \quad \theta^r(a''x+b''y+c''z)\right], \quad r = 1, 2, 3,$$

correspond to one and the same non-homogeneous substitution.

The sub-group $\qquad (\theta^r x, \ \theta^r y, \ \theta^r z), \qquad r = 1, 2, 3,$

of the homogeneous group, being permutable with every one of its substitutions, is a self-conjugate sub-group, and the homogeneous group is merihedrically isomorphous to the non-homogeneous group, in such a way that to the identical substitution of the latter corresponds the above self-conjugate sub-group of order 3 of the former. The homogeneous group, moreover, contains no sub-group which is holohedrically isomorphous to the non-homogeneous group. For, if it did, of the three substitutions

$$\begin{aligned} &(x, \quad \theta y, \quad \theta^2 z), \\ &(\theta x, \quad \theta^2 y, \quad z), \\ &(\theta^2 x, \quad y, \quad \theta z), \end{aligned}$$

one only would belong to the sub-group; but the two latter are obtained from the former by transforming it by (y, z, x) and (z, x, y).

It would, however, be most inconvenient to use the non-homogeneous forms throughout the following discussions, and there will be no difficulty or confusion in still using the homogeneous form with the understanding that the substitutions

$$\left[\theta^r(ax+by+cz),\ \theta^r(a'x+b'y+c'z),\ \theta^r(a''x+b''y+c''z)\right],\ r=1,2,3,$$

are not to be regarded as distinct.

This is the same as regarding the three sets of multipliers

$$\lambda_1, \lambda_2, \lambda_3;\quad \theta\lambda_1, \theta\lambda_2, \theta\lambda_3;\quad \theta^2\lambda_1, \theta^2\lambda_2, \theta^2\lambda_3$$

as equivalent; or, in other words, the three characteristic congruences

$$f(\lambda)\equiv 0,\quad f(\theta\lambda)\equiv 0,\quad f(\theta^2\lambda)\equiv 0$$

as equivalent.

It may be shown at once, precisely as in the former case, that any two substitutions which have equivalent characteristic congruences with unequal roots are conjugate to each other, and the reduction of any substitution to a typical form may be carried out exactly as in the former case.

If, now, the characteristic congruence has for its roots $\lambda, \lambda^p, \lambda^{p^2}$, where λ is a primitive root of

$$\lambda^{p^2+p+1}-1\equiv 0,$$

which again will always be the case for some suitably chosen substitution, this substitution in its typical form will be

$$\xi'\equiv\lambda\xi,\quad \eta'\equiv\lambda^p\eta,\quad \zeta'\equiv\lambda^{p^2}\zeta,$$

and its order m will be the least integer for which

$$\lambda^m=\lambda^{mp}\equiv\lambda^{mp^2}.$$

In this case 3 is the only common factor of $p-1$ and p^2+p+1, and therefore the order of this substitution is $\frac{1}{3}(p^2+p+1)$. The order, then, of every substitution whose characteristic congruence is irreducible is $\frac{1}{3}(p^2+p+1)$ or a factor of this number.

If, again, μ is a primitive root of the congruence

$$\mu^{p^3-1}-1\equiv 0,$$

there must be substitutions, whose characteristic congruences have an irreducible quadratic factor, of the type

$$\xi' \equiv \mu\xi, \quad \eta' \equiv \mu^p\eta, \quad z' \equiv \mu^{-(p+1)}z.$$

The order n of such a substitution is the least integer for which

$$\mu^n \equiv \mu^{np} \equiv \mu^{-n(p+1)},$$

and this is $\frac{1}{3}(p^2-1)$. Every substitution, then, whose characteristic congruence contains an irreducible quadratic factor has for its order $\frac{1}{3}(p^2-1)$ or a factor of this number.

Of the remaining types

$$\begin{array}{lllll}
\text{(i)} & x' \equiv \alpha x, & y' \equiv \alpha^r y, & z' \equiv \alpha^{-(r+1)}z, \\
\text{(ii)} & x' \equiv \alpha x, & y' \equiv \alpha y, & z' \equiv \alpha^{-2}z, \\
\text{(iii)} & x' \equiv \alpha^{-2}x, & y' \equiv \alpha y, & z' \equiv \alpha(y+z), \\
\text{(iv)} & x' \equiv x, & y' \equiv y+z, & z' \equiv z, \\
\text{(v)} & x' \equiv x, & y' \equiv y+x, & z' \equiv z+y,
\end{array}$$

where the coefficients are real, the orders may be determined at once by inspection. Thus in (i) the order is equal to or a factor of $p-1$; in (ii) equal to or a factor of $\frac{1}{3}(p-1)$; in (iii) equal to or a factor of $\frac{1}{3}p(p-1)$; in (iv) and (v) equal to p.

Hence the order of every substitution contained in the group must be equal to or a factor of one of the numbers $\frac{1}{3}(p^2+p+1)$, $\frac{1}{3}(p^2-1)$, $\frac{1}{3}p(p-1)$, p, $p-1$, while, on the other hand, the group contains substitutions whose orders are actually equal to each one of these numbers. Also every substitution whose order is a factor of $\frac{1}{3}(p^2+p+1)$ must be a power of a substitution whose order is $\frac{1}{3}(p^2+p+1)$, and a similar property holds for a substitution whose order is a factor of $\frac{1}{3}(p^2-1)$ other than $p-1$ or its factors.

The number of cyclical sub-groups of each type and their distribution in conjugate sets may now be investigated.

Order $\frac{1}{3}(p^2+p+1)$. Exactly as in the corresponding order of the former case, it may be shown that a substitution S of order $\frac{1}{3}(p^2+p+1)$ is permutable only with its own powers, and therefore forms one of a set of $\dfrac{N}{\frac{1}{3}(p^2+p+1)}$ conjugate substitutions, where N is the order of the main group.

Now, the only powers of S which have equivalent multipliers with S are S^p and S^{p^2}, and hence to each set of substitutions such as S^r, S^{rp}, S^{rp^2}, there corresponds such another set of $\frac{N}{\frac{1}{3}(p^2+p+1)}$ conjugate substitutions.

There are, therefore, in all $\frac{\frac{1}{3}(p^2+p+1)-1}{p^2+p+1} N$ such substitutions, whose orders are $\frac{1}{3}(p^2+p+1)$ or one of its factors, and these form $\frac{N}{p^2+p+1}$ conjugate cyclical sub-groups of order $\frac{1}{3}(p^2+p+1)$, each of which must therefore be contained self-conjugately in a sub-group of order p^2+p+1.

Order $\frac{1}{3}(p^2-1)$. Exactly as in the corresponding order of the previous case, it may again be shown here that there are $\frac{N}{\frac{2}{3}(p^2-1)}$ conjugate cyclical sub-groups of order $\frac{1}{3}(p^2-1)$, each being self-conjugate in a group of order $\frac{2}{3}(p^2-1)$; and that these sub-groups contain in all $\frac{Np}{2(p+1)}$ different substitutions whose orders are equal to or factors of $\frac{1}{3}(p^2-1)$ without at the same time being factors of $\frac{1}{3}(p-1)$.

Order $\frac{1}{3}(p^2-p)$. By similar reasoning to that used in the former case, it may be shown that there are $\frac{N}{\frac{1}{3}p(p-1)^2}$ conjugate cyclical sub-groups of order $\frac{1}{3}(p^2-p)$, so that each is contained self-conjugately in a sub-group of order $\frac{1}{3}p(p-1)^2$; also that these sub-groups contain $\frac{p-4}{p(p-1)} N$ different substitutions, whose orders are neither p nor $\frac{1}{3}(p-1)$ nor one of its factors.

Order p. For cyclical sub-groups of order p arising from a substitution of the form

$$(x+z,\ y,\ z),$$

it may be shown by a slight modification of the former method of proof that the main group contains a single conjugate set of $\frac{N}{\frac{1}{3}(p-1)^2p^3}$ sub-groups, so that each such sub-group is self-conjugate in a group of order $\frac{1}{3}(p-1)^2p^3$. These conjugate cyclical sub-groups contain in all $\frac{3N}{p^3(p-1)}$ different substitutions of order p.

For cyclical sub-groups arising from a substitution of the form

$$(x+y,\ y+z,\ z),$$

it will be found again, as before, that there is a single conjugate set of $\frac{3N}{(p-1)p^2}$ in the main group, so that each is self-conjugate in a sub-group of order $\frac{1}{3}(p-1)p^2$, while the whole set contain $\frac{3N}{p^2}$ different substitutions of order p.

Order $p-1$. It is no longer the case here that every substitution whose order is a factor of $p-1$ is the power of substitution whose order is $p-1$. If, α being a primitive root mod. p, α^r, α^s, α^{-r-s} are the multipliers of a substitution, it is still a necessary condition in order that the order of the substitution may be $p-1$ that the highest common factor of r and s should be relatively prime to $p-1$. But this condition is not now sufficient, for, if the difference of r and s is a multiple of 3, the order of the substitution is only $\frac{p-1}{3}$, and it is easy to see that the substitution is not the third power of a substitution of order $p-1$.

It is not difficult to modify the result of the previous case for the number of conjugate sets of cyclical sub-groups of order $p-1$, so as to obtain the numbers of conjugate sets in this case of cyclical sub-groups of orders $p-1$ and $\frac{p-1}{3}$, but the result is rather complicated, and it will be replaced here by a determination of the number of conjugate sets of substitutions, and the number in each set.

For this purpose, consider the congruence

$$\alpha\beta\gamma \equiv 1 \pmod{p}.$$

It has $(p-1)^2$ different solutions; in three of which α, β, γ are equal to each other, while in $3(p-4)$ of the remainder two only of the three quantities α, β, γ are equal. There are, therefore,

$$(p-1)^2-3(p-4)-3$$

solutions in which the three quantities are unequal, and therefore, allowing for the six permutations of α, β, γ among themselves, there are

$$\frac{(p-1)^2-3(p-4)-3}{6}$$

distinct sets of unequal multipliers of substitutions whose orders are factors of $p-1$ in the homogeneous group. Of these the set $1, \theta, \theta^2$ is the only one which is equivalent to itself; α, β, γ; $\theta\alpha, \theta\beta, \theta\gamma$; $\theta^2\alpha, \theta^2\beta, \theta^2\gamma$, being, as before defined, three equivalent sets of multipliers. The number of equivalent sets of unequal multipliers, *i.e.*, the number of sets of unequal multipliers, in the non-homogeneous group is therefore

$$1+\tfrac{1}{3}\left(\frac{(p-1)^2-3(p-4)-3}{6}-1\right),$$

or

$$1+\frac{(p-1)(p-4)}{18}.$$

Allowing for permutations among α, β, γ, the $3(p-4)$ solutions of the above congruence in which two of the three quantities are equal give $p-4$ sets of multipliers in the homogeneous group, and $\frac{p-4}{3}$ sets of multipliers in the non-homogeneous group. To each of these sets of multipliers corresponds a single conjugate set of substitutions. Now, a substitution

$$(x,\ \theta y,\ \theta^2 z)$$

is permutable with the group arising from

$$\begin{aligned}&(ax,\ by,\ cz), \qquad abc \equiv 1,\\ &(y,\ z,\ x),\\ &(-y,\ x,\ z),\end{aligned}$$

which generate in the homogeneous group a sub-group of order $6(p-1)^2$, to which corresponds a sub-group of order $2(p-1)^2$ in the non-homogeneous group. There is, therefore, a conjugate set of $\frac{N}{2(p-1)^2}$ substitutions with multipliers $1, \theta, \theta^2$. Every other substitution with 3 unequal multipliers is permutable only with the group

$$(ax,\ by,\ cz), \qquad abc \equiv 1;$$

and therefore gives rise to a conjugate set of $\frac{N}{\frac{1}{3}(p-1)^2}$ substitutions in the non-homogeneous group.

Finally, a substitution $(ax,\ ay,\ a^{-2}z)$

is permutable in the homogeneous group, as in the former case, with a sub-group of order $p(p+1)(p-1)^2$, and is therefore in the non-

homogeneous group one of a set of $\frac{N}{\frac{1}{3}p(p+1)(p-1)^2}$ substitutions. Hence the total number of substitutions in the group whose orders are equal to or factors of $p-1$ is

$$\frac{N}{2(p-1)^2} + \frac{N(p-4)}{6(p-1)} + \frac{N(p-4)}{p(p+1)(p-1)^2}.$$

On adding together the numbers of substitutions of the different types that have thus been obtained with an additional unity for the identical substitution the sum will be found to be N, as it should be.

It is not necessary to go again through the discussion of sub-groups containing substitutions of order p.

The result, exactly as in the former case, is that a sub-group, containing operations of order p, and neither contained in the sub-group of order $\frac{1}{3}p^3(p-1)^2(p+1)$ which keeps one symbol fixed, nor in the isomorphous sub-group which interchanges the p^2+p+1 symbols in two transitive sets of p^2 and $p+1$, must be transitive, while its order must be divisible by p^3.

Now, it has been seen that every cyclical sub-group of order $\frac{1}{3}(p^2+p+1)$ is contained self-conjugately in a sub-group of order p^2+p+1. This sub-group is not, however, transitive in all the symbols, but interchanges them transitively in sets of $\frac{1}{3}(p^2+p+1)$ each. Suppose now that a transitive sub-group g exists of order $\frac{1}{3}(p^2+p+1)m$, containing cyclical sub-groups of order $\frac{1}{3}(p^2+p+1)$. Since the sub-group is transitive, m must be divisible by 3, and the group must contain either

$$\left\{\tfrac{1}{3}(p^2+p+1)-1\right\}m \quad \text{or} \quad \left\{\tfrac{1}{3}(p^2+p+1)-1\right\}\tfrac{1}{3}m$$

substitutions displacing all the symbols, leaving over either m or $m\left\{\frac{2}{9}(p^2+p+1)+\frac{1}{3}\right\}$ substitutions.

The first supposition is clearly impossible, and the latter gives, as in the former case,

$$\frac{m}{3} \equiv 1, \qquad \text{mod.}\ \tfrac{1}{3}(p^2+p+1).$$

This again leads, according as m is or is not divisible by p, either to

$$m = 3p^3$$

or

$$m = (p-1)^2(p+1),$$

and it may be again shown here that neither of the corresponding types of group exists.

The former reasoning may also be repeated to show that there can be no transitive sub-group which contains substitutions of the orders p_1 and p_2, where p_1, p_2 are the two different prime factors of $\frac{1}{3}(p^2+p+1)$, this number being supposed not to be a prime, without containing substitutions of the order $p_1 p_2$; so that, finally, the group contains in this case no transitive sub-groups. The possibility occurs in this case of an intransitive sub-group containing substitutions of order $\frac{1}{3}(p^2+p+1)$, but a consideration of the sets in which such a substitution would displace the symbols immediately shows that no such type can exist with the exception of the above mentioned sub-groups of order p^2+p+1. The previous reasoning applies to all other types of intransitive sub-group without modification, and leads to the same result, viz., that every intransitive sub-group, other than those whose orders are equal to or factors of p^2+p+1, is contained either in the sub-group of order $\frac{1}{3}p^3(p-1)^2(p+1)$ that keeps one symbol fixed, or in the isomorphous group that displaces the symbols in two transitive sets of p^2 and $p+1$, or, finally, in the sub-group of order $2(p-1)^2$, arising from

$$\left[(y,\ z,\ x),\quad (-y,\ x,\ z),\quad (ax,\ by,\ cz)\right].$$

It may be noticed that the intransitive sub-group of the homogeneous group which keeps one symbol fixed contains a sub-group of order $\frac{1}{3}p^3(p-1)^2(p+1)$, viz.,

$$(ax+by+cz,\ a'x+b'y+c'z,\ c''z),\quad (ab'-a'b)\,c''\equiv 1,\quad c''^{\frac{1}{3}(p-1)}\equiv 1,$$

which is holohedrically isomorphous with the corresponding sub-group of the non-homogeneous group.

9. *On the Group G for $p=2$ and $p=3$.*

When $p=2$, the order of the main group is 168. The only simple group of this order is the known group of the modular equation for transformation of the seventh order of elliptic functions; so that this case does not require separate discussion.

It may be noticed that the sub-group of order p^3 or 8 in this case contains substitutions of order p^2 or 4, whereas in all other cases the substitutions of the sub-groups of order p^3 are all of order p.

When $p=3$, the order of the main group is 5616 or $13\,.\,3^3\,.\,2^4$. A consideration of the multipliers of a substitution of order 13 shows, as before, that every cyclical sub-group of this order is contained self-conjugately in a sub-group of order 39. If, now, there were any

other sub-groups containing substitutions of order 13, and therefore of order $13m$, either m or $\frac{m}{3}$ must, by Sylow's theorem, be congruent to unity mod. 13. But the only factors of $3^3 . 2^4$ which are congruent to unity mod. 13 are 3^3 and $3^2 . 2^4$. Now sub-groups of orders $13 . 3^3$ and $13 . 3^2 . 2^4$, if they existed, would be transitive in 13 symbols, and would at the same time contain $12 . 3^3$ and $12 . 3^2 . 2^4$ substitutions respectively of order 13; but this is impossible. The only transitive sub-groups, therefore, are those of orders 13 and 39.

The intransitive sub-groups, finally, will come under the same three heads as in the two general cases already discussed.

Thursday, January 10th, 1895.

Major MACMAHON, R.A., F.R.S., President, in the Chair.

Mr. Ernest Frederick John Love, M.A., Queen's College, Carlton, Melbourne, Victoria, was elected a member, and Mr. J. H. Hooker was admitted into the Society.

The Chairman gave a short obituary account of Mr. A. Cowper Ranyard's work and connexion with the Society.

The following communications were made:—

On Fundamental Systems for Algebraic Functions: Mr. H. F. Baker.

On the Expansion of Functions: Mr. E. T. Dixon.

Some Properties of a Generalized Brocard Circle: Mr. J. Griffiths.

Electrical Distribution on Two Intersecting Spheres: Mr. H. M. Macdonald.

The Dynamics of a Top: Prof. Greenhill.

The following presents were received:—

"Calendar of Queen's College, Cork," 1894–5; Cork, 1894.

"Journal of the Institute of Actuaries," Vol. xxxi., Pt. 5; October, 1894.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. i., No. 3; New York, 1894.

Issaly, M. l'Abbé.—"Optique Géométrique," pamphlet, 8vo; Bordeaux.

"Berichte über die Verhandlungen der Koniglich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," ii., 1894.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. VIII., No. 4.

"Bulletin des Sciences Mathématiques," Tome XVIII., December, 1894; Paris.

"Bulletin de la Société Mathématique de France," Tome XXII., No. 9.

"Rendiconti del Circolo Matematico di Palermo," Tomo VIII., Fasc. 6; Nov.-Dec., 1894.

"Atti della Reale Accademia dei Lincei—Rendiconti," 2 Sem., Vol. III., Fasc. 10; Roma, 1894.

"Educational Times," January, 1895.

"Annals of Mathematics," Vol. IX., No. 1; November, 1894, Virginia.

"Indian Engineering," Vol. XVI., Nos. 21–24; Nov. 24–Dec. 15, 1894.

A bound volume of letters from Prof. De Morgan and his son G. C. De Morgan, to A. C. Ranyard, bearing upon the foundation of The London Mathematical Society, and a letter from Mrs. De Morgan.

Tracts by Professor De Morgan :—

i. "On the Mode of using the Signs + and − in Plane Geometry."
ii. (i. *continued*) "and on the Interpretation of the Equation of a Curve."
iii. "On the word 'Αριθμός."
iv. "On a Property of Mr. Gompertz's Law of Mortality."
v. "Remark on Horner's Method of Solving Equations."
vi. "Contents of the Correspondence of Scientific Men of the Seventeenth Century."
vii. "On Ancient and Modern Usage in Reckoning."
viii. "On the Difficulty of Correct Description of Books."
ix. "On the Progress of the Doctrine of the Earth's Motion, between the times of Copernicus and Galileo."
x. "On the Early History of Infinitesimals in England."

These two volumes were left by will, by Mr. Ranyard, for the acceptance of the Council.

On Fundamental Systems for Algebraic Functions. *By* H. F. Baker. Read January 10th, 1895. Received, in abbreviated form, 18th February, 1895.

In a note which has appeared in the *Math. Annal.*, Vol. XLV., p. 118, it is verified that certain forms for Riemann's integrals, given by Herr Hensel for integrals of the first kind, and deduced by him algebraically from quite fundamental considerations, can be very briefly obtained on the basis of Riemann's theory. But a desire to dispense with the homogeneous variables used by Herr Hensel has

led me to under-estimate the necessity for formally establishing one particularity implied in that representation. The assumption is made (see p. 121, bottom) that a set of fundamental integral functions, such as those of which general forms are given by Kronecker, can be taken so that in the expression of an integral function by them no redundant terms, or terms of higher infinity than that of the function, need be employed. It is obvious, of course, that not all fundamental sets have this property—for instance, when $p = 2$ and λ is great enough $(1, y+x^\lambda)$ is a fundamental set not having it (as suggested to me by Herr L. Baur). But it is easy, nevertheless, to specify such a system (having, in common with all fundamental systems, the property of not being connected by any integral relation) —as in § 2 below. This forms the main object of the present note. The following paragraphs are intended to prove, what would seem to be nearly obvious, how a fundamental system for homogeneous *forms* may be thence deduced. The actual algebraic determination of a fundamental system for an arbitrary algebraic equation, which gives such interest to Herr Hensel's papers, is not considered here.

I wish to add that the matrix Ω, occurring page 123 of my note, may not without proof be assumed to have all the elements on the right of the diagonal zero, or g_i to have the simple form there indicated—an unessential modification—nor is there need, as stated on p. 129, to take $\zeta = 0$ to be a place where $F'(\eta)$ is not zero. § 6 below is added to explain the solution given on that page (*cf.* Hensel, *Math. Annal.*, XLV., 598).

I hope I shall be allowed to supply in this connexion the reference —Christoffel, "Ueber die canonische Form der Riemannschen Integrale erster Gattung," *Annali di Mat.*, 2me serie, Tom IX., 1878–1879.

[The remarks included in square brackets were added since the paper was submitted to the referees.]

1. Suppose that for rational functions whose only infinities are at the (n or less) places $x = a$ of a Riemann's surface we can construct a set $h_1, \ldots h_{n-1}$ of functions, also only infinite at these places, such that every function whose only infinities are at these places can be expressed in the form

$$F = \left(1, \frac{1}{x-a}\right)_\lambda + \left(1, \frac{1}{x-a}\right)_{\lambda_1} h_1 + \left(1, \frac{1}{x-a}\right)_{\lambda_2} h_2 + \ldots,$$

in such a way that the lowest integral power of $x-a$, *say* $(x-a)^D$, *for*

which $(x-a)^D F$ *is finite at all places* $x=a$, *is sufficient, used as a multiplier, to make every term on the right finite at* $x=a$.

Let $\tau_1+1, \ldots \tau_{n-1}+1$ be the lowest integers such that

$$g_i = (x-a)^{\tau_i+1} h_i$$

is finite at all the places $x=a$. Then g_i will be an integral function on the surface, and in the sense employed in the note referred to (*Math. Annal.*, XLV., 119) will be of rank τ_i. We shall also speak of it as of dimension* τ_i+1 (see below, § 4). D is the dimension of the integral function $(x-a)^D F$ and $D-1$ its rank.

Let, now, f be any integral function of rank τ, so that $(x-a)^{-(\tau+1)} f$ is only infinite at the places $x=a$, while one at least of the values of $(x-a)^{-\tau} f$ at ∞ will be ∞. By the assumption we can write

$$(x-a)^{-(\tau+1)} f = \left(1, \frac{1}{x-a}\right)_\lambda + \left(1, \frac{1}{x-a}\right)_{\lambda_1} h_1 + \ldots,$$

and the D used in stating the assumption is $\tau+1$, while

$$(x-a)^{\tau+1} \left(1, \frac{1}{x-a}\right)_{\lambda_i} h_i = (x-a)^{\tau+1} \left(1, \frac{1}{x-a}\right)_\lambda \frac{g_i}{(x-a)^{\tau_i+1}}$$

is finite at $x=a$; thus

$$\tau+1 \gtreqless \lambda_i+\tau_i+1.$$

Hence we can write

$$f = (x-a, 1)_\lambda (x-a)^{\tau+1-\lambda} + \ldots + (x-a, 1)_{\lambda_i} (x-a)^{\tau+1-\lambda_i-(\tau_i+1)} g_i + \ldots,$$

namely, $(1, g_1, g_2, \ldots g_{n-1})$ are a fundamental system for all integral functions f, such that in using them no terms occur on the right whose rank is greater than that of the function to be expressed.

2. Consider, then, the construction of such a system as $h_1, h_2, \ldots h_{n-1}$. We may assume that the places $x=a$ are none of them branch points —that is, there are n places, and that in each sheet $x-a$ is zero of the first order—and may write

$$x-a = \frac{1}{\xi},$$

and afterwards $\xi = x$, and so are reduced to finding a fundamental

* Though the dimension thus defined will depend on the values of the coefficients in the expression of the function as well as on the algebraic form.

system for integral functions when the places $x = \infty$ are distinct. We may further assume that the derived function $F'(\eta)$ does not vanish.

The orders of infinity of all integral functions in the various sheets will therefore be all expressible by integral powers of x.

Any integral function may be represented by $(R_1, R_2, \ldots R_n)$, where $R_1, \ldots, R_n$, which we shall call the suffixes, are the orders of infinity in the various sheets. By subtracting a suitable polynomial in x of degree R_n, we may express the function in the form

$$(R_1, R_2, \ldots R_n) = (x, 1)_{R_n} + (S_1, S_1, \ldots S_{n-1}, 0) \quad \ldots\ldots\ldots\text{(i.)}.$$

Consider, then, all integral functions of which the n^{th} suffix is zero. It is possible to construct such a one with given suffixes, provided the sum of the suffixes be $p+1$; otherwise it is possible with a sum less than $p+1$ [as follows immediately from the Riemann theory]. Starting with a set of suffixes $(p+1, 0, 0, \ldots 0)$, consider how far the first suffix can be reduced by increasing the $2, 3, \ldots (n-1)^{\text{th}}$ suffixes. In constructing the successive functions with smaller first suffix, it will be necessary in the most general case to increase some of the other $2^{\text{nd}}, 3^{\text{rd}}, \ldots (n-1)^{\text{th}}$ suffixes, and there will be a certain arbitrariness as to the way in which this shall be done. But, if we consider only those functions of which the sum of the suffixes is less than $p+2$, there will be only a finite number possible for which the first suffix has a given value. There will, therefore, only be a finite number of functions of the kind considered for which the further condition is satisfied *that the first suffix is the least possible such that it is not less than any of the others.* Let this least value be r_1, and suppose there are k_1 functions satisfying this condition. Call them the reduced functions of the first class, and in general let any function whose n^{th} suffix is zero be said to be of the first class when its first suffix is greater or not less than its other suffixes. In the same way reckon as functions of the second class all those (with n^{th} suffix zero) whose second suffix is greater than the first suffix, and greater than or equal to the following suffixes. Let the functions whose second suffix has the least value consistently with this condition be called the reduced functions of the second class, k_2 in number suppose, the value of their second suffix being r_2. Then r_2 is the least *dimension* occurring among functions of the second class. In general, reckon to the i^{th} class $(i<n)$ all those functions, with n^{th} suffix zero, whose i^{th} suffix is greater than the preceding suffixes and not less

than the succeeding suffixes; let these be k_i reduced functions of this class with i^{th} suffix equal to r_i. Clearly none of $r_1, \ldots r_{n-1}$ is zero.

Let now $(S_1 \ldots S_{i-1} R_i S_{i+1} \ldots S_{n-1}, 0)$ be any function of the i^{th} class other than a reduced function of this class,

$$(R_i > S_1, > S_2, \ldots, > S_{i-1}, \geqq S_{i+1}, \ldots \geqq S_{n-1}, R_i > r_i),$$

and let $(s_1 \ldots s_{i-1} r_i s_{i+1}, \ldots s_{n-1}, 0)$ be a selected one of the reduced functions of the i^{th} class

$$(r_1 > s_1, \ldots > s_{i-1} \geqq s_{i+1}, \ldots \geqq s_{n-1}).$$

Any one of the k_i reduced functions may be chosen. Then, by choice of a proper constant coefficient λ, we can write

$$(S_1 S_2 \ldots S_{i-1} R_i S_{i+1} \ldots S_{n-1}, 0) - \lambda x^{R_i - r_i} (s_1 s_2 \ldots s_{i-1} r_i s_{i+1} \ldots s_{n-1}, 0)$$
$$= (T_1 \ldots T_{i-1}, R_i', T_{i+1}, \ldots T_{n-1}, R_i - r_i) \quad \ldots\ldots\ldots\ldots\text{(ii.)},$$

where $R_i' < R_i$.

T_1 may be as great as the greater of $S_1, R_i - (r_i - s_1)$, but is certainly less than R_i, and, similarly, $T_2, \ldots T_{i-1}$ are certainly less than R_i.

T_{i+1} may be as great as the greater of $S_{i+1}, R_i - (r_i - s_{i+1})$, and is therefore not greater than R_i, and, similarly, $T_{i+2}, \ldots T_{n-1}$ are certainly not greater than R_i.

Further, if $(x, 1)_{R_i - r_i}$ be a suitable polynomial of order $R_i - r_i$, we can write

$$(T_1 \ldots T_{i-1} R_i' \ldots T_{n-1}, R_i - r_i) - (x, 1)_{R_i - r_i}$$
$$= (S_1' \ldots S_{i-1}' R_i'' S_{i+1}' \ldots S_{n-1}', 0) \ldots\ldots\ldots\text{(iii.)},$$

where R_i'' may be as great as the greater of $R_i', R_i - r_i$, and is certainly less than R_i.

S_1' may be as great as the greater of $T_1, R_i - r_i$, and is certainly less than R_i, and so for $S_2' \ldots S_{i-1}'$.

S_{i+1}' may be as great as the greater of $T_{i+1}, R_i - r_i$, and is certainly not greater than R_i, and so for $S_{i+2}' \ldots S_{n-1}'$.

Now, there are two possibilities:

Either $(S_1' \ldots S_{i-1}' R_i'' S_{i+1}' \ldots S_{n-1}', 0)$ is still of the i^{th} class, namely,

$$R_i'' > S_1, \ldots > S_{i-1}' \geqq S_{i+1}' \ldots \geqq S_{n-1}',$$

and in this case it is of lower dimension (R_i'') than

$$(S_1 \ldots S_{i-1} R_i S_{i+1} \ldots S_{n-1}, 0).$$

Or it is a function of another class, of dimension possibly as great as before (though not greater), but such that the number of suffixes whose value is this dimension is at least one less than before.

[And, so far as these are essential to the remainder of the argument, these possibilities can be stated in the simpler forms:

(1) Either $(S_1' \dots S_{i-1}' R_i'' S_{i+1}' \dots S_{n-1}', 0)$ is of lower dimensions than $(S_1 \dots S_{i-1} R_i S_{i+1} \dots S_{n-1}, 0)$;

(2) Or it is of the same dimension, and then belongs to a more advanced class {that is to the $(i+k)^{\text{th}}$ class, where $k>0$}.]

In the same way, if $(t_1 \dots t_{i-1} r_i t_{i+1} \dots t_{n-1}, 0)$ be a reduced function of the i^{th} class other than $(s_1 \dots s_{i-1} r_i s_{i+1} \dots s_{n-1}, 0)$, we can, by choice of a suitable coefficient μ, write

$$(t_1 \dots t_{i-1} r_i t_{i+1} \dots t_{n-1}, 0) - \mu (s_1 \dots s_{i-1} r_i s_{i+1} \dots s_{n-1}, 0) = (t_1' \dots t_{i-1}' r_i' t_{i+1}' \dots t_{n-1}', 0) \quad \text{(iv.)},$$

where $r_i' < r_i$, $t_1', \dots t_{i-1}'$ may be respectively as great as the greater of the pairs $(t_1, s_1) \dots (t_{i-1}, s_{i-1})$, but are each certainly less than r_1, while, similarly, none of $t_{i+1}', \dots t_{n-1}'$ is greater than r_i.

The function $(t_1' \dots t_{i-1}' r_i' t_{i+1}' \dots t_{n-1}', 0)$ cannot be of the i^{th} class, since no function of the i^{th} class has its suffix less than r_i, and, whatever class it belongs to, though its dimensions may be as great as before, the number of suffixes having a value equal to this dimension is at least one less than before.

Hence, selecting $n-1$ reduced functions, one from each class, say $g_1, g_2, \dots g_{n-1}$, *any* function whatever, of dimension R_i, can be expressed as a sum of

(1) Powers of x; (2) one of $g_1 \dots g_{n-1}$ multiplied by powers of x; (3) a function which is either of lower dimension, or, if of the same dimension, has not as many suffixes reaching the dimension as the function expressed.

In none of the equations (ii.), (iii.), (iv.) does there occur any term of dimension greater than R_i; and in equation (i.) no term on the right is of higher dimension than the left.

Hence any function can be expressed in the form

$$f = (x, 1)_\lambda + (x, 1)_\mu g_1 + \dots + (x, 1)_\nu g_{n-1} + F,$$

where F is a function of lower dimension than that of f, and no terms occur of higher dimension than that of f.

Hence any function can be expressed

$$f = (x, 1)_{\lambda} + (x, 1)_{\mu} g_1 + \ldots + (x, 1)_{\nu_1} g_{n-1} + F_1,$$

where F_1 is one of the $k_1 + \ldots + k_{n-1}$ reduced functions, and hence in the form

$$(x, 1)_{\lambda_2} + (x, 1)_{\mu_2} g_1 + \ldots + (x, 1)_{\nu_2} g_{n-1} + F_2,$$

where F_2 is a reduced function of the lowest dimension occurring, and thence, since there are no functions of dimension less than those of the lowest reduced function, can be expressed in the form

$$f = (x, 1)_L + (x, 1)_M g_1 + \ldots + (x, 1)_N g_{n-1},$$

and none of the terms on the right are of higher dimension than that of f.

[It is easy to see that the resulting system of fundamental functions is practically independent of the order in which the sheets are arranged. Moreover, the theorem can be stated more generally, having regard to functions infinite in all but one of the (simple) poles of *any* algebraic function of the surface.]

3. As an example we may give (see next page) the specifications of the reduced functions for a surface of four sheets in the case in which no integral function exists of aggregate order less than $p+1$.

Taking the standard reduced functions to be those which are here first written (at random), we may exemplify the way in which others are expressible by them, in two cases:

(*a*) When $p+1 = 3M$,

$$(M, M+1, M-1, 0) - \lambda\,(M-2, M+1, M+1, 0) = \{M, M, M+1, 0\},$$

the right hand denoting a function whose infinity orders are at least not higher than those marked $-$, while

$$\{M, M, M+1, 0\} - \lambda_1\,(M-1, M, M+1, 0)$$
$$= \{M, M, M, 0\} = A\,(M, M, M, 0) + B$$

(since a function of $p+1$ poles has two constants, as here). Hence we have the expression

$$(M, M+1, M-1, 0) = \lambda g_2 + \lambda_1 g_1 + A g_1 + B.$$

(*b*) When $p+1 = 3P+1$, we obtain

$$(P+1, P+1, P-1, 0) = \lambda g_1 + A\,(P, P+1, P, 0) + B$$
$$= \lambda g_1 + A\,[\lambda_1 g_2 + C g_3 + D] + B.$$

	Reduced Functions of			Respectively Rank	Whose Sum is
	First Class	Second Class	Third Class		
$p+1=3M$	$(M, M, M, 0)$	$(M-2, M+1, M+1, 0)$ $(M-1, M+1, M, 0)$ $(M, M+1, M-1, 0)$	$(M-1, M, M+1, 0)$	$M-1, M, M$	$3M-1=p$
$p+1=3N-1$	$(N, N, N-1, 0)$ $(N, N-1, N, 0)$	$(N-1, N, N, 0)$	$(N-1, N-1, N+1, 0)$	$N-1, N-1, N$	$3N-2=p$
$p+1=3P+$	$(P+1, P, P, 0)$ $(P+1, P+1, P-1, 0)$ $(P+1, P-1, P+1, 0)$	$(P-1, P+1, P+1, 0)$ $(P, P+1, P, 0)$	$(P, P, P+1, 0)$	P, P, P	$3P=p$

[The specification of all reduced forms of given class for any number of sheets is clearly an easy arithmetical problem. A set of reduced forms when

$$p+1 = (n-1)\,k-r,$$

where $r<n-1$, is clearly given by

$$g_1, \dots g_{r+1}$$

$$= (k, \dots k, k-1, \dots k-1, 0)(k-1, k, \dots k, k-1, \dots k-1, 0) \dots$$

$$\dots (k-1, \dots k-1, k, \dots k, 0),$$

$$g_{r+2}, \dots g_{n-1}$$

$$= (k-1, \dots k-1, k+1, k, \dots k, 0)(k-1, \dots k-1, k, k+1, k, \dots k, 0) \dots$$

$$\dots (k-1, \dots k-1, k, \dots k, k+1, 0),$$

wherein in the first row there are r numbers $k-1$ in each symbol, and in the second row there are $r+1$ numbers $k-1$ in each symbol; in each case $k, \dots k$ denotes a set of numbers k, and $k-1, \dots k-1$ denotes a set of numbers $k-1$. The ranks of the functions $g, \dots g_{r+1}$ are each $k-1$, and of the symbols $g_{r+2}, \dots g_{n-1}$ are each k; their sum is

$$(r+1)(k-1)+(n-r-2)\,k = (n-1)\,k-r-1 = p.]$$

4. Passing from the theory of fundamental systems, I proceed to consider the connexion between the dimension of an integral function, as hitherto defined, and the dimension of an integral *form*, as defined by Herr Hensel.

Let an integral algebraic function satisfy the equation

$$f^\nu + \dots + f^{\nu-1}(x, 1)_{\lambda_i} + \dots = 0.$$

Put $f = x^D K$, giving

$$K^\nu + \dots + K^{\nu-i}\left(\frac{1}{x}\right)^{iD-\lambda}\left(1, \frac{1}{x}\right)^{\lambda_i} + \dots = 0.$$

Let D be taken to be the least positive integer, such that K is finite at all the places $x=\infty$, so that $D-1$ is the rank of f. Then D is the integer actually, and just less than the greatest of the quantities $\frac{\lambda_i}{i}+1$.

Putting $x = \omega/\zeta$, $\zeta^D f = F$, we have the equation

$$F^\nu + \ldots + F^{\nu-1}\,\zeta^{iD+\lambda_i}(\omega, \zeta)_{\lambda_i} + \ldots = 0.$$

The effect of writing here $t\omega$, $t\zeta$ for ω, ζ is to multiply F by t^D. Hence we may speak of F as a homogeneous form of dimension D; integral in the sense that it does not become infinite for finite values of ω and ζ. It appears, then, that D is what Herr Hensel calls the dimension of the form F (see *Crelle*, CIX., pp. 7 and 9; *Math. Ann.*, LXV., 599).

5. We can show now how to form a fundamental system for the expression of homogeneous integral forms.

Let $g_1, \ldots g_{n-1}$ be integral functions for the surface

$$y^n + \ldots + y^{n-i}(x, 1)_{t_i} + \ldots = 0,$$

such that every integral function can be written in the form

$$f = (x, 1)_\mu + (x, 1)_{\mu_1 g_1} + \ldots,$$

the right hand containing no terms of higher rank than that of f. Namely, if D be the dimension of f, $\tau_i + 1$ that of g_i, $D \overline{=}\!> \tau_i + 1$.

Let F be any integral form which in Hensel's sense is of dimension D; that is, a form satisfying an equation of the form

$$F^\nu + \ldots + F^{\nu-i}(\omega, \zeta)_{iD} + \ldots = 0,$$

wherein the coefficient of $F^{\nu-i}$ is homogeneous in ω, ζ of degree iD, so that the form F is changed to $t^D F$, when ω, ζ are changed to $t\omega$, $t\zeta$. We consider how far the process of § 4 can be inverted. Let

$$(\omega, \zeta)_{iD} = \zeta^{iD-\lambda^i}(\omega, \zeta)_{\lambda_i},$$

the general case being when

$$\lambda_i = iD.$$

Then $y^{-D}F$, which we denote by f, satisfies an equation

$$f^\nu + \ldots + f^{\nu-i}(x, 1)_{\lambda_i} + \ldots = 0.$$

If no one of λ_i be so small as $iD-i$, the integer just less than the greatest of the quantities $\frac{\lambda_i}{i} + 1$ will be as great as D, and will therefore be D, since $\lambda_i \not> iD$. If, however, every one of λ_i is as small as $id-ri$, say

$$\lambda_i = i\,(D-r) - L_i,$$

so that L_i is zero or a greater integer, but every value of L_i is not as great as i, the greatest of the quantities

$$\frac{\lambda_i}{i}+1 = D-r+\left(1-\frac{L_i}{i}\right)$$

is greater than $D-r$, and less than or equal to $D-r+1$. Hence the rank of f is $D-r+1$. In this case, however, $G=\zeta^{-r}F$ satisfies an equation

$$G^{r}+\ldots+G^{r-i}\,\zeta^{L_i}\,(\omega,\,\zeta)_{\lambda_i}+\ldots=0,$$

and is an integral form of dimension $D-r$. So that, though an integral function f of rank $D-1$ necessarily leads to an integral form of dimension D, $\zeta^{D}f$, an integral form F of dimension D will lead to an integral function $f=\zeta^{-D}F$, whose rank is $D-r-1$, r being the greatest integer such that $\zeta^{-r}F$ is an integral form.

Since f is of rank $D-r-1$, it can be written

$$f=(x,\,1)_{\mu_0}+(x,\,1)_{\mu_1}g_1+\ldots,$$

where

$$D-r-(\mu_i+\tau_i+1)\geqq 0.$$

Therefore $G=\zeta^{D-r}f$ can be written

$$G=\ldots+(\omega,\,\zeta)_{\mu_i}\,\zeta^{D-r-(\mu_i+\tau_i+1)}\,\gamma_i+\ldots,$$

where $\gamma_i=\zeta^{\tau_i+1}g_i$ is an integral form of dimension τ_i+1.

Therefore every integral form $F(=\zeta^{r}G)$ can be written in terms of the fundamental system γ_i in the form

$$F=\ldots+(\omega,\,\zeta)_{k_i}\gamma_1+\ldots,$$

where $k_i=D-(\tau_i+1)$ is positive.

Conversely, every expression of this form is an integral form of dimension D.

Hence $\quad 1,\ \zeta^{\tau_1+1}g_1,\ \zeta^{\tau_2+1}g_2,\ \ldots\ \zeta^{\tau_{n-1}+1}g_{n-1}$

form a basis for the representation of integral forms.

6. If, for a surface

$$f(y,\,x)=y^{n}+Q_1y^{n-1}+\ldots+Q_n=0,$$

we are given a fundamental system of integral forms γ_1, γ_2, ... (such as found in § 5), we can determine their dimensions by expressing them in terms of the variables ω, ζ, η, where ($D-1$ being the rank of y), $y=\zeta^{-D}\eta$, $x=\omega/\zeta$; or by forming the equations satisfied by

them. If, however, we do not use homogeneous variables and are given a fundamental system of integral functions $g_1, g_2, \ldots$, and require to determine the ranks of them, we may either consider their orders of infinity at $x = \infty$, or form the equation satisfied by them (using then § 4). But, in a certain case, we can more simply use the remark "When the functions $g_1, g_2, \ldots$ are of the form

$$g_i = \frac{y^i + y^{i-1} Q_1 + \ldots + Q_i}{x^{m_i} (x, 1)^{n_i}},$$

which is a common case, or, more generally, of the form

$$g_i = \frac{y^i + y^{i-1} R_1 + \ldots + R_i}{x^{m_i} (x, 1)^{n_i}},$$

wherein $R_1, \ldots R_i$ are such polynomials in x that, by the substitution $y = \zeta^{-D}\eta$, $x = 1/\zeta$, the numerator becomes changed to

$$\zeta^{-iD} [\eta^i + \eta^{i-1} S_1 + \ldots + S_i],$$

wherein $S_1, \ldots S_i$ are *integral* polynomials in ζ, and the equation is such that

$$F'(\eta) = \frac{\partial}{\partial \eta} F(\eta, \zeta) = \frac{\partial}{\partial \eta} \left[\zeta^{nD} f(\eta\zeta^{-D}, \zeta^{-1}) \right]$$

does not vanish at any place $\zeta = 0$, *then* the effect of substituting in g_i for y, x respectively $y = \eta\zeta^{-D}$, $x = 1/\zeta$ is of the form

$$g_i = \zeta^{-(\tau_i + 1)} \frac{\eta^i + \eta^{i-1} \overline{Q}_1 + \ldots}{(1, \zeta)_{n_i}},$$

wherein $\overline{Q}_1, \ldots \overline{Q}_i$ are *integral* polynomials in ζ. Namely, the rule for the rank (τ_i) of an integral function g_i whose expression is given is to notice the power $\tau_i + 1$ of ζ which disengages itself under the specified substitution."

In fact, this equation shows (1) that $\zeta^{\tau_i + 1} g_i$ is finite when $\zeta = 0$ (since η is finite when $\zeta = 0$), and (2) that $\zeta^{\tau_i + 1 - \epsilon} g$ (ϵ a positive integer), which is equal to

$$\zeta^{-\epsilon} \frac{\eta^i + \eta^{i-1} \overline{Q}_i + \ldots}{(1, \zeta)_{n_i}},$$

cannot be finite at every place $\zeta = 0$, unless $\eta^i + \eta^{i-1} \overline{Q}_i + \ldots$ vanish at every such place. But, since $F'(\eta)$ does not vanish, there are n distinct values of η at $\zeta = 0$, and this polynomial, of order less than n, cannot vanish for every one of them.

When $F'(\eta)$ is zero at $\zeta = 0$, and there are only k distinct values of η there, we can still use this remark to determine the rank of $g_1, g_2, \ldots g_{k-1}$.

For instance, when the surface is

$$y^3 + y^2 (x, 1)_2 + yx (x, 1)_1 + Ax^2 = 0,$$

y is of rank 1, and a fundamental set of integral functions is $1, \frac{x}{y}, \frac{x^2}{y^2}$. These are respectively replaceable by $1, \frac{y^2 + yQ_1 + Q_2}{x}$, $y + Q_1$, and these lead, by $y = \eta\zeta^{-2}$, $x = 1/\zeta$, to

$$g_1 = \zeta^{-2}\left[\eta + (1, \zeta)_2\right],$$

$$g_2 = \zeta^{-3}\left[\eta^2 + \eta (1, \zeta)_2 + \zeta^2 (1, \zeta)_1\right].$$

At $\zeta = 0$ there are two distinct values of η, the rank of g_1 is correctly given, but that of g_2 incorrectly, by noticing the power of ζ which disengages itself.

It is easy to see that $\zeta g_2 = -A\zeta \frac{x}{y}$ is finite at $\zeta = 0$, and that $1, \zeta \frac{x}{y}, \zeta^2 \frac{x^2}{y^2}$ form a fundamental system of integral forms. (*Cf.* Hensel, *Math. Ann.*, XLV., 599, and *Math. Ann.*, XLV., 129. Herr Hensel's fundamental system $(1, \eta, \eta^2)$ should be printed $(1, \eta_1, \eta_1^2)$.)

Electric Vibrations in Condensing Systems. *By* J. LARMOR.

Read December 13th, 1894. Received December 22nd, 1894.

1. In forming a theory of rapid electric vibrations, the first point to settle is as to the conditions that may be taken to hold at the boundary of the dielectric medium, where it abuts on a good conductor like a metal. It is well known that when the vibrations are of such short period as free vibrations usually are, the currents in the conductor are confined to mere sheets on the surface. Inside these surface sheets the electric force is null and the magnetic force is null; for if any such forces, of alternating character, existed, there would be currents induced by them, contrary to the fact. The cir-

cumstances are thus practically the same as if the conductors were of perfect conducting quality, that is, as if they formed simple cavities devoid of elasticity in the active dielectric medium, with proper boundary conditions over their surfaces. It is now easy to infer what these boundary conditions must be. On the surface there exists an electric current sheet, and also such a free electric charge and such a layer of magnetic poles as are required to satisfy the necessary conditions as to continuity of the fluxes and forces involved in the problem; but there cannot be any electric double sheet on the surface, except the permanent one of chemical origin which enters into the explanation of voltaic potential difference; nor can there ever be a magnetic double sheet. Now by means of these surface layers the actual electric force in the dielectric is to be made to correspond with the null electric force in the conductor. A density of free charge on the surface can always adjust the normal components of electric displacement into agreement; but, in the absence of a special double electric sheet, the tangential components of the electric force must be continuous across the surface, and therefore the tangential force at this boundary in the dielectric must be null. Again, as regards magnetic force, the normal component of magnetic induction must be continuous, by its fundamental character as a flux, therefore the normal component of magnetic force at this boundary of the dielectric must be null; on the other hand the discrepancy in the tangential components of the magnetic force on the two sides of the interface merely determines the intensity of the current sheet that flows there. This condition of continuity of normal magnetic induction across the surface is not of course an additional one, but is derived at once from the previous condition of continuity of tangential electric force by application of Ampère's circuital relation.

In the equations which follow, dissipation of the skin currents into heat does not therefore appear. That is not because there is no such dissipation, but because their being confined to the outer skin arises from inductance being much more influential than dissipation, owing to the high period. The vibrations will thus be prolonged for a very large number of periods, but will not, even theoretically, go on for ever, although there will be no radiation when the dielectric is completely surrounded by a conducting medium.

2. Suppose now we attempt to form an analogy by taking the magnetic force to represent the velocity in an elastic medium; as

regards its bodily elastic equations this medium will have the properties of an incompressible elastic solid, or of a rotationally elastic incompressible perfect fluid,* for there is no difference between the two except in the formulæ for the tractions on an interface, and therefore in the boundary conditions. The actual boundary conditions in the present problem are that the normal component of the velocity of the medium, and therefore of the displacement, must vanish, while the tangential components are unrestricted.

Thus, keeping to the elastic solid analogy as the most vivid for the moment, though as we shall see presently not the real representation, the circumstances of electric vibration in the active medium are of similar type to those of elastic vibrations in an incompressible solid whose boundaries (where the dielectric abuts on conductors) are free to move tangentially but not normally. There are of course no free boundaries because the dielectric extends to infinity all round; but its effective elastic properties may change at an interface where its material constitution changes; at such an interface the elastic solid analogy breaks down, and we must fall back upon the rotationally elastic incompressible fluid which completely represents all the electrical conditions under the most general circumstances.

3. Suppose now we examine the character of the electric vibrations in a simple condenser, consisting of a thin plate of dielectric material, plane or curved, of thickness uniform or varying, separating two conducting bodies. The case is somewhat analogous to that of the vibrations of an elastic plate of the same form, which is constrained at the edge by the thick compact mass of the surrounding elastic medium; but at its faces, and therefore practically throughout its breadth when that is small compared with its radii of curvature, the movement is confined to be tangential; so that the vibrations are of extensional, but not at all of flexural, character.

As the elastic solid of the analogy is incompressible, on account of the circuital character of the magnetic induction, we may represent the purely tangential displacements of our problem by means of a stream function ψ, which we may consider as belonging to the mean surface of the dielectric plate; in the general problem, in which the plate is not of uniform thickness, it is displacement multiplied by its thickness τ that is derived from the stream function.

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* *Phil. Trans.*, 1894; or *Proc. Roy. Soc.*, 1893–4, "On a Dynamical Theory of the Electric and Luminiferous Medium."

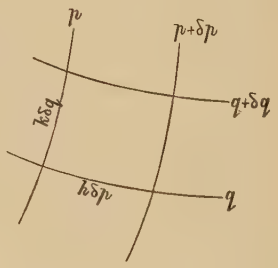
For the general case of a curved sheet, it is clearly proper to employ Gaussian orthogonal coordinates p, q, so that the elementary length on the surface is given by the formula

$$\delta s^2 = h^2\delta p^2 + k^2\delta q^2,$$

where the parameters h, k are functions of position on the surface. The elements of length along the coordinate curves $q =$ constant, $p =$ constant, are

$$\delta x = h\delta p, \quad \delta y = k\delta q,$$

as in the diagram.



The magnetic force in the sheet is tangential, and its components are thus

$$\frac{1}{\mu\tau}\frac{d}{dy}\frac{d\psi}{dt}, \quad -\frac{1}{\mu\tau}\frac{d}{dx}\frac{d\psi}{dt}, \quad 0;$$

that is,

$$\frac{1}{\mu\tau k}\frac{d}{dq}\frac{d\psi}{dt}, \quad -\frac{1}{\mu\tau p}\frac{d}{dp}\frac{d\psi}{dt}, \quad 0.$$

On the other hand the electric force (P, Q, R) is purely normal at the two faces of the plate, and therefore practically throughout its thickness; thus P and Q are each null. By Faraday's circuital law (law of induced electric force) the time-rate of decrease of the magnetic induction is equal to the curl of the electric force; so that the components of the magnetic force are also expressible in the form

$$-\frac{1}{\mu}\left(\frac{d}{dt}\right)^{-1}\frac{dR}{dy}, \quad \frac{1}{\mu}\left(\frac{d}{dt}\right)^{-1}\frac{dR}{dx}, \quad 0;$$

that is

$$-\frac{1}{\mu k}\left(\frac{d}{dt}\right)^{-1}\frac{dR}{dq}, \quad \frac{1}{\mu h}\left(\frac{d}{dt}\right)^{-1}\frac{dR}{dp}, \quad 0.$$

While by the other circuital law, that of Ampère, we have the current multiplied by 4π equal to the curl of the magnetic force, so that

$$K\frac{dR}{dt}\delta x\delta y = \frac{d}{dx}\left(-\frac{1}{\mu\tau}\frac{d}{dx}\frac{d\psi}{dt}\delta y\right)\delta x - \frac{d}{dy}\left(\frac{1}{\mu\tau}\frac{d}{dy}\frac{d\psi}{dt}\delta x\right)\delta y;$$

that is

$$R = -\frac{1}{Khk}\left(\frac{d}{dp}\frac{k}{\mu\tau h}\frac{d}{dp} + \frac{d}{dq}\frac{h}{\mu\tau k}\frac{d}{dq}\right)\psi.$$

Equating the two expressions thus obtained for the magnetic force, we have

$$R = -\frac{1}{\tau}\frac{d^2\psi}{dt^2};$$

thus, working with the two circuital electrodynamic laws, there was no occasion to introduce the function ψ,—these laws showing at once that R serves as a stream function for the magnetic force, so that the magnetic equipotential lines are the curves along which R is constant. The function ψ will however be essential presently, when we examine the purely dynamical aspect of the problem.

Eliminating R, the differential equation for ψ is

$$\frac{d^2\psi}{dt^2} = \frac{\tau}{Khk}\left(\frac{d}{dp}\,\frac{k}{\mu\tau h}\,\frac{d}{dp} + \frac{d}{dq}\,\frac{h}{\mu\tau k}\,\frac{d}{dq}\right)\psi,$$

representing vibratory motion in the condenser layer, unaccompanied by dissipation, as it ought under the circumstances to do.

Before proceeding to a discussion of the types and periods of the vibrations in condenser layers of simple forms, we have still to formulate the mathematical conditions which obtain round the edge of such a layer where it merges in the mass of the surrounding dielectric medium. If we held to the imperfect elastic solid analogy, we should infer that the edge is maintained fixed by the mass of dielectric beyond ; and that would give ψ constant and $d\psi/dn$ null along the edge, which are more conditions than can be satisfied by the solution of a vibrational equation of the second order. As however we have seen already, the elastic solid analogy does not extend to the formulæ for the tractions in the medium, so that we cannot apply it in this way. Similar difficulties attach† to the complete representation by means of a rotationally elastic fluid æther, as well as those associated with the unusual character of the elasticity ; at the present stage it is simpler and safer to employ immediate electrical considerations. We therefore direct our attention to the current sheets on the two opposed faces of the condenser, and consider the magnetic field as that due to these currents. At all parts of the plate the currents on its two faces are equal and opposite, and so neutralize each others' effects except in the contiguous part of the plate, when the plate is very thin ; at the edge the currents are tangential,* therefore the magnetic force near the edge of the plate

* Any tendency to flow over, across the edge, to the other face of the coating, would be effectually resisted, on account of the very great increase of electric energy that such a flow would produce. In illustration of the case where the dielectric plate does not come abruptly to a sharp boundary, but gradually widens out into the surrounding medium, the experiments of Righi (*Rend. dei Lincei*, 1893) with two equal spheres close together and sparking into each other may be noticed. The observed values of the wave-length indicate that the adjacent points of the spheres are antinodes and the remote points nodes in the principal electric vibration which surges over their surfaces. The closeness of the spheres, here also, implies a fair amount of capacity and consequent potential energy, and therefore tolerable persistence of the vibrations.

† *Cf. infra*, § 11.

has only a very small, practically vanishing, component in the tangential direction. Thus we may take the condition at the edge to be that the tangential magnetic force is null; that is, $d\psi/dn$ is to be null along the edge. When the plate is not very thin, there will be a correction to be made to solutions thus obtained, which might be calculated according to the same principles as the well-known correction for the open ends of organ pipes in the theory of acoustical vibrations.

The general scheme of vibration at which we have arrived is of course in keeping with the characteristic of electric undulations in general, that the electric force and the magnetic force are in the plane of the wave-front and at right angles to each other. Thus here the undulations advance along the dielectric plate, and the electric force is across it; therefore the magnetic force is tangential, as we have seen. We do not consider the other type of vibration in which the wave sways across the plate from one face to the other, and for which the periods would of course be extremely high when the plate is thin.

A very striking result of the theory is that the types and periods of the electric vibrations in a condenser layer are unaffected by any possible deformation of the layer by bending, which does not involve stretching, and does not interfere with the condition of freedom at the edges or the distribution of thickness in the layer. This proposition forms a vivid illustration of Maxwell's fundamental position, that in the analytical formulation of electric phenomena no considerations of action across a distance need enter.

4. The equations of vibration at which we have arrived are the same as those for the vibrations of a sheet of air or gas of the same form and law of thickness as the condenser plate, bounded on each side by rigid walls and with a rigid boundary round its edge, ψ now representing the velocity potential of the air. We can accordingly at once utilize in electric theory the results obtained in the discussion of this problem for spherical sheets by Lord Rayleigh, *Theory of Sound*, Vol. II., Chapter XVIII.*

* In a similar manner, the magnetic transverse vibrations of a cylindrical system correspond to the acoustical vibrations of a uniform plate of air of the same form of section, but with an open edge at which the pressure remains constant. As in the above, the electric force is everywhere in a constant direction parallel to the axis of the cylinder, so that it satisfies the same equation as the velocity potential in the motion of the air; while at a conducting boundary its value is null.

There is another type of electric vibration in cylindrical systems in which the electric force is transverse; the magnetic force is now longitudinal and satisfies the equation of a velocity potential in the acoustical problem, where the edge must now be a fixed boundary, as in the text.

In actual problems it is always possible and usually easy to employ a system of conformal coordinates, so that

$$h = k, \quad \text{and} \quad ds^2 = h^2 (dp^2 + dq^2) ;$$

the equation of vibration becomes

$$\frac{d^2\psi}{dt^2} = \frac{\tau}{Kh^2}\left(\frac{d}{dp}\frac{1}{\mu\tau}\frac{d}{dp} + \frac{d}{dq}\frac{1}{\mu\tau}\frac{d}{dq}\right)\psi ;$$

it reduces to its simplest form when $\mu\tau$ is constant over the sheet, and then represents waves travelling with velocity $(K\mu)^{-\frac{1}{2}}$, which is of course the velocity of radiation in the medium of which the plate is composed.

We proceed to some examples of this type, $\mu\tau$ constant, which includes the case of sheets of uniform thickness and uniform magnetic quality ; if we take the electric coefficient K also uniform, the velocity of the waves will be uniform all over the sheet. The equation is now

$$\frac{d^2\psi}{dt^2} = \frac{c^2}{h^2}\left(\frac{d^2\psi}{dp^2} + \frac{d^2\psi}{dq^2}\right),$$

with $d\psi/dn$ null along the edge ; where $c^{-2} = K\mu$.

(i.) For a flat condenser

$$\frac{d^2\psi}{dt^2} = c^2\left(\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2}\right).$$

If it is of rectangular form with the origin of the rectangular coordinates (x, y) at one corner, and its sides of lengths a and b, the vibration is of type given by

$$\psi = A \cos px \cos qy \cos (rt + \gamma),$$

where

$$r^2 = c^2 (p^2 + q^2),$$

and the remaining part of the condition at the edge, $d\psi/dx$ null when $x = a$, $d\psi/dy$ null when $y = b$, gives

$$pa = m\pi, \quad qb = n\pi,$$

when m and n are integers.

Thus $\psi = A \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \cos \left\{ \pi c \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{\frac{1}{2}} t + \gamma \right\}$;

so that the period for the type of vibration in which there are $m - 1$

nodal lines in the plate parallel to the side b, and $n-1$ parallel to the side a, excluding the edges themselves, is

$$\frac{2}{c}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-\frac{1}{2}}.$$

(ii.) For a cylindrical condenser of any form of section, which we may always bend into a circular section without altering the problem, we have

$$ds^2 = a^2 d\theta^2 + dz^2,$$

so that $$p = a\theta, \quad q = z, \quad h = 1;$$

thus $$\frac{d^2\psi}{dt^2} = c^2\left(\frac{1}{a^2}\frac{d^2\psi}{d\theta^2}+\frac{d^2\psi}{dz^2}\right),$$

with $d\psi/dz$ null at the two ends of the cylinder.

Hence we can have standing vibrations round the perimeter of the section, of wave-length a sub-multiple of this perimeter; and standing vibrations along the length of the cylinder, with the two ends both nodes in the electric vibration, so that the wave-length is a sub-multiple of half the length l of the cylinder. For the general type, we may take

$$\psi = A\cos pa\theta\cos qz\cos(rt+\gamma),$$

where $$r^2 = c^2(p^2+q^2),$$

and $$pa = m, \quad ql = n\pi,$$

where m and n are integers. Thus

$$\psi = A\cos m\theta\cos\frac{n\pi}{l}z\cos\left\{c\left(m^2+\frac{n^2\pi^2}{l^2}\right)^{\frac{1}{2}}t+\gamma\right\},$$

so that the period for the type in which there are $2m$ longitudinal nodal lines and $n-1$ transverse nodal circles, is

$$\frac{2}{c}\left(\frac{m^2}{\pi^2a^2}+\frac{n^2}{l^2}\right)^{-\frac{1}{2}}.$$

5. When the dielectric plate is of uniform material but varying thickness,

$$\frac{d^2\psi}{dt^2} = c^2\frac{\tau}{h^2}\left(\frac{d}{dp}\frac{1}{\tau}\frac{d\psi}{dp}+\frac{d}{dq}\frac{1}{\tau}\frac{d\psi}{dq}\right).$$

For example, if τ is a function of p only, we may write

$$\psi = \chi e^{i(nq+rt)},$$

and χ is to be determined by the equation

$$\tau \frac{d}{dp} \frac{1}{\tau} \frac{d\psi}{dp} + \left(\frac{r^2}{c^2} h^2 - n^2 \right) \chi = 0.$$

(i.) Thus if the flat coatings of a plate condenser are slightly inclined to each other, we may take the line of intersection of their planes for the axis of y, and

$$x \frac{d}{dx} \frac{1}{x} \frac{d\psi}{dx} + \left(\frac{r^2}{c^2} - n^2 \right) \psi = 0.$$

This equation is the same as

$$\frac{d^2\psi}{dx^2} - \frac{1}{x} \frac{d\psi}{dx} + \left(\frac{r^2}{c^2} - n^2 \right) \psi = 0,$$

which reduces to Bessel's form by the substitution $\psi = x^\kappa \chi$. It turns out that $\kappa = 1$, and then

$$\frac{d^2\chi}{dx^2} + \frac{1}{r} \frac{d\chi}{dx} + \left(\frac{r^2}{c^2} - n^2 - \frac{2}{x^2} \right) \chi = 0,$$

so that $$\chi = AJ_{\sqrt{2}} \left(\frac{r^2}{c^2} - n^2 \right)^{\frac{1}{2}} x + BJ_{-\sqrt{2}} \left(\frac{r^2}{c^2} - n^2 \right)^{\frac{1}{2}} x.$$

It would be however simpler to adopt independent treatment, which is quite straightforward, on the analogy of the Bessel equation of zero order.

(ii.) In the case of circular plates, the appropriate coordinates are polar, and

$$ds^2 = dr^2 + r^2 d\theta^2$$

$$= r^2 (r^{-2} dr^2 + d\theta^2),$$

so that $$p = \log r, \quad q = \theta, \quad h^2 = r^2;$$

hence $$\frac{d^2\psi}{dt^2} = c^2 \frac{\tau}{r^2} \left(r \frac{d}{dr} \frac{r}{\tau} \frac{d}{dr} + \frac{d}{d\theta} \frac{1}{\tau} \frac{d}{d\theta} \right) \psi,$$

which may be discussed after the same manner as the cases next following.

(iii.) In a cylindrical condenser with two coatings of circular section but slightly eccentric,

$$\tau = a (\beta + \cos \theta),$$

so that $$(\beta+\cos\theta)\frac{d}{d\theta}\frac{1}{\beta+\cos\theta}\frac{d\psi}{d\theta}+a^2\left(\frac{r^2}{c^2}-n^2\right)\psi=0,$$

that is, $$\frac{d^2\psi}{d\theta^2}+\frac{\sin\theta}{\beta+\cos\theta}\frac{d\psi}{d\theta}+a^2\left(\frac{r^2}{c^2}-n^2\right)\psi=0.$$

(iv.) If the condenser forms a portion of a spherical surface, on which θ is co-latitude and ω longitude,

$$ds^2=a^2d\vartheta^2+a^2\sin^2\theta\,d\omega^2$$

$$=a^2\sin^2\theta\left(\frac{d\vartheta^2}{\sin^2\theta}+d\omega^2\right),$$

so that $$p=\log\tan\tfrac{1}{2}\theta,\quad q=\omega,\quad h^2=a^2\sin^2\theta\,;$$

hence $$\frac{d^2\psi}{dt^2}=\frac{c^2\tau}{a^2\sin^2\theta}\left(\sin\theta\frac{d}{d\vartheta}\frac{\sin\theta}{\tau}\frac{d}{d\theta}+\frac{d}{d\omega}\frac{1}{\tau}\frac{d}{d\omega}\right)\psi.$$

If in this case the law of thickness is

$$\tau=\tau_0\sin\theta,$$

then $$\frac{d^2\psi}{dt^2}=\frac{c^2}{a^2}\left(\frac{d^2\psi}{d\theta^2}+\frac{1}{\sin^2\theta}\frac{d^2\psi}{d\omega^2}\right).$$

The vibration-type is clearly

$$\psi=\chi e^{is\omega+int},$$

where $$\frac{d^2\chi}{d\theta^2}+\left(\frac{a^2n^2}{c^2}-\frac{s^2}{\sin^2\theta}\right)\chi=0.$$

In particular, for the meridianal type of vibrations $s=0$, and they take place just as in a flat condensing strip, the wave-length measured along the curved plate being uniform. The two coatings would come into contact at the poles of the spherical surface; but the conditions there, if such a point were included in the system, would be indeterminate. If the sheet form a zone of breadth l measured along the surface, the period when there are $k-1$ nodal parallels in addition to the free edges, is $2l/kc$, where as above

$$c^{-2}=K\mu.$$

When the coatings form portions of two spherical surfaces, which touch at a pole, $\tau=\tau_0(1-\cos\theta)$, and the equation becomes

$$\frac{1-\eta^2}{\eta}\frac{d}{d\eta}\eta\frac{d\psi}{d\eta}+\frac{1}{\eta^2(1-\eta^2)}\frac{d^2\psi}{d\omega^2}=\frac{a^2}{c^2}\frac{d^2\psi}{dt^2},$$

where $\eta = \cos\frac{1}{2}\theta$. For the purely radial types of period $2\pi/n$, we have, writing κ^2 for a^2n^2/c^2,

$$(\eta^{-2}-1)\left(\eta\frac{d}{d\eta}\right)^2\psi+\kappa^2\psi=0;$$

so that, attempting a serial solution

$$\psi = \ldots+A_r\eta^r+A_{r+2}\eta^{r+2}+\ldots,$$

we have $$(r+2)^2A_{r+2}-r^2A_r+\kappa^2A_r=0,$$

with A_0 arbitrary. Thus we obtain a solution

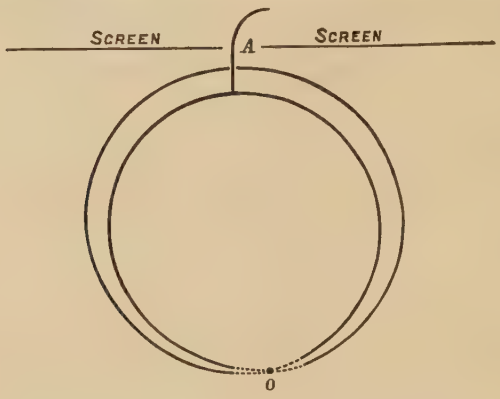
$$\psi = 1-\frac{\kappa^2}{2^2}\eta^2+\frac{\kappa^2\cdot\kappa^2-2^2}{2^2\cdot4^2}\eta^4-\frac{\kappa^2\cdot\kappa^2-2^2\cdot\kappa^2-4^2}{2^2\cdot4^2\cdot6^2}\eta^6+\ldots,$$

which converges everywhere except at the point of contact of the spherical surfaces ($\eta=1$), where the divergence may be considered to express that the contact is equivalent to an undefined sink of electric motions.

The condition along the edge is $d\psi/d\eta$ null, which will give the values of κ, and therefore the periods, by successive approximation; the other pole of the sheet is clearly a node, as it ought to be. For complete spherical surfaces, there are definite vibrations with free periods only when the divergence is avoided by the series terminating, that is when κ is an even integer, say $2m$; the periods are therefore the times required by radiation to traverse an even sub-multiple ($1/2m$) of the circumference of a great circle. An open shell bounded by a nodal line of a free electric vibration in the complete spherical sheet will have the same periods. For $m=2$, there are no nodal lines except the pole; for $m=4$ the circle $\cos\frac{1}{2}\theta=\cdot25$ is nodal; for $m=6$, the circles $\cos\frac{1}{2}\theta=\cdot92$ or $\cdot60$ are nodal.

The annexed diagram (p. 130) represents a condenser with spherical surfaces which, when completed, would come into contact at the pole O. At the opposite point A a wire is connected with the inner coating but insulated from the outer one. If this wire is connected, after passing through a hole in a metal screen, with one of the knobs of a sparking apparatus, and the outer coating of the condenser is connected similarly with the other knob, free electric oscillations will be set up in the condenser by each spark, and they will be persistent because they can be radiated away but slowly at the edge of the dielectric plate. An arrangement of this kind is not therefore, like

the ordinary Hertzian vibrators, a rapidly damped system, but is



rather analogous to a pipe or cord in acoustics which maintains its vibrations, once they get started, for a long series of periods without important damping. It would perhaps be better to connect the coatings with induction plates influenced by the sparking knobs, instead of the knobs themselves. If the free period of the exciter is nearly the same as that of the condenser, it is conceivable that a steady permanent state of vibration might be established,* at any rate if the exciting sparks could be made to follow each other with sufficient rapidity.

6. There is also the usual class of spherical condensers, which is amenable to more complete treatment, viz., those in which the thickness of the dielectric is uniform and the aperture, if any, is very small. The differential equation is, writing μ for $\cos\theta$,

$$\frac{d}{d\mu}(1-\mu^2)\frac{d\psi}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2\psi}{d\omega^2}=\frac{a^2}{c^2}\frac{d^2\psi}{dt^2};$$

* In this mode of excitation by spark discharge in an influencing system, the exciter, consisting of the condenser *plus* connecting wires *plus* sparker and other accessories, has free, though transitory, periods of its own; which are not to be confounded with those of the dielectric plate, or what is the same thing, those of the swayings of the electric charges on its coatings. For these free swayings the pole A is nodal as well as the edge; and the vibrations are not transitory, for it is only at the edge that they can radiate away. The analogy is with an organ pipe excited by an air blast across a wooden lip, rather than with a pipe excited by a vibrating reed. The condenser might also be excited by waves of proper period travelling across the surrounding medium, like Helmholtz's acoustical resonators, provided the waves were very steady. The intensity of the charge of the condenser affects the strength but not the period of the vibrations.

or on substituting as before

$$\psi = \chi e^{i s\omega + int},$$

$$\frac{d}{d\mu}(1-\mu^2)\frac{d\chi}{d\mu} + \left(\frac{a^2n^2}{c^2} - \frac{s^2}{1-\mu^2}\right)\chi = 0,$$

which is the equation of tesseral spherical harmonics. It is known* that there is no solution which remains finite all over the complete sphere unless

$$a^2n^2/c^2 = \alpha\,(\alpha+1),$$

where α is an integer, in which case one of the solutions in series reduces to a finite number of terms. Whether α is integral or not, the solution is

$$\chi = (1-\mu^2)^{\frac{1}{2}s}\left(\frac{d}{d\mu}\right)^s P,$$

where P is the solution of a binomial equation, and thus easily expressible by series.

When α is an integer, one of the two forms of P is the finite series representing the zonal harmonic or Legendre's function of order α.

Thus for the case of a complete sphere, the periods are

$$2\pi a/c\sqrt{(1\,.\,2)},\quad 2\pi a/c\sqrt{(2\,.\,3)},\ \ldots\ 2\pi a/c\sqrt{(\alpha\,.\,\alpha+1)},\ \ldots,$$

whether the vibrations are of zonal or tesseral type. But the nodal lines of the electric vibration ($d\psi/dn$ null) are different in the two cases. For the zonal vibrations involving P_α there are $\alpha-1$ nodal parallels, including the poles, determined by $(d/d\mu)\,P_\alpha = 0$; for the tesseral vibrations with s nodal meridian circles, there are only $\alpha-s-1$ nodal parallels determined by

$$(d/d\mu)^{s+1}\,P_\alpha = 0.$$

We might consider the condenser as terminated by any of these nodal series, so that solutions are thus obtained immediately for a great variety of cases.

For example in the case $P_2 = \frac{1}{2}(3\mu^2-1)$, the single nodal line is the diametral circle, and the period of the lowest radial vibration in a hemispherical condenser is thus $2\pi a/c\sqrt{6}$. The other radial

* Lord Rayleigh, *Theory of Sound*, Vol. II., Ch. XVIII.

periods are $2\pi a/c\sqrt{(4.5)}$, $2\pi a/c\,(6.7)$, ..., viz. the alternate ones of the complete set for a sphere. Again, we have types suitable to a hemisphere which involve an even number of nodal meridian circles, and whose periods are $2\pi a/c\,\sqrt{(2a.2a+1)}$ where $2a-1$ is the number of nodal lines on the complete sphere corresponding to the type. And we have also vibration-types with an odd number of nodal meridian circles of which the periods are $2\pi a/c\,\sqrt{(2a+1.2a+2)}$, where $2a$ is the total number of nodal lines corresponding to the type on the complete sphere; the largest period of this kind is $\pi a/\sqrt{3}$, corresponding to one nodal meridian circle, and no nodal parallel except the edge of the hemisphere, and this is the lowest free period belonging to the hemispherical condenser.

Now suppose that our spherical condenser is not quite complete, but that there is a small aperture at the opposite end of the diameter from A, as in the diagram of the previous case. The effect of this aperture is merely to make its boundary a node instead of the point O, so that the periods are not perceptibly altered if the aperture is small.

7. The solutions hitherto obtained for incomplete spherical surfaces have been special ones derived from the nodal lines of the complete sphere. To attack the general problem we must transform the independent variable in the equation of vibration so as to obtain general solutions, finite at that pole of the sphere which belongs to the condenser; these will necessarily be infinite at the other pole, which is outside the condenser, unless in the special case above considered where the series terminates.

To derive series which will be convergent over the whole spherical surface up to the other pole, we write $z = \frac{1}{2}(1-\mu)$, thus obtaining

$$\psi = (1-\mu^2)^{\frac{1}{2}s}\left(\frac{d}{d\mu}\right)^s P,$$

where
$$z\,(1-z)\,\frac{d^2P}{dz^2} + (1-2z)\,\frac{dP}{dz} + a\,(a+1)\,P = 0,$$

$a\,(a+1)$ representing as before n^2a^2/c^2, but a not now being integral.

The solution which remains finite when $z = 0$ is Murphy's form,*

* Murphy, *On Electricity*, 1833, Preliminary Propositions.

analogous to the series in § 5 (iv.),

$$P = 1 - \frac{a \cdot a+1}{1 \cdot 1}\frac{1-\mu}{2} + \frac{a-1 \cdot a \cdot a+1 \cdot a+2}{2!\,2!}\left(\frac{1-\mu}{2}\right)^2 + \ldots$$

$$\ldots + (-)^r \frac{a-r+1 \cdot a-r+2 \ldots a+r}{r!\,r!}\left(\frac{1-\mu}{2}\right)^r + \ldots .$$

The equation for the period is obtained by making $d\psi/dn$ null at the edge of the sheet. Unless the sheet is more than a hemisphere, a rapid approximation to the periods may be made, as the series that occurs in the equation converges fairly. But when the sheet is nearly a complete sphere the convergence is very slow, and we must either attempt to transform P into a semi-convergent series by the method of Kummer and Stokes, or else we may work with a definite integral type of solution. The former method seems to be inapplicable, or at any rate it is not easy to transform the equation to a binomial type, after the intrinsic singularity has been reduced by introducing a logarithmic factor multiplying the dependent variable. For attempting the latter procedure, we have available C. Neumann's solution

$$P = \int_0^\infty \frac{\cosh\,(a+\tfrac{1}{2})\,\phi}{(\mu+\cosh\,\phi)^{\frac{1}{2}}}\,d\phi,$$

as this is finite when $\mu = +1$ and infinite when $\mu = -1$.

8. For the important case of flat circular plates, including implicitly semicircular plates and various sectors, and also conical sheets bounded by circles, the analysis is given with full numerical results by Lord Rayleigh in his treatment of the identical problems of the vibrations of a circular sheet of air, and the oscillations under gravity of water in a cylindrical vessel.* The wave-length of the vibrations in the circular dielectric plate is its circumference divided by x, where x is a root of $J_n'(z) = 0$; and a numerical table of these roots is given. The longest wave-length corresponds to $x = 1{\cdot}841$, and represents a swaying backwards and forwards along the direction of a diameter; the next, $x = 3{\cdot}054$, is a vibration of the same kind, with an antinode in the middle instead of a node; the next, $x = 3{\cdot}832$, is a radial vibration. In all cases, in passing from the dielectric plate to the surrounding atmosphere, the wave-length is increased in

* *Theory of Sound*, Vol. II., § 339; "Water-waves in Cylinders," *Phil. Mag.*, April, 1876; also *Nature*, July 29th, 1875.

the ratio of the velocity of propagation in air to that in the plate, that is of unity to $K^{\frac{1}{2}}$ when the plate is non-magnetic.

The plates of a condenser may be divided across, along nodal lines, without interfering with the types or periods. A condenser with a guard ring will thus vibrate as two separate condensers, without sensible interference between them.

9. One application of the present formulæ might be to the determination of the effective value of K for vibrations of the period under consideration, by tuning condensers with various dielectrics so as to be in unison.

From this point of view the case in which the dielectric in a horizontal plate condenser is a composite one, consisting say of a layer of alcohol or water below and a layer of air above, merits consideration, as on it might perhaps be based a method of measurement of the dielectric constants of badly conducting electrolytes. If we try to proceed as in § 3, the electric force R will not be constant across the plate, so we must employ instead the electric displacement $\mathrm{P}\,(=KR/4\pi)$ which is constant. It will be found however that the conditions cannot be all satisfied, so that the magnetic force cannot now be tangential right across the plate.

We must therefore introduce a coordinate to represent depth in the plate, and employ the ordinary three-dimensional equations. But we may still escape from the analytical intricacies of the general problem if we confine ourselves to plane waves in a flat rectangular condenser, or indeed to any other case of symmetry in which the magnetic induction may be specified by means of a stream function.

Let us consider these plane waves in a rectangular condenser; let τ_1 be the thickness of the part of the dielectric which is of inductive capacity K_1, τ_2 that of K_2, so that the thickness of the plate is $\tau = \tau_1 + \tau_2$; we may take the value of μ to be uniform throughout, the materials being non-magnetic. Let the two velocities of propagation be c_1, c_2, so that $c_1^{-2} = K_1\mu$, $c_2^{-2} = K_2\mu$; we have, in each medium, equations of the type

$$\frac{d^2\alpha}{dt^2} = c^2\nabla^2\alpha, \quad \frac{d^2\gamma}{dt^2} = c^2\nabla^2\gamma,$$

where

$$\alpha = \frac{d\psi}{dz}, \quad \gamma = -\frac{d\psi}{dx};$$

the tangential magnetic force and normal magnetic induction are continuous at the interface, that is $d\psi/dx$ and $d\psi/dz$ are so; while the

normal magnetic force is null at each boundary, that is $d\psi/dx$ is so. Thus measuring z from the interface

$$\frac{d^2\psi}{dt^2} = c^2\left(\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dz^2}\right);$$

let $$\psi = \chi e^{-int+imx},$$

then $$\frac{d^2\chi}{dz^2} - \left(m^2 - \frac{n^2}{c^2}\right)\chi = 0;$$

so that for a plane wave we may take, if

$$m^2 - n^2/c^2 = \kappa^2,$$

$$\psi = \sin(mx - nt)(Ae^{\kappa z} + Be^{-\kappa z}).$$

At the interface $$A_1 + B_1 = A_2 + B_2,$$

$$(A_1 - B_1)\kappa_1 = (A_2 - B_2)\kappa_2.$$

At the faces of the plate,

$$A_1 e^{\kappa_1 \tau_1} - B_1 e^{-\kappa_1 \tau_1} = 0,$$

$$A_2 e^{-\kappa_2 \tau_2} - B_2 e^{\kappa_2 \tau_2} = 0.$$

Thus from the latter pair of equations,

$$A_1 = \lambda_1 e^{-\kappa_1 \tau_1}, \quad B_1 = \lambda_1 e^{\kappa_1 \tau_1}, \quad A_2 = \lambda_2 e^{\kappa_2 \tau_2}, \quad B_2 = \lambda_2 e^{-\kappa_2 \tau_2};$$

where, from the former pair,

$$\lambda_1 \cosh \kappa_1 \tau_1 = \lambda_2 \cosh \kappa_2 \tau_2,$$

$$\kappa_1 \lambda_1 \sinh \kappa_1 \tau_1 = -\kappa_2 \lambda_2 \sinh \kappa_2 \tau_2,$$

so that $$\kappa_1 \tanh \kappa_1 \tau_1 + \kappa_2 \tanh \kappa_2 \tau_2 = 0,$$

or, written at length,

$$(m^2 - n^2 c_1^{-2})^{\frac{1}{2}} \tanh (m^2 - n^2 c_1^{-2})^{\frac{1}{2}} \tau_1 + (m^2 - n^2 c_2^{-2})^{\frac{1}{2}} \tanh (m^2 - n^2 c_2^{-2})^{\frac{1}{2}} \tau_2 = 0,$$

an equation to determine the period $2\pi/n$ of plane waves of length $2\pi/m$, travelling along a composite dielectric plate.

If either of the quantities represented by κ_1 and κ_2 is imaginary, there will be nodal planes parallel to the plate; the lowest period of a given plate has of course no such nodes.

10. The scope of the present method may also be illustrated by applying it to the case of an ordinary Leyden, with flat base of thick-

ness τ_1 and radius a, and cylindrical sides of thickness τ_2 and length l. If the transition between the thicknesses τ_1 and τ_2 is not very abrupt compared with either of them, there will be no important disturban by reflexion or otherwise at that place. In the circular base

$$\frac{d^2\psi}{dt^2} = \frac{c^2}{r^2}\left(r\frac{d}{dr}\,r\,\frac{d}{dr} + \frac{d^2}{d\theta^2}\right)\psi\,;$$

in the cylindrical sides

$$\frac{d^2\psi}{dt^2} = \left(\frac{d^2}{dr^2} + \frac{1}{a^2}\,\frac{d^2}{d\theta^2}\right)\psi\,;$$

at the junction between them $d\psi/dr$ and $d\psi/rd\theta$ must be continuous; and at the free edge $d\psi/dr$ must be null.

Let $$\psi = \chi e^{int+im\theta}\,;$$

then in the base $$\frac{d^2\chi_1}{dr^2} + \frac{1}{r}\,\frac{d\chi_1}{dr} + \left(\frac{n^2}{c^2} - \frac{m^2}{r^2}\right)\chi_1 = 0,$$

in the sides $$\frac{d^2\chi_2}{dr^2} + \left(\frac{n^2}{c^2} - \frac{m^2}{a^2}\right)\chi_2 = 0.$$

Thus $$\chi_1 = AJ_m\left(\frac{nr}{c}\right),$$

there being no source at the origin; and

$$\chi_2 = B\cos\left(\frac{n^2}{c^2} - \frac{m^2}{a^2}\right)^{\frac{1}{2}}(r-a-l)$$

with the conditions at the junction,

$$AJ_m\left(\frac{na}{c}\right) = B\cos\left(\frac{n^2}{c^2} - \frac{m^2}{a^2}\right)^{\frac{1}{2}} l,$$

$$A\,\frac{n}{c}\,J'_m\left(\frac{na}{c}\right) = -B\left(\frac{n^2}{c^2} - \frac{m^2}{a^2}\right)^{\frac{1}{2}}\sin\left(\frac{n^2}{c^2} - \frac{m^2}{a^2}\right)^{\frac{1}{2}} l.$$

Hence the free periods are $2\pi/n$, where

$$J_m\left(\frac{na}{c}\right)\left(\frac{n^2}{c^2} - \frac{m^2}{a^2}\right)^{\frac{1}{2}}\tan\left(\frac{n^2}{c^2} - \frac{m^2}{a^2}\right)^{\frac{1}{2}} l + \frac{n}{c}\,J'_m\left(\frac{na}{c}\right) = 0,$$

$2m$ being the number of radial nodal lines.

11. The investigation (§ 3) of the differential equations of the problem of a condenser plate above considered can be put into a purely dynamical form, which will conveniently illustrate the dynamical aspect of the electromotive equations in the theory of electromagnetism. We assume as *data*, as in fact kinematic relations of the system, that the magnetic induction is circuital, and so is derived in the present problem from a stream function ψ, and that the electric current is derived from the magnetic induction by Ampère's law. The other circuital law of Faraday should now follow as a dynamical consequence of Maxwell's expressions for the kinetic and potential energies,

$$T = \iint \frac{1}{8\pi\mu\tau} \left\{ \frac{1}{k^2} \left(\frac{d^2\psi}{dq\,dt} \right)^2 + \frac{1}{h^2} \left(\frac{d^2\psi}{dp\,dt} \right)^2 \right\} hk\,dp\,dq,$$

$$W = \iint \frac{\tau}{8\pi K} \frac{1}{h^2k^2} \left\{ \frac{d}{dp} \left(\frac{k}{\mu\tau h} \frac{d\psi}{dp} \right) + \frac{d}{dq} \left(\frac{h}{\mu\tau k} \frac{d\psi}{dq} \right) \right\}^2 hk\,dp\,dq,$$

and should form a confirmation of the validity of these expressions. The vibrations of the dielectric layer are to be derived from these expressions by the dynamical principle of least action

$$\delta \int (T - W)\,dt = 0,$$

in which the symbol of variation δ refers to ψ alone. On making the variation, integrating by parts in the usual manner to get rid of differential coefficients of $\delta\psi$ except at the limits of the integral, and equating to zero the coefficient of this purely arbitrary variation $\delta\psi$ in the resulting form, we have the equation of vibration

$$\frac{d^2}{dt^2} \left(\frac{d}{dp} \frac{k}{\mu\tau h} \frac{d}{dp} + \frac{d}{dq} \frac{h}{\mu\tau k} \frac{d}{dq} \right) \psi$$

$$= \left(\frac{d}{dp} \frac{k}{\mu\tau h} \frac{d}{dp} + \frac{d}{dq} \frac{h}{\mu\tau k} \frac{d}{dq} \right) \frac{\tau}{Khk} \left(\frac{d}{dp} \frac{k}{\mu\tau h} \frac{d}{dp} + \frac{d}{dq} \frac{h}{\mu\tau k} \frac{d}{dq} \right) \psi,$$

wherein K, μ, τ may all be any functions of position on the mean surface of the condenser. This equation may now be split into two, and simplified by the introduction of a subsidiary independent variable, such as R in § 3.

We thus obtain

$$\chi = -\frac{\tau}{Khk} \left(\frac{d}{dp} \frac{k}{\mu\tau h} \frac{d}{dp} + \frac{d}{dq} \frac{h}{\mu\tau k} \frac{d}{dq} \right) \psi,$$

and $$\frac{d^2\chi}{dt^2} = \frac{\tau}{Khk}\left(\frac{d}{dp}\frac{k}{\mu\tau h}\frac{d}{dp} + \frac{d}{dq}\frac{h}{\mu\tau k}\frac{d}{dq}\right)\chi.$$

On substitution for χ from the first of these equations in the left side of the second, and integration, there follows

$$\psi = -\left(\frac{d}{dt}\right)^{-2}\chi + F(p, q),$$

where $F(p, q)$ denotes some function of the coordinates that would be a stream function, for steady electric flow without internal sources, on either coating of the condenser. Thus, the value of χ being derived as above from the solution of the second equation, the value of ψ deduced from it will involve in addition this function $F(p, q)$.

The dynamical problem is therefore more general than the present electrical one.* For instance, if we take T and W to represent the energy of a rotationally elastic fluid æther, the additional term $F(p, q)$ will represent an irrotational flow in the æther, which as we know will excite no elastic reactions and so will not interfere with the equation of vibrations. If, on the other hand, we imagine the æther to be an elastic solid, its potential energy W will include other terms in addition to those above expressed; but as they can be integrated into an expression relating only to the boundary, they will not affect the equation of propagation; in this case $F(p, q)$ will represent an irrotational strain. We might by aid of this function satisfy in either case the conditions necessary to make the edge of the plate fixed, viz. $d\psi/dn$ null and also ψ constant all along it; for the case of a cylindrical condenser of length l the periods of the vibrations which have $2m$ longitudinal nodal lines would then come out to be

$$\frac{2}{c}\left(\frac{m^2}{\pi^2 a^2} + \frac{n^2}{l^2}\right)^{-\frac{1}{2}},$$

where n is a root of the equation

$$\tanh ml \tan nl = \frac{2mn}{m^2 - n^2},$$

* [It is not implied here that the representation by means of a rotationally elastic fluid æther is in any way defective. It it were possible practically to prescribe the velocity of the æther round the edge of the plate, this analysis would just be sufficiently wide to determine the resulting motions; and these would consist of an elastic vibrational part and an irrotational fluid motion. But it is not possible to impose a velocity on the æther at the edge except by introducing an extraneous magnetic field; and the analysis simply states that throughout the plate this field is superposed on the electromagnetic vibrations if it is steady, and modifies them in the ordinary manner if it is variable.]

instead of being an integer, thus differing from the result in § 4 (ii.) to which the direct electric equations led. Thus the ordinary electric equations imply that in a rotational fluid æther no irrotational flow is associated with electric vibrations relating to fixed conductors; they also imply, on this and many other grounds, that the æther is not of elastic solid constitution. On the rotational theory, flow of the æther is of course associated with a steady magnetic field; the question as to whether it is actually influenced by the movements of matter must be decided, at the present time, on grounds of optical theory and observation.*

The problem of electric vibrations is thus only a special problem in the dynamical theory of the electrical and optical medium; the latter is wider, because, not to mention the electromagnetic forcive on material bodies, it must include the theory of the electrics and optics of moving media, and possibly the theory of the atoms of matter themselves considered as intrinsic singularities, of motion or strain or both together, existing in the fundamental structureless medium.

It may be noticed, that the elastic solid problem, whose solution has just been stated for the case of a cylinder, is identical with the problem of the extensional, wholly non-flexural, vibrations of a plane or curved elastic plate, which has been treated by Mr. A. E. H. Love† under the much more difficult circumstances which obtain when the material is not incompressible. Such vibrations are too high in pitch and too difficult of excitation to be of acoustical importance; but if the æther were like an elastic incompressible solid they would be the ones we should have to deal with here, except that the more usual case would involve as above the simpler conditions appertaining to a fixed edge in place of those of a free one.

This mode of forming a concrete representation of electric vibrations is explained at some length, in *Proc. Camb. Phil. Soc.*, Vol. VI., 1890, "On a Mechanical Representation of a Vibrating Electric System and its Radiation." The analogy there developed is however the *conjugate* one in which electric force represents the

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* The term $F(p, q)$ enters in the analysis as representing a part of the magnetic force which is independent of, and not excited by, the electric force,—in other words a part of the motion of the æther which is not directly traceable to electric strain. It implies, and is derived from, activity of a hydrostatic pressure in the æther, which is probably operative only in molecular problems, the æther in bulk being on this theory stationary except in a magnetic field. *Cf. Phil. Trans.*, 1894, A, p. 790.

† *Phil. Trans.*, 1888.

velocity in the elastic-solid medium; the surface conditions at a conductor are then clearly that the horizontal velocity vanishes while the normal movement is unrestricted. Though that scheme is more remote from the actual electrical conditions, it has the advantage for intuitional purposes that the surface relations are completely represented by the presence of a very thin skin of much more powerful elasticity, supposed to exist on the elastic solid of the analogy. The fact that we can always obtain conjugate representations, of the same scope but of totally different character, by replacing magnetic force by electric force as the independent variable, is very fundamental in this kind of theory; and use has been made of it in various ways by Willard Gibbs, Drude, and other writers. It was clearly indicated and utilized in Maxwell's original paper "On the Electromagnetic Field," *Phil. Trans.*, 1864.

12. The propagation of electrical waves in the dielectric of a long cylindrical condenser like a submarine electric cable may be discussed in a similar manner. The lines of electric force are transverse, passing radially from the core to the outer sheath across the dielectric. If the frequency of the vibrations is sufficiently great, the currents will not penetrate far into the conductors, and there will also be little viscous loss of energy; so that the dielectric layer will act like a speaking tube in acoustics, maintaining the vibrations in force, as they travel along it.

This propagation is of course very different from the ordinary working of submarine cables, when the alternations are so slow that electric inertia hardly counts, and the viscous forces of conduction, which are the only ones in action, make the propagation of diffusive type like the conduction of heat, instead of undulatory type as here.

It might be supposed that long submarine cables would possibly be adaptable for telephonic purposes by making the incident sound waves act, in the manner of a relay, on a spark gap, and so set loose electrical vibrations which would be propagated along the cable and received at the other end of it; but the discussion which follows negatives such an idea.

In examining this point we shall also incidentally observe the order of frequency which is necessary to restrict the importance of the dissipative terms, so that the conclusions of the above discussion, in which they do not appear, may be valid. When as here the pene-

tration into the conductor is to a small depth compared with its radius of curvature, its surface may be treated as plane.

The equations of propagation, for conductivity κ', are

$$\left(K\frac{d^2}{dt^2}+4\pi\kappa'\frac{d}{dt}\right)(P, Q, R)=\mu^{-1}\nabla^2(P, Q, R).$$

It will suffice to consider the case in which Q vanishes; there is then a current function ψ, so that, x being parallel and z perpendicular to the interface, and the dielectric sheet being then,

$$(P, R)=\left(\frac{d}{dz}, -\frac{d}{dx}\right)\psi;$$

and the characteristic equation of the problem is

$$\left(c^{-2}\frac{d^2}{dt^2}+4\pi\kappa\frac{d}{dt}\right)\psi=\left(\frac{d^2}{dx^2}+\frac{d^2}{dz^2}\right)\psi,$$

where c^{-2} stands for $K\mu$ and κ for $\kappa'\mu$. In non-magnetic matter such as the dielectric μ is unity; and c is very great, being of the order of the velocity of radiation.

For a wave of period $2\pi/n$ travelling along the dielectric layer,

$$\psi=e^{\iota m'x-\iota nt}\chi,$$

where m' is now complex, as we suppose the wave to be damped by conduction in the metals. We have therefore

$$\frac{d^2\chi}{dz^2}-(m^2-4\pi\kappa n\iota)\chi=0,$$

where

$$m^2=m'^2-n^2/c^2.$$

If the plane of xy be taken along the middle of the dielectric layer, of small breadth $2a$, then by symmetry we have in the dielectric

$$\chi_1=A(e^{mz}+e^{-mz})$$

and in the upper conductor

$$\chi_2=B'e^{-rz},$$

where

$$r^2=m_2^2-4\pi\kappa n\iota;$$

so that very approximately

$$r=(2\pi\kappa n)^{\frac{1}{2}}\left(1-\iota+\frac{m_2^2}{8\pi\kappa n}(1+\iota)+\ldots\right),$$

in which the last term may also be neglected because n/m_2 is of the order of the velocity of radiation.

Thus in the dielectric plate

$$\psi_1 = A\,(e^{mz}+e^{-mz})\,e^{\iota m'x-\iota nt};$$

in the upper conductor

$$\psi_2 = Be^{-(2\pi\kappa n)^{\frac{1}{2}}(1-\iota)(z-a)}\,e^{\iota m'x-\iota nt}.$$

The constants are to be determined by the conditions that at the interface $\left(\frac{K}{4\pi}\frac{d}{dt}+\kappa'\right)\psi$ and also $d\psi/dz$ are to be continuous. Thus we have exactly, κ' being null in the dielectric,

$$-\iota nKA\,(e^{ma}+e^{-ma}) = 4\pi\kappa' B,$$

$$mA\,(e^{ma}-e^{-ma}) = -rB;$$

and the equation for m is therefore

$$ma\frac{e^{ma}-e^{-ma}}{e^{ma}+e^{-ma}} = \frac{nKra}{4\pi\kappa'}\,\iota.$$

If there were no dissipation m would vanish; so that to our order of approximation m is small and c is the velocity of propagation. Thus

$$(ma)^2 = \pi a\left(\frac{\mu}{c\kappa'\lambda^3}\right)^{\frac{1}{2}}(1+\iota),$$

so that

$$\left(\frac{m'\lambda}{\pi}\right)^2 = 4+\frac{1}{\pi a}\left(\frac{\mu\lambda}{c\kappa'}\right)^{\frac{1}{2}}(1+\iota),$$

or

$$m' = \frac{2\pi}{\lambda}\left\{1+\frac{1}{8\pi a}\left(\frac{\mu\lambda}{c\kappa'}\right)^{\frac{1}{2}}(1+\iota)\right\}.$$

The exponential coefficient of decay along the wave-train is therefore

$$\frac{1}{4a}\left(\frac{\mu}{c\kappa'\lambda}\right)^{\frac{1}{2}};$$

it is diminished equally by increase of conductivity κ' or diminution of magnetic permeability μ in the metal, or by increase of the wave-length, which however must not be too small if this analysis is to apply.

For copper, $\kappa' = 1600^{-1}$, and if λ is one metre and the thickness $2a$ of the dielectric sheet one centimetre, the amplitude would thus be reduced in the ratio of 1 to e after travelling about 100 metres, that

is after 100 vibrations. Thus oscillations of this kind, though enormously persistent compared with ordinary Hertzian waves, are nothing like so persistent as ordinary sound waves; nor can they be transmitted very far along a dielectric cylinder without sensible loss.

The exponential coefficient of penetration into the metal is the real part of r, which is $2\pi (c\mu\kappa'/\lambda)^{\frac{1}{2}}$, being independent of the thickness of the dielectric plate; this is about 3×10^3 under the above circumstances, so that the current is reduced in the ratio of 1 to e at a depth of about 3×10^{-4} centimetres, which of course amply justifies the procedure of the previous part of this paper.

13. It may be convenient to briefly treat on these lines the problem of electric vibrations in other systems than dielectric shells, a subject already referred to in the footnote to § 4.

The periods of a circular cylinder of dielectric, of radius a, with a conducting boundary, are easily expressed; when the electric force is longitudinal and the magnetic force transverse, they are given by the roots of $J_0(na/c) = 0$; when the magnetic force is longitudinal and the electric force transverse, by the roots of $J_0'(na/c) = 0$. The wave-lengths in free æther of the two types are the diameter of the cylinder divided by ·765, 1·757, 2·754, ... and 1·220, 2·233, 3·238, ... respectively (Stokes, *Camb. Trans.*, IX.; Rayleigh, *Sound*, § 206). Both classes of vibrations should be excited by sparking between two knobs in the middle of a hollow metal cylinder whose length is several times its diameter; and in analogy (*supra*) with the air vibrations excited in a pipe by tapping it, they will last a considerable number of periods before disappearing by radiation from the open ends. On the same analogy also, shortening the cylinder should somewhat increase the periods; for it diminishes the constraint.

The more general problem of the vibrations excited in a dielectric region bounded by a conducting surface of revolution is also amenable to similar treatment. There are two types of vibration symmetrical with respect to the axis; in one the lines of magnetic force are circles round the axis, and the lines of electric force are curves in the meridian planes; in the other *vice versa*.

In the first case if H is the intensity of the magnetic force, the component, parallel to the axis, of the electric displacement is $dH/4\pi d\rho$ where ρ denotes distance from the axis. Thus the equation satisfied by H is

$$\frac{d^2H}{dt^2} = c^2\nabla^2 H,$$

and the condition of continuity of the tangential electric force at the boundary makes dH/dn null at the boundary. The vibration types and periods are thus precisely the same as those of a gas in a rigid envelope of the same form, H corresponding to the velocity potential of the gas. The periods are well known (Rayleigh, *Sound*, § 331) for the case of a spherical envelope, including a hemispherical envelope and various other types as sub-cases; and by combining symmetrical vibrations relating to different axes the most general class of vibrations in spheres is arrived at, the magnetic lines of force always lying on concentric spherical surfaces, and the periods being of course unaltered. If the exciting spark-gap lies along a radius, the vibrations excited in the dielectric sphere will be of this type, symmetrical round that radius; and the boundary condition will not be essentially modified if an aperture is made in the conducting boundary at either pole; thus the period will be practically unaltered while the vibrations are slowly radiated away through this aperture.

In the second type of vibrations the intensity E of the electric force will satisfy the equation

$$\frac{d^2E}{dt^2} = c^2\nabla^2 E,$$

while over the boundary E will be constant. In a spherical boundary the periods are thus easily determined by the same analysis as applies to the previous case; they will be intermediate between those of the previous set, the lowest corresponding to a wave-length 1·4 times the radius, instead of 3·02 times the radius which gives the lowest period of the previous set. These vibrations will not be sensibly excited when the spark-gap lies along a radius; and their periods will not be sensibly altered if an aperture is made in the conducting boundary at an antinode of the electric force so as to admit of radiation into the outside space.

On certain Definite ϑ-Function Integrals. *By* L. J. Rogers.

Received and read December 13th, 1894.

1. If c_m is such a function of $\sqrt{-1}$ that c_{-m} is the same function of $-\sqrt{-1}$, then the series $\sum_{m=-\infty}^{\infty} c_m e^{mui}$ is real.

Let
$$\vartheta(u) \sum_{m=-\infty}^{\infty} c_{2m} e^{2mui} = \sum_{r=-\infty}^{\infty} \kappa_{2r} e^{2rui} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Then κ_{2r} is the coefficient of e^{2rui} in the product

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nui} \times c_{2m} e^{2mui}$$

$$= \sum_{r=-\infty}^{\infty} \left\{ e^{2rui} \sum_{m=-\infty}^{\infty} (-1)^{r-m} q^{(r-m)^2} c_{2m} \right\};$$

therefore
$$\kappa_{2r} = q^{r^2} \sum_{m=-\infty}^{\infty} (-1)^{r-m} q^{m(m-2r)} c_{2m} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Suppose
$$c_{2m} = \frac{2q^m e^{vi}}{1+q^{2m}e^{2vi}},$$

so that
$$c_{-2m} = \frac{2q^m e^{-vi}}{1+q^{2m}e^{-2vi}};$$

then the condition for the series being real is satisfied.

From (2), moreover, we have

$$\kappa_{2r} q^{-r^2} - \kappa_{2r-2} q^{-(r-1)^2} e^{2vi} = \sum_{m=-\infty}^{\infty} (-1)^{r-m} q^{m(m-2r)} (1+q^{2m}e^{2vi}) c_{2m}$$

$$= 2e^{vi} \sum_{m=-\infty}^{\infty} (-1)^{r-m} q^{m(m-2r+1)}$$

$$= 0 \quad \text{identically.}$$

Hence
$$\kappa_{2r} q^{-r^2} = \kappa_{2r-2} q^{-(r-1)^2} e^{2vi} = \kappa_0 e^{2rvi}.$$

Moreover,
$$\kappa_0 = \sum_{m=-\infty}^{\infty} (-1)^m q^{m^2} \frac{2q^m e^{vi}}{1+q^{2m}e^{2vi}},$$

which, by a theorem of Jacobi's,

$$= \frac{\vartheta_1'(0)}{\vartheta_2(v)}.$$

Hence
$$\vartheta(u) \sum_{m=-\infty}^{\infty} \frac{2q^m e^{vi}}{1+q^{2m}e^{2vi}} e^{2mui}$$
$$= \frac{\vartheta_1'(0)}{\vartheta_2(v)} \{1+2q\cos 2(u+v)+2q^4\cos 4(u+v)+\ldots\}$$
$$= \frac{\vartheta_1'(0)\,\vartheta_3(u+v)}{\vartheta_2(v)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

2. We may derive a remarkable algebraic fact from the foregoing result.

Suppose the values of c_{2m} and κ_{2m} in (1) quite general, and let b_{2m} be such a function of u that

$$\vartheta(v) \sum_{m=-\infty}^{\infty} b_{2m} e^{2mvi} = \vartheta_3(u+v).$$

Then, as in § 1, (2),

$$q^{r^2} e^{2rui} = \sum_{m=-\infty}^{\infty} (-1)^{r-m} q^{(r-m)^2} b_{2m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

But b_{2m} is the coefficient of e^{2mvi} in

$$\frac{\vartheta_2(u)}{\vartheta_1'(0)} \sum_{m=-\infty}^{\infty} \frac{2q^m e^{ui}}{1+q^{2m}e^{2ui}},$$

by interchanging u and v in § 1, (3).

Hence
$$\frac{\vartheta_2(u)}{\vartheta_1'(0)} \sum_{m=-\infty}^{\infty} \kappa_{2m} \frac{2q^m e^{ui}}{1+q^{2m}e^{2ui}} = \sum_{m=-\infty}^{\infty} \kappa_{2m} b_{2m}$$
$$= \sum_{m=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} (-1)^{r-m} q^{(r-m)^2} c_{2r} b_{2m}, \quad \text{by § 1, (1)},$$
$$= \sum_{r=-\infty}^{\infty} c_{2r} q^{r^2} e^{2rui}, \quad \text{by (1)}.$$

We see then that, *if* c_{2m} *and* κ_{2m} *are so related that*

$$\vartheta(u) \sum_{m=-\infty}^{\infty} c_{2m} e^{2mui} = \sum_{m=-\infty}^{\infty} \kappa_{2m} e^{2mui}$$

then these coefficients are also so related that

$$\vartheta_2(u) \sum_{m=-\infty}^{\infty} \kappa_{2m} \frac{2q^m e^{ui}}{1+q^{2m}e^{2ui}} = \vartheta_1'(0) \sum_{m=-\infty}^{\infty} c_{2m} q^{m^2} e^{2mui} \quad \ldots\ldots(2).$$

By changing e^{vi} into $q^{\frac{1}{2}}e^{vi}$ in § 1, (3), so that $\vartheta_2(v)$ becomes $q^{-\frac{1}{4}}e^{-vi}\vartheta_3(v)$ and $\vartheta_3(u+v)$ becomes $q^{-\frac{1}{4}}e^{-(u+v)i}\vartheta_2(u+v)$, we get

$$\vartheta(u)\sum_{m=-\infty}^{\infty}\frac{2q^{m+\frac{1}{2}}e^{vi}}{1+q^{2m+1}e^{2vi}}=\frac{\vartheta_1'(0)\,\vartheta_2(u+v)}{\vartheta_3(v)} \quad \text{.....(3),}$$

whence, in the same manner as above, we find that, if

$$\vartheta(u)\sum_{m=-\infty}^{\infty}c_{2m+1}e^{(2m+1)ui}=\sum_{m=-\infty}^{\infty}\kappa_{2m+1}e^{(2m+1)ui},$$

then $$\vartheta_3(u)\sum_{m=-\infty}^{\infty}\kappa_{2m+1}\frac{2q^{m+\frac{1}{2}}e^{ui}}{1+q^{2m+1}e^{2ui}}=\vartheta_1'(0)\sum_{m=-\infty}^{\infty}c_{2m+1}q^{(m+\frac{1}{2})^2}e^{(2m+1)ui}$$

.....(4).

Moreover, since $$\int_0^{\pi}e^{2mti}\,dt=0,$$

when $$m\neq 0,$$

we see that

$$\left.\begin{aligned}\sum_{m=-\infty}^{\infty}c_{2m}q^{m^2}e^{2mui}&=\frac{1}{\pi}\int_0^{\pi}\sum_{m=-\infty}^{\infty}c_{2m}e^{2m(u+t)i}\,\vartheta_3(t)\,dt\\ \text{and}\quad\sum_{m=-\infty}^{\infty}c_{2m+1}q^{(m+\frac{1}{2})^2}e^{(2m+1)ui}&=\frac{1}{\pi}\int_0^{\pi}\sum_{m=-\infty}^{\infty}c_{2m+1}e^{(2m+1)(u+t)i}\,\vartheta_2(t)\,dt\end{aligned}\right\}$$

.....(5).

3. In § 2, (2), let $$c_{2m}=\frac{2q^m e^{vi}}{1+q^{2m}e^{2vi}},$$

so that, by § 1, (3), $$\kappa_{2m}=\frac{\vartheta_1'(0)}{\vartheta_2(v)}q^{m^2}e^{2mvi}.$$

We see then that

$$\vartheta_2(u)\sum_{m=-\infty}^{\infty}\frac{2q^m e^{ui}}{1+q^{2m}e^{2ui}}q^{m^2}e^{2mvi}=\vartheta_2(v)\sum_{m=-\infty}^{\infty}\frac{2q^m e^{vi}}{1+q^{2m}e^{2vi}}q^{m^2}e^{2mui} \quad \text{...(1),}$$

so that either side is symmetrical in u and v.

Similarly, from § 2, (2) and (4), we get that

$$\vartheta_3(u)\sum_{m=-\infty}^{\infty}\frac{2q^{m+\frac{1}{2}}e^{ui}}{1+q^{2m+1}e^{2ui}}q^{(m+\frac{1}{2})^2}e^{(2m+1)vi} \quad \text{.....(2)}$$

is symmetrical in u and v.

Either side of (1) will be denoted by $M_3(u, v)$, while (2) will be written $M_2(u, v)$, so that

$$M_3(u, v)=M_3(v, u) \quad \text{and} \quad M_2(u, v)=M_2(v, u).$$

Moreover, it is easy to see that

$$M_2(u, -v) = M_2(-u, v) = M_2(v, -u),$$

and

$$M_3(u, -v) = M_3(v, -u).$$

It will be convenient to write $\Lambda_2(u, v)$, $\Lambda_3(u, v)$ for $M_2(u, -v)$, $M_3(u, -v)$, so that the Λ-functions are also symmetrical in u and v. By adding and subtracting, we easily establish the symmetry in u and v of the four following expressions

$$\left.\begin{aligned}
&\vartheta_2(v)\left\{\frac{1}{\cos v}+\frac{4q(1+q^2)\cos v}{1+2q^2\cos 2v+q^4}q\cos 2u+\frac{4q^2(1+q^4)\cos v}{1+2q^4\cos 2v+q^8}q^4\cos 4u+\ldots\right\}\\
&\vartheta_2(v)\left\{\frac{4q(1-q^2)\sin v}{1+2q^2\cos 2v+q^4}q\sin 2u+\frac{4q^2(1-q^4)\sin v}{1+2q^4\cos 2v+q^8}q^4\sin 4u+\ldots\right\}\\
&\vartheta_3(v)\left\{\frac{4q^{\frac{1}{2}}(1+q)\cos v}{1+2q\cos 2v+q^2}q^{\frac{1}{2}}\cos u+\frac{4q^{\frac{3}{2}}(1+q^3)\cos v}{1+2q^3\cos 2v+q^6}q\cos 3u+\ldots\right\}\\
&\vartheta_3(v)\left\{\frac{4q^{\frac{1}{2}}(1-q)\sin v}{1+2q\cos 2v+q^2}q^{\frac{1}{2}}\sin u+\frac{4q^{\frac{3}{2}}(1-q^3)\cos v}{1+2q^3\cos 2v+q^6}q^{\frac{3}{2}}\sin 3u+\ldots\right\}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

When $v = 0$, the first and third of these become, respectively,

$$\vartheta_2(0)\left\{1+\frac{4q^2}{1+q^2}\cos 2u+\frac{4q^6}{1+q^4}\cos 4u+\frac{4q^{12}}{1+q^6}\cos 6u+\ldots\right\}$$

and $\vartheta_3(0)\left\{\frac{4q^{\frac{1}{2}}}{1+q}\cos u+\frac{4q^{\frac{3}{2}}}{1+q^3}\cos 3u+\ldots\right\}$,

which will be called $\Lambda_3(u)$ and $\Lambda_2(u)$, respectively.

Expressing (1) and (2) in terms of definite integrals by the help of § 1, (3), and § 2, (3) and (5), we see that

$$\int_0^\pi \frac{\vartheta_3(u+v+t)\,\vartheta_3(t)}{\vartheta(u+t)}\,dt = \int_0^\pi \frac{\vartheta_3(u+v+t)\,\vartheta_3(t)}{\vartheta(v+t)}\,dt = \frac{M_3(u, v)}{\vartheta_1'(0)}$$

and

$$\int_0^\pi \frac{\vartheta_2(u+v+t)\,\vartheta_2(t)}{\vartheta(u+t)}\,dt = \int_0^\pi \frac{\vartheta_2(u+v+t)\,\vartheta_2(t)}{\vartheta(v+t)} = \frac{M_2(u, v)}{\vartheta_1'(0)}.$$

4. It may now be shown that $\Lambda_2(u, v)$ may be expressed in terms of $\Lambda_2(u+v)$, and $\Lambda_3(u, v)$ in terms of $\Lambda_3(u+v)$.

Suppose that

$$\vartheta_3(u)\sum_{m=-\infty}^{\infty} c_{2m}q^{m^2}e^{2mui} - \vartheta_2(u)\sum_{m=-\infty}^{\infty} c_{2m+1}q^{(m+\frac{1}{2})^2}e^{(2m+1)ui} = \sum_{r=-\infty}^{\infty} \kappa^{2r}e^{2rui}.$$

Then the left-hand side

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ c_{2m} q^{m^2+n^2} e^{2(m+n)ui} - c_{2m+1} q^{(m+\frac{1}{2})^2+(n-\frac{1}{2})^2} e^{2(m+n)ui} \right\},$$

so that, putting $m+n=r$, we see that

$$\kappa_{2r} = q^{r^2} \sum_{m=-\infty}^{\infty} \left\{ q^{2m(m-r)} c_{2m} - q^{(2m+1)(m-r+\frac{1}{2})} c_{2m+1} \right\}$$

$$= q^{r^2} \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(m-2r)} c_m.$$

Hence, just as in § 1, when c_m has the same value,

$$\kappa_{2r} q^{-r^2} + \kappa_{2r-2} q^{-(r-1)^2} e^{2vi} = 0,$$

while $$\kappa_0 = \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m} c_m = \frac{\vartheta_1'(0, q^{\frac{1}{2}})}{\vartheta_2(v, q^{\frac{1}{2}})} = \frac{\vartheta_1'(0)\,\vartheta(0)}{\vartheta_2(v)\,\vartheta_3(v)},$$

so that $$\kappa_{2r} = \frac{\vartheta_1'(0)\,\vartheta(0)}{\vartheta_2(v)\,\vartheta_3(v)}\,\vartheta(u+v).$$

We have therefore the following linear relation connecting $M_2(u, v)$ and $M_3(u, v)$,

$$\vartheta_3(u)\,\vartheta_3(v)\,M_3(u, v) - \vartheta_2(u)\,\vartheta_2(v)\,M_2(u, v) = \vartheta_1'(0)\,\vartheta(0)\,\vartheta(u+v) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Changing v into $-v$, we have, moreover,

$$\vartheta_3(u)\,\vartheta_3(v)\,\Lambda_3(u, v) - \vartheta_2(u)\,\vartheta_2(v)\,\Lambda_2(u, v) = \vartheta_1'(0)\,\vartheta(0)\,\vartheta(u-v) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

These equations may be immediately obtained by dividing the known equation

$$\vartheta_3(u+v+t)\,\vartheta_3(t)\,\vartheta_3(u)\,\vartheta_3(v) - \vartheta_2(u+v+t)\,\vartheta_2(t)\,\vartheta_2(u)\,\vartheta_2(v)$$
$$= \vartheta(0)\,\vartheta(u+v)\,\vartheta(u+t)\,\vartheta(v+t)$$

by $\vartheta(u+t)$ and integrating for t between limits π and 0.

5. If $a = -i \log q$, so that $e^{(v+a)i} = q^{\frac{1}{2}} e^{vi}$, we see that, by adding a to u and to v,

$c_{2m} q^{m^2} e^{2mui}$ becomes $c_{2m+1} q^{m^2+m} e^{2mui} = e^{-ui} q^{-\frac{1}{4}} c_{2m+1} q^{(m+\frac{1}{2})^2} e^{(2m+1)ui}$.

Hence $$M_3(u+\tfrac{1}{2}a,\ v+\tfrac{1}{2}a) = \vartheta_2(v+\tfrac{1}{2}a)\, e^{-ui} q^{-\frac{1}{4}} \frac{M_2(u, v)}{\vartheta_3(v)}$$
$$= e^{-(u+v)i} q^{-\frac{1}{4}} M_2(u, v).$$

Thus, writing $v-\frac{1}{2}a$ for v, we have

$$M_3(u+\tfrac{1}{2}a, v) = e^{-(u+v)i} M_2(u-\tfrac{1}{2}a, v),$$

so that, changing v into $-v$,

$$\Lambda_3(u+\tfrac{1}{2}a, v) = e^{-(u-v)i} \Lambda_2(u-\tfrac{1}{2}a, v),$$

and replacing u by $u+\frac{1}{2}a$,

$$\Lambda_3(u+a, v) = e^{-(u-v)i} q^{-\frac{1}{4}} \Lambda_2(u, v),$$

so that
$$\begin{aligned} e^{2ui}\Lambda_3(u+a, v) &= e^{(u+v)i} q^{\frac{1}{4}} \Lambda_2(u, v) \\ &= e^{2vi} \Lambda_3(u, v+a), \text{ by symmetry.} \end{aligned}$$

Similarly,
$$\begin{aligned} e^{4ui}\Lambda_3(u+2a, v) &= e^{2(u+v)i} \Lambda_3(u+a, v+a) \\ &= e^{4vi} \Lambda_3(u, v+2a), \text{ by symmetry.} \end{aligned}$$

Proceeding in this way, we easily get

$$e^{2rui}\Lambda_3(u+ra, v) = e^{2rvi}\Lambda_3(u, v+ra) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

We may notice, moreover, that $\Lambda_3(u)$ and $\Lambda_2(u)$ have a kind of imaginary period, for

$$\Lambda_3(u+\tfrac{1}{2}a) + \Lambda_3(u-\tfrac{1}{2}a)$$

$$= \vartheta_2(0) \left\{ 1 + \frac{2q^2}{1+q^2}(q^{-1}+q)\cos 2u + \frac{2q^6}{1+q^4}(q^{-2}+q^2)\cos 4u + \ldots \right\}$$

$$= \vartheta_2(0)\, \vartheta_3(u),$$

while
$$\Lambda_2(u+\tfrac{1}{2}a) + \Lambda_2(u-\tfrac{1}{2}a) = \vartheta_3(0)\, \vartheta_2(u).$$

6. Let us now expand

$$\vartheta_3(u+x) \sum_{m=-\infty}^{\infty} q^{m^2} c_{2m} e^{2m(u-x)i} + \vartheta_2(u+x) \sum_{m=-\infty}^{\infty} q^{(m+\frac{1}{2})^2} c_{2m+1} e^{(2m+1)(u-x)i} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in the form
$$\Sigma A_{2r} e^{2rxi}.$$

The proposed expansion is

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} q^{m^2+n^2} c_{2m} e^{2(m+n)ui} e^{2(n-m)xi}$$

$$+ \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} q^{(m+\frac{1}{2})^2+(n+\frac{1}{2})^2} c_{2m+1} e^{2(m+n+1)ui} e^{2(n-m)xi},$$

so that, if $n = m+r$, it becomes

$$\sum_{r=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ q^{r^2+2m(m+r)} c_{2m} e^{2(2m+r)ui} e^{2rxi} + q^{r^2+(2m+1)(m+r+\frac{1}{2})} c_{2m+1} e^{2(2m+1+r)} e^{2rxi} \right\},$$

and $$A_{2r} = q^{r^2} \sum_{m=-\infty}^{\infty} q^{\frac{1}{2}m(m+2r)} c_m e^{2mui} e^{2rui}.$$

Now, if we change q into q^2 in A_{2r}, we get

$$q^{2r^2} e^{2rui} \sum_{m=-\infty}^{\infty} q^{m^2} c_{2m} e^{2mui} q^{2mr} = q^{2r^2} e^{2rui} \sum_{m=-\infty}^{\infty} q^{m^2} c_{2m} e^{2m(u+x)i}$$

$$= q^{2r^2} e^{2rui} \frac{M_3(u+ra, v)}{\vartheta_2(v)}.$$

We see, then, that, if v were changed to $-v$, this coefficient after multiplying by $\vartheta_2(v)$ would be symmetrical in u and v, by § 5.

Hence (1) is symmetrical in u and v after multiplying by $\vartheta_2(v, q^{\frac{1}{2}})$, and changing v into $-v$.

But $$\vartheta_2(v, q^{\frac{1}{2}})\, \vartheta_2(0, q^{\frac{1}{2}}) = 2\vartheta_2(v)\, \vartheta_3(v),$$

so that we finally get

$$\vartheta_3(u+x)\, \vartheta_3(v)\, \Lambda_3(u-x, v) + \vartheta_2(u+x)\, \vartheta_2(v)\, \Lambda_2(u-x, v)$$

$$= \vartheta_3(v+x)\, \vartheta_3(u)\, \Lambda_3(v-x, u) + \vartheta_2(v+x)\, \vartheta_2(u)\, \Lambda_2(v-x, u) \ldots (2).$$

Eliminating $\Lambda_3(u-x, v)$ by § 4, we see that

$$\frac{\vartheta_3(u+x)}{\vartheta_3(u-x)} \left[\Lambda_2(u-x, v)\, \vartheta_2(u-x)\, \vartheta_2(v) + \vartheta_1'(0)\, \vartheta(0)\, \vartheta(u-v-x) \right]$$

$$+ \vartheta_2(u+x)\, \vartheta_2(v)\, \Lambda_2(u-x, v)$$

is symmetrical in u and v.

But $$\vartheta_3(u+x)\, \vartheta_2(u-x) + \vartheta_2(u+x)\, \vartheta_3(u-x) = \vartheta_2(u, q^{\frac{1}{2}})\, \vartheta_2(x, q^{\frac{1}{2}}),$$

so that

$$\vartheta_2(u, q^{\frac{1}{2}})\, \vartheta_2(x, q^{\frac{1}{2}})\, \vartheta_2(v, q^{\frac{1}{2}})\, \vartheta(0, q^{\frac{1}{2}}) \left\{ \frac{\Lambda_2(u-x, v)}{\vartheta_3(u-x)\, \vartheta_3(v)} - \frac{\Lambda_2(v-x, u)}{\vartheta_3(v-x)\, \vartheta_3(u)} \right\}$$

$$= 2\vartheta_1'(0)\, \vartheta(0) \left\{ \frac{\vartheta_3(v+x)\, \vartheta(u-v+x)}{\vartheta_3(v-x)} - \frac{\vartheta_3(u+x)\, \vartheta(u-v-x)}{\vartheta_3(u-x)} \right\}.$$

But

$$\vartheta_3(u+x)\, \vartheta_3(v-x)\, \vartheta(u-v-x)\, \vartheta(x) - \vartheta_3(v+x)\, \vartheta_3(u-x)\, \vartheta(u-x+x)\, \vartheta(x)$$

$$= \vartheta_1(v-u)\, \vartheta_1(2x)\, \vartheta_2(u)\, \vartheta_2(v),$$

and $$\vartheta(0)\,\vartheta_1(2x)\,\vartheta_2(0)\,\vartheta_3(0) = 2\vartheta(x)\,\vartheta_1(x)\,\vartheta_2(x)\,\vartheta_3(x),$$

so that, finally, the above equation reduces to

$$\vartheta_3(v-x)\,\vartheta_3(u)\,\Lambda_2(u-x, v) - \vartheta_3(u-x)\,\vartheta_3(v)\,\Lambda_2(v-x, u) = \vartheta_1'(0)\,\vartheta_1(u-v)\,\vartheta_1(x) \quad \ldots\ldots(3).$$

Changing u into $u+v$, and x into v, this becomes

$$\vartheta_3(u+v)\,\vartheta_3(0)\,\Lambda_2(u, v) = \vartheta_3(u)\,\vartheta_3(v)\,\Lambda_2(u+v) + \vartheta_1'(0)\,\vartheta_1(u)\,\vartheta_1(v) \quad \ldots\ldots(4).$$

Similarly, by § 4, (2), we may eliminate the Λ_2-functions from (3), and obtain the relation

$$\vartheta_2(v-x)\,\vartheta_2(u)\,\Lambda_3(u-x, v) - \vartheta_2(u-x)\,\vartheta_2(v)\,\Lambda_3(v-x, u) = \vartheta_1'(0)\,\vartheta_1(v-u)\,\vartheta_1(x) \quad \ldots\ldots(5),$$

or, with the same change as before,

$$\vartheta_2(u+v)\,\vartheta_2(0)\,\Lambda_3(u, v) = \vartheta_2(u)\,\vartheta_2(v)\,\Lambda_3(u+v) - \vartheta_1'(0)\,\vartheta_1(u)\,\vartheta_1(v) \quad \ldots\ldots(6).$$

We have, then, formulæ for expressing $\Lambda_2(u, v)$ and $\Lambda_3(u, v)$ in terms of the simple functions $\Lambda_2(u+v)$, $\Lambda_3(u+v)$ and ϑ-functions.

Again, by the equation

$$\vartheta_3(u+v)\,\vartheta_3(0)\,\Lambda_3(u+v) - \vartheta_2(u+v)\,\vartheta_2(0)\,\Lambda_2(u+v) = \vartheta_1'(0)\,\vartheta(0)\,\vartheta(u+v),$$

which is easily obtained from § 4, (2), we may eliminate $\Lambda_3(u+v)$ from (6), and obtain

$$\vartheta_3(u+v)\,\vartheta_3(0)\,\Lambda_3(u, v) = \vartheta_2(u)\,\vartheta_2(v)\,\Lambda_2(u+v) + \vartheta_1'(0)\,\vartheta(u)\,\vartheta(v) \quad \ldots\ldots(7).$$

Changing u, v into $u+\frac{\pi}{2}$, $v+\frac{\pi}{2}$, and noticing that

$$\Lambda_2(u+v+\pi) = -\Lambda_2(u+v),$$

we have, from (4) and (7),

$$\vartheta_3(u+v)\,\vartheta_3(0)\,\Lambda_2\left(u+\frac{\pi}{2}, v+\frac{\pi}{2}\right) = -\vartheta(u)\,\vartheta(v)\,\Lambda_2(u+v) + \vartheta_1'(0)\,\vartheta_2(u)\,\vartheta_2(v) \quad \ldots\ldots(8),$$

and $\vartheta_3(u+v)\,\vartheta_3(0)\,\Lambda_3\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)$

$$= -\vartheta_1(u)\,\vartheta_1(v)\,\Lambda_2(u+v)+\vartheta_1'(0)\,\vartheta_3(u)\,\vartheta_3(v)\ldots\ldots(9).$$

Moreover, since

$$\vartheta_3^2(u)\,\vartheta_3^2(v)-\vartheta_2^2(u)\,\vartheta_2^2(v)+\vartheta_1^2(u)\,\vartheta_1^2(v)-\vartheta^2(u)\,\vartheta^2(v)=0,$$

we get, from (4), (7), (8), and (9),

$$\Lambda_3^2(u,v)-\Lambda_2^2(u,v)+\Lambda_2^2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)-\Lambda_3^2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)=0.$$

Similarly, by changing v into $-v$, and seeing that

$$M_2\left(u+\frac{\pi}{2},\ v-\frac{\pi}{2}\right)=-M_2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right),$$

we have

$$M_3^2(u,v)-M_2^2(u,v)+M_2^2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)-M_3^2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)=0.$$

The corresponding M-equations are, in fact,

$$\vartheta_3(u-v)\,\vartheta_3(0)\,M_2(u,v)=\vartheta_3(u)\,\vartheta_3(v)\,\Lambda_2(u-v)-\vartheta_1'(0)\,\vartheta_1(u)\,\vartheta_1(v),$$

$$\vartheta_3(u-v)\,\vartheta_3(0)\,M_3(u,v)=\vartheta_2(u)\,\vartheta_2(v)\,\Lambda_2(u-v)+\vartheta_1'(0)\,\vartheta(u)\,\vartheta(v),$$

$$\vartheta_3(u-v)\,\vartheta_3(0)\,M_2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)$$

$$=-\vartheta_1(u)\,\vartheta(v)\,\Lambda_2(u-v)+\vartheta_1'(0)\,\vartheta_2(u)\,\vartheta_2(v),$$

$$\vartheta_3(u-v)\,\vartheta_3(0)\,M_3\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)$$

$$=-\vartheta_1(u)\,\vartheta_1(v)\,\Lambda_2(u-v)-\vartheta_1'(0)\,\vartheta_3(u)\,\vartheta_3(v),$$

whence also

$$\Lambda_3(u,v)\,M_3(u,v)-\Lambda_2(u,v)\,M_2(u,v)$$

$$+\Lambda_2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)M_2\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)$$

$$-\Lambda_3\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)M_3\left(u+\frac{\pi}{2},\ v+\frac{\pi}{2}\right)=2\vartheta_1'(0)^2.$$

7. By putting $v = 0$ in various relations obtained above, we get the following relations between the simple series.

From § 4, (1),

$$\vartheta_3(u)\left\{1+\frac{4q^2}{1+q^2}\cos 2u+\frac{4q^6}{1+q^4}\cos 4u+\ldots\right\}$$

$$-\vartheta_2(u)\left\{\frac{4q^3}{1+q}\cos u+\frac{4q^{3\cdot 5}}{1+q^3}\cos 3u+\ldots\right\}=\vartheta(0)^2\,\vartheta(u).$$

By changing q into $-q$,

$$\vartheta(u)\left\{1+\frac{4q^2}{1+q^2}\cos 2u+\frac{4q^6}{1+q^4}\cos 4u+\ldots\right\}$$

$$+\vartheta_2(u)\left\{\frac{4q^3}{1-q}\cos u-\frac{4q^{3\cdot 5}}{1-q^3}\cos 3u+\ldots\right\}=\vartheta_3(0)^2\,\vartheta_3(u).$$

By eliminating $1+\frac{4q^2}{1+q^2}\cos 2u+\ldots$ from these two equations,

$$\vartheta(u)\left\{\frac{4q^3}{1+q}\cos u+\frac{4q^{3\cdot 5}}{1+q^3}\cos 3u+\ldots\right\}$$

$$+\vartheta_3(u)\left\{\frac{4q^3}{1-q}\cos u-\frac{4q^{3\cdot 5}}{1-q^3}\cos 3u+\ldots\right\}=\vartheta_2(0)^2\,\vartheta_2(u).$$

Changing v into $-v$ in § 6, (6), and subtracting, we have, moreover,

$$\frac{2}{\vartheta_2(u)}\left\{\frac{4q(1-q^2)\sin v}{1+2q^2\cos 2v+q^4}q\sin 2u+\frac{4q^2(1-q^4)}{1+2q^4\cos 2v+q^8}q^4\sin 4u+\ldots\right\}$$

$$=\frac{\Lambda_3(u+v)}{\vartheta_2(u+v)\,\vartheta_2(0)}-\frac{\Lambda_3(u-v)}{\vartheta_2(u-v)\,\vartheta_2(0)}$$

$$-\frac{\vartheta_1'(0)\,\vartheta_1(u)\,\vartheta_1(v)}{\vartheta_2(0)\,\vartheta_2(u)\,\vartheta_2(v)}\left\{\frac{1}{\vartheta_2(u+v)}+\frac{1}{\vartheta_2(u-v)}\right\}.$$

When $v=\frac{\pi}{2}$, this becomes

$$\vartheta_1(u)\left\{\frac{4q^2}{1-q^2}\sin 2u+\frac{4q^6}{1-q^4}\sin 4u+\ldots\right\}$$

$$+\vartheta_2(u)\left\{1-\frac{4q^2}{1+q^2}\cos 2u+\frac{4q^6}{1+q^4}\cos 4u-\ldots\right\}=\vartheta_1'(u).$$

When $v = 0$, we get also

$$\vartheta_2(0)\left\{\frac{4q^2(1-q^2)}{(1+q^2)^2}\sin 2u + \frac{4q^6(1-q^4)}{(1+q^4)^2}\sin 4u + \dots\right\}$$

$$= \vartheta_2(u)\frac{d}{du}\frac{\Lambda_3(u)}{\vartheta_2(u)} - \frac{\vartheta_1'(0)^2\,\vartheta_1(u)}{\vartheta_2(0)\,\vartheta_2(u)}.$$

Similarly,

$$\vartheta_3(0)\left\{\frac{4q^{\frac{1}{2}}(1-q)}{(1+q)^2}\sin u + \frac{4q^{\frac{3}{2}}(1-q^3)}{(1+q^3)^2}\sin 3u + \dots\right\}$$

$$= \vartheta_3(u)\frac{d}{du}\frac{\Lambda_2(u)}{\vartheta_2(u)} + \frac{\vartheta_1'(0)^2\,\vartheta_1(u)}{\vartheta_3(0)\,\vartheta_3(u)}.$$

8. The series $\sum_{m=-\infty}^{\infty}\frac{2q^m e^{vi}}{1+q^{2m}e^{2vi}}q^{m^2}e^{2mui}$ is easily seen to satisfy a partial differential equation.

For the operation of $q\frac{d}{dq}$ on the general terms is equal to that of

$$-\frac{1}{2}\frac{d^2}{du\,dv} - \frac{1}{4}\frac{d^2}{du^2}.$$

Thus $q\frac{d}{dq}\frac{M_3(u,v)}{\vartheta_2(v)} = -\frac{1}{2}\frac{d^2}{du\,dv}\frac{M_3(u,v)}{\vartheta_2(v)} - \frac{1}{4}\frac{d^2}{du^2}\frac{M_3(u,v)}{\vartheta_2(v)}$.

If we write X_3 for $M_3(u,v)/\vartheta_2(u)\,\vartheta_2(v)$, this equation becomes

$$q\frac{dX_3}{dq} = -\frac{1}{2}\frac{d^2X_3}{du\,dv} - \frac{1}{4}\frac{d^2X_3}{du^2} - \frac{1}{2}\frac{\vartheta_2'(u)}{\vartheta_2(u)}\left(\frac{dX_3}{du}+\frac{dX_3}{dv}\right) \quad \dots(1).$$

But, by § 6, (6),

$$X_3 = \frac{\Lambda_3(u-v)}{\vartheta_2(u-v)\,\vartheta_2(0)} - \frac{\vartheta_1'(0)\,\vartheta_1(u)\,\vartheta_1(v)}{\vartheta_2(u-v)\,\vartheta_2(0)\,\vartheta_2(u)\,\vartheta_2(v)}.$$

Hence $\left(\frac{d}{du}+\frac{d}{dv}\right)X_3$ is purely elliptic, since the differential operator annihilates all functions of $u-v$, and by ordinary ϑ-function formulæ, including

$$\vartheta(u)\,\vartheta_3(u)\,\vartheta_1(v)\,\vartheta_2(v) + \vartheta(v)\,\vartheta_3(v)\,\vartheta_1(u)\,\vartheta_2(u)$$

$$= \vartheta_3(0)\,\vartheta(0)\,\vartheta_1(u+v)\,\vartheta_2(u-v),$$

we have

$$\left(\frac{d}{du}+\frac{d}{dv}\right)X_3 = \frac{\{\vartheta_1'(0)\}^2\,\vartheta_1(u+v)}{\vartheta_2^2(u)\,\vartheta_2^2(v)} \quad \dots\dots\dots\dots(2).$$

This relation will help to reduce the unsymmetric equation (1) to the symmetrical form

$$4q\frac{dX_3}{dq}+\frac{d^2X_3}{du\,dv}=-\frac{\{\vartheta_1'(0)\}^2\vartheta_1'(u+v)}{\vartheta_2^2(u)\,\vartheta_2^2(v)} \quad \text{.....(3).}$$

Moreover, eliminating $\frac{d}{dv}$ by (2), we get

$$4q\frac{dX_3}{dq}-\frac{d^2X_3}{du^2}=-2\frac{\{\vartheta_1'(0)\}^2}{\vartheta_2(u)\,\vartheta_2(v)}\frac{d}{du}\frac{\vartheta_1(u+v)}{\vartheta_2(u)}.$$

When $v=0$, this becomes

$$\left(4q\frac{d}{dq}-\frac{d^2}{du^2}\right)\frac{\Lambda_3(u)}{\vartheta_2(0)\,\vartheta_2(u)}=-2\{\vartheta_1'(0)\}^2\frac{\vartheta(u)\,\vartheta_3(u)}{\vartheta_2^3(u)}.$$

In a similar manner, we obtain

$$\left(4q\frac{d}{dq}-\frac{d^2}{du^2}\right)\frac{\Lambda_2(u)}{\vartheta_3(0)\,\vartheta_3(u)}=-2\{\vartheta_1'(0)\}^2\frac{\vartheta(u)\,\vartheta_3(u)}{\vartheta_3^3(u)}.$$

The Electrical Distribution on a Conductor bounded by Two Spherical Surfaces cutting at any Angle. *By* H. M. Macdonald. Read January 10th, 1895. Received January 17th, 1895.

In Maxwell's *Electricity and Magnetism*, Vol. I., §§ 165, 166, the problem of the distribution of electricity induced by an electrified point placed between them on two planes cutting at an angle which is a submultiple of two right angles, and the inverse problem of the conductor formed by two spherical surfaces cutting at such an angle (the angle referring to the dielectric), are solved by the method of point images. This method is inapplicable when the (dielectric) angle is not a submultiple of two right angles, as has been shown by W. D. Niven, *Proc. Lond. Math. Soc.*, Vol. XXI., p. 27. The only other case which has been hitherto solved is, as far as I know, that of the spherical bowl (Lord Kelvin, *Papers on Electrostatics and Magnetism*, p. 178). In the paper by W. D. Niven mentioned above

an attempt is made to deduce the capacity of such a conductor from the solution of a functional equation for a particular value of one of the variables, but the result obtained does not seem in the case of the spherical bowl to agree with Lord Kelvin's. The results obtained hereafter also differ from those given by Niven.

The object of this paper is to obtain the solution in the general case. To effect this the functional image of a point placed between two planes intersecting at any angle is obtained in the form of a definite integral, § 1. In the next few paragraphs the reduction of this integral to known forms is effected in certain cases, and it is shown that the integration can be performed when the angle of intersection is any submultiple of four right angles; the case in which it is reducible to elliptic functions is also discussed.

In § 5 the functional image of a line of uniform density parallel to the intersection of the planes is deduced. In § 6 a certain definite integral is discussed shortly. In § 7 the potential due to a freely charged conductor bounded by two spherical surfaces cutting at any angle is obtained, and some particular cases discussed. The capacity of such a conductor is obtained in § 8 in finite terms, and some particular cases are discussed in § 9; one of the most interesting of these is the capacity of a hemisphere, which is found to be nearly seventeen-twentieths that of the complete sphere. Some cases are mentioned in the last paragraph which could be deduced from the results of the preceding ones.

1. *The Functional Image of a Point placed between Two Planes intersecting at any Angle.*

To obtain the potential produced by an electrified point situated in the space between two conducting planes intersecting at any angle. Consider a prism whose cross-section is a sector of a circle radius b and angle α, and having its ends perpendicular to the axis of the prism, distance h from each other. Take the origin of coordinates at the centre of the circle the sector of which forms one of the ends of the prism; then, using cylindrical coordinates, V the potential at any point inside the prism satisfies the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} + 4\pi\rho = 0 \quad \ldots\ldots\ldots\ldots (1),$$

where ρ is the density at the point r, θ, z, and $V = 0$ all over the

boundary. Assume

$$V = \Sigma W_n \sin \frac{n\pi\theta}{\alpha},$$

where n is an integer, for $V = 0$ when $\theta = 0$ or $\theta = \alpha$; then (1) becomes

$$\Sigma \left\{ \frac{\partial^2 W_n}{\partial r^2} + \frac{1}{r} \frac{\partial W_n}{\partial r} - \frac{n^2\pi^2}{\alpha^2 r^2} W_n + \frac{\partial^2 W_n}{\partial z^2} \right\} \sin \frac{n\pi\theta}{\alpha} + \pi\rho = 0 \quad \ldots(2).$$

Put

$$W_n = \Sigma J_{n\pi/\alpha}(\kappa r)\, U_\kappa,$$

where the condition that $V = 0$, when $r = b$, makes κ a root of

$$J_{n\pi/\alpha}(\kappa b) = 0,$$

and the summation is taken for all the roots of this equation; then (2) becomes

$$\Sigma\Sigma \left\{ \frac{\partial^2 U_\kappa}{\partial z^2} - \kappa^2 U_\kappa \right\} J_{n\pi/\alpha}(\kappa r) \sin \frac{n\pi\theta}{\alpha} + 4\pi\rho = 0 \quad \ldots\ldots\ldots\ldots(3).$$

Hence

$$4\pi\rho = \Sigma\Sigma\Sigma A_{\kappa\lambda n} (\lambda^2 + \kappa^2) J_{n\pi/\alpha}(\kappa r) \sin \frac{n\pi\theta}{\alpha} \sin \lambda z \quad \ldots\ldots\ldots\ldots(4),$$

where, since $V = 0$, when $z = 0$ and $z = h$, $\lambda = \frac{m\pi}{h}$, m being an integer. From (4), by an application of Fourier's theorem,

$$4\pi \int_0^h \rho' \sin \lambda z'\, dz' = \frac{h}{2} \Sigma\Sigma A_{\kappa\lambda n} (\lambda^2 + \kappa^2) J_{n\pi/\alpha}(\kappa r) \sin \frac{n\pi\theta}{\alpha},$$

whence

$$4\pi \int_0^\alpha \int_0^h \rho' \sin \lambda z' \sin \frac{n\pi\theta'}{\alpha}\, d\theta' dz' = \frac{\alpha h}{4} \Sigma A_{\kappa\lambda n} (\lambda^2 + \kappa^2) J_{n\pi/\alpha}(\kappa r).$$

Multiplying both sides by $r' J_{n\pi/\alpha}(\kappa r')$, and integrating from 0 to b,

$$4\pi \int_0^b \int_0^\alpha \int_0^h \rho' \sin \lambda z' \sin \frac{n\pi\theta'}{\alpha} J_{n\pi/\alpha}(\kappa r')\, r'\, dr'\, d\theta'\, dz' = \frac{\alpha h b^2}{8} J'^2_{n\pi/\alpha}(\kappa b) A_{\kappa\lambda n} (\lambda^2 + \kappa^2);$$

therefore, for a charge q at the point r', θ', z',

$$\frac{\alpha h b^2}{8} J'^2_{n\pi/\alpha}(\kappa b) A_{\kappa\lambda n} (\lambda^2 + \kappa^2) = 4\pi q \sin \lambda z' \sin \frac{n\pi\theta'}{\alpha} J_{n\pi/\alpha}(\kappa r'),$$

and

$$V = \frac{32\pi q}{\alpha h b^2} \Sigma\Sigma\Sigma \frac{\sin \lambda z' \sin \frac{n\pi\theta'}{\alpha} J_{n\pi/\alpha}(\kappa r')}{(\lambda^2 + \kappa^2) J'^2_{n\pi/\alpha}(\kappa b)} \sin \lambda z \sin \frac{n\pi\theta}{\alpha} J_{n\pi/\alpha}(\kappa r).$$

Now,
$$\frac{1}{2\kappa}+\Sigma\frac{\kappa\cos\lambda\zeta}{\lambda^2+\kappa^2}=\frac{h}{2}\,\frac{\cosh\kappa\,(h-\zeta)}{\sinh\kappa h},$$

if $2h>\zeta>0$; therefore

$$2\Sigma\frac{\sin\lambda z\sin\lambda z'}{\lambda^2+\kappa^2}=\frac{h}{2}\,\frac{\cosh\kappa\,(h-\overline{z-z'})-\cosh\kappa\,(h-\overline{z+z'})}{\kappa\sinh\kappa h},$$

when $z>z'$. Hence

$$V=\frac{8\pi q}{ab^2}\Sigma\Sigma\frac{\sin\frac{n\pi\theta'}{a}\,J_{n\pi/a}(\kappa r')}{J'^2_{n\pi/a}(\kappa b)}\,\frac{\cosh\kappa\,(h-\overline{z-z'})-\cosh\kappa\,(h-\overline{z+z'})}{\kappa\sinh\kappa h}\times\sin\frac{n\pi\theta}{a}\,J_{n\pi/a}(\kappa r).$$

Change the origin to the point z' on the axis, and make both ends move off to an infinite distance; then the potential, due to an electrified point q at the point r', θ', of the plane $z=0$, at any point of the space bounded by the infinitely long cylindrical surface, whose cross-section is a sector radius b and angle a, is given by

$$V=\frac{8\pi q}{ab^2}\Sigma\Sigma\frac{\sin\frac{n\pi\theta'}{a}\,J_{n\pi/a}(\kappa r')}{\kappa J'^2_{n\pi/a}(\kappa b)}\,e^{-\kappa z}\sin\frac{n\pi\theta}{a}\,J_{n\pi/a}(\kappa r),$$

when z is positive; when z is negative, $e^{\kappa z}$ has to be written for $e^{-\kappa z}$.

To obtain V when b is made indefinitely great it is necessary to evaluate $b^2J'^2_{n\pi/a}(\kappa b)$, subject to $J_{n\pi/a}(\kappa b)=0$, when b is infinite;

$$J_{n\pi/a}(\kappa b)=\sqrt{\frac{2}{\pi\kappa b}}\cos\left\{\frac{n\pi^2}{2a}+\frac{\pi}{4}-\kappa b\right\},$$

when b is very great;

$$J'_{n\pi/a}(\kappa b)=\sqrt{\frac{2}{\pi\kappa b}}\sin\left\{\frac{n\pi^2}{2a}+\frac{\pi}{4}-\kappa b\right\},$$

when $J_{n\pi/a}(\kappa b)=0$ and b is infinite.

If κ and $\kappa+\delta\kappa$ are consecutive roots of

$$J_{n\pi/a}(\kappa b)=0,$$

$$b\,\delta\kappa=\pi;$$

therefore
$$b^2J'^2_{n\pi/a}(\kappa b)=\frac{2b}{\pi\kappa}=\frac{2}{\kappa\delta\kappa}.$$

Hence the potential at the point r, θ, z of the dielectric bounded by two conducting infinite planes intersecting at an angle α, due to a charge q at the point r', θ', 0, is given by

$$V = \frac{4\pi q}{\alpha} \sum_{n=1}^{n=\infty} \int_0^\infty e^{-\kappa z}\, d\kappa \sin\frac{n\pi\theta'}{\alpha} \sin\frac{n\pi\theta}{\alpha} J_{n\pi/\alpha}(\kappa r') J_{n\pi/\alpha}(\kappa r),$$

when z is positive, and by

$$V = \frac{4\pi q}{\alpha} \sum_{n=1}^{n=\infty} \int_0^\infty e^{\kappa z}\, d\kappa \sin\frac{n\pi\theta'}{\alpha} \sin\frac{n\pi\theta}{\alpha} J_{n\pi/\alpha}(\kappa r') J_{n\pi/\alpha}(\kappa r),$$

when z is negative.

It has been proved by Sonnine, *Math. Ann.*, XVI., that

$$\int_0^\infty J_{n\pi/\alpha}(\kappa r) J_{n\pi/\alpha}(\kappa r')\, d\kappa = \frac{2}{\pi (rr')^{n\pi/\alpha}} \int_0^{r'} \frac{x^{2n\pi/\alpha}\, dx}{\sqrt{(r^2-x^2)(r'^2-x^2)}},$$

where $r > r'$.

This latter integral can be transformed into

$$\frac{1}{\pi\sqrt{2rr'}} \int_\eta^\infty \frac{e^{-(n\pi\zeta)/\alpha}\, d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}},$$

for all values of r and r', where η is to be taken positive, and

$$2\cosh\eta = \frac{r^2+r'^2}{rr'}.$$

Hence, when $z = 0$, the potential is given by

$$V = \frac{4q}{\alpha\sqrt{2rr'}} \sum_{n=1}^{n=\infty} \int_\eta^\infty \frac{e^{-(n\pi\zeta)/\alpha}\, d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}} \sin\frac{n\pi\theta}{\alpha} \sin\frac{n\pi\theta'}{\alpha},$$

$$V = \frac{2q}{\alpha\sqrt{2rr'}} \int_\eta^\infty \frac{d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}} \times \sum_{n=1}^{n=\infty} e^{-(n\pi\zeta)/\alpha} \left\{ \cos\frac{n\pi}{\alpha}(\theta-\theta') - \cos\frac{n\pi}{\alpha}(\theta+\theta') \right\}.$$

Therefore

$$V = \frac{q}{\alpha\sqrt{2rr'}} \int_\eta^\infty \frac{d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}} \times \left\{ \frac{\sinh\frac{\pi\zeta}{\alpha}}{\cosh\frac{\pi\zeta}{\alpha} - \cos\frac{\pi(\theta-\theta')}{\alpha}} - \frac{\sinh\frac{\pi\zeta}{\alpha}}{\cosh\frac{\pi\zeta}{\alpha} - \cos\frac{\pi(\theta+\theta')}{\alpha}} \right\}.$$

This may be extended to any point by making η satisfy

$$2\cosh\eta = \frac{r^2+r'^2+z^2}{rr'}.$$

2. *Form of the Function when α is a Submultiple of π.*

When π/α is an integer, the expression for V may be reduced to the known form and expressed as the sum of the reciprocals of the distances from a series of points. Let $\pi/\alpha = p$; then, when p is an integer,

$$\frac{p\sinh p\zeta}{\cosh p\zeta - \cos p\phi} = \sum_{k=0}^{k=p-1} \frac{\sinh\zeta}{\cosh\zeta - \cos\left(\phi + \frac{2k\pi}{p}\right)},$$

and

$$\int_\eta^\infty \frac{d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}} \frac{\sinh\zeta}{\cosh\zeta - \cos\phi} = \frac{\pi}{\sqrt{\cosh\eta - \cos\phi}};$$

hence

$$V = \frac{q}{\sqrt{2rr'}} \sum_{k=0}^{k=p-1} \left\{ \frac{1}{\sqrt{\cosh\eta - \cos\left(\theta - \theta' + \frac{2k\pi}{p}\right)}} - \frac{1}{\sqrt{\cosh\eta - \cos\left(\theta + \theta' + \frac{2k\pi}{p}\right)}} \right\}$$

that is,

$$V = q \sum_{k=0}^{k=p-1} \left\{ \frac{1}{\sqrt{r^2+r'^2+z^2-2rr'\cos\left(\theta - \theta' + \frac{2k\pi}{p}\right)}} - \frac{1}{\sqrt{r^2+r'^2+z^2-2rr'\cos\left(\theta + \theta' + \frac{2k\pi}{p}\right)}} \right\},$$

which is the expression obtained by the method of images.

3. *Form of the Function when α is an Odd Submultiple of 2π.*

When

$$\alpha = \frac{2\pi}{2m+1},$$

m being a positive integer, the potential can be expressed as the sum

of a finite number of terms involving arctangents; for then

$$V=\frac{2m+1}{2\pi}\frac{q}{\sqrt{2rr'}}\int_\eta^\infty\frac{d\zeta}{\sqrt{\cosh\zeta-\cosh\eta}}$$

$$\times\left\{\frac{\sinh\frac{2m+1}{2}\zeta}{\cosh\frac{2m+1}{2}\zeta-\cos\frac{2m+1}{2}(\theta-\theta')}-\frac{\sinh\frac{2m+1}{2}\zeta}{\cosh\frac{2m+1}{2}\zeta-\cos\frac{2m+1}{2}(\theta+\theta')}\right\},$$

that is,

$$V=\frac{q}{2\pi\sqrt{2rr'}}\sum_{k=0}^{k=2m}\int_\eta^\infty\frac{d\zeta}{\sqrt{\cosh\zeta-\cosh\eta}}$$

$$\times\left\{\frac{\sinh\frac{\zeta}{2}}{\cosh\frac{\zeta}{2}-\cos\left(\frac{\theta-\theta'}{2}+\frac{2k\pi}{2m+1}\right)}-\frac{\sinh\frac{\zeta}{2}}{\cosh\frac{\zeta}{2}-\cos\left(\frac{\theta+\theta'}{2}+\frac{2k\pi}{2m+1}\right)}\right\},$$

whence

$$V=\frac{2q}{\pi\sqrt{2rr'}}\sum_{k=0}^{k=2m}$$

$$\times\left\{\frac{1}{\sqrt{\cosh\eta-\cos\left(\theta-\theta'+\frac{4k\pi}{2m+1}\right)}}\tan^{-1}\sqrt{\frac{\cosh\frac{\eta}{2}+\cos\left(\frac{\theta-\theta'}{2}+\frac{2k\pi}{2m+1}\right)}{\cosh\frac{\eta}{2}-\cos\left(\frac{\theta-\theta'}{2}+\frac{2k\pi}{2m+1}\right)}}\right.$$

$$\left.-\frac{1}{\sqrt{\cosh\eta-\cos\left(\theta+\theta'+\frac{4k\pi}{2m+1}\right)}}\tan^{-1}\sqrt{\frac{\cosh\frac{\eta}{2}+\cos\left(\frac{\theta+\theta'}{2}+\frac{2k\pi}{2m+1}\right)}{\cosh\frac{\eta}{2}-\cos\left(\frac{\theta+\theta'}{2}+\frac{2k\pi}{2m+1}\right)}}\right\}.$$

This includes as a particular case half an infinite plane under the action of an electrified point; here $m=0$, and therefore

$$V=\frac{2q}{\pi\sqrt{2rr'}}\left\{\frac{1}{\sqrt{\cosh\eta-\cos(\theta-\theta')}}\tan^{-1}\sqrt{\frac{\cosh\frac{\eta}{2}+\cos\frac{\theta-\theta'}{2}}{\cosh\frac{\eta}{2}-\cos\frac{\theta-\theta'}{2}}}\right.$$

$$\left.-\frac{1}{\sqrt{\cosh\eta-\cos(\theta+\theta')}}\tan^{-1}\sqrt{\frac{\cosh\frac{\eta}{2}+\cos\frac{\theta+\theta'}{2}}{\cosh\frac{\eta}{2}-\cos\frac{\theta+\theta'}{2}}}\right\},$$

or, in terms of the cylindrical coordinates,

$$V=\frac{2q}{\pi}\left\{\frac{\tan^{-1}\sqrt{\left(\sqrt{\overline{r+r'}^2+z^2}+2\sqrt{rr'}\cos\frac{\theta-\theta'}{2}\right)\Big/\left(\sqrt{\overline{r+r'}^2+z^2}-2\sqrt{rr'}\cos\frac{\theta-\theta'}{2}\right)}}{\sqrt{r^2+r'^2+z^2-2rr'\cos(\theta-\theta')}}\right.$$

$$\left.-\frac{\tan^{-1}\sqrt{\left(\sqrt{\overline{r+r'}^2+z^2}+2\sqrt{rr'}\cos\frac{\theta+\theta'}{2}\right)\Big/\left(\sqrt{\overline{r+r'}^2+z^2}-2\sqrt{rr'}\cos\frac{\theta+\theta'}{2}\right)}}{\sqrt{r^2+r'^2+z^2-2rr'\cos(\theta+\theta')}}\right\}$$

Thus the potential can be expressed in terms of algebraic and circular functions when α is a submultiple of 2π.

4. *Form of the Function in other Cases.*

In all other cases the reduction of the expression for the potential involves transcendental functions: *e.g.*, when

$$\alpha=\frac{3\pi}{3m\pm1},$$

it can be expressed in terms of complete elliptic integrals of the first and third kinds; for then, as above,

$$V=\frac{q}{3\pi\sqrt{2rr'}}\sum_{k=0}^{k=3m\text{ or }3m-2}\int_\eta^\infty\frac{d\zeta}{\sqrt{\cosh\zeta-\cosh\eta}}$$

$$\times\left\{\frac{\sinh\frac{\zeta}{3}}{\cosh\frac{\zeta}{3}-\cos\left(\frac{\theta-\theta'}{3}+\frac{2k\pi}{3m\pm1}\right)}-\frac{\sinh\frac{\zeta}{3}}{\cosh\frac{\zeta}{3}-\cos\left(\frac{\theta+\theta'}{3}+\frac{2k\pi}{3m\pm1}\right)}\right\}.$$

Now,
$$\tfrac{1}{3}\int_\eta^\infty\frac{d\zeta}{\sqrt{\cosh\zeta-\cosh\eta}}\,\frac{\sinh\frac{\zeta}{3}}{\cosh\frac{\zeta}{3}-\gamma}$$

$$=\tfrac{1}{2}\int_{x_0}^\infty\frac{dx}{(x-\gamma)\sqrt{(x-x_0)(x^2+xx_0+x_0^2-\frac{3}{4})}}$$

$$=2\sqrt{\lambda}\,\frac{\lambda+x_0-\gamma}{\lambda-x_0+\gamma}\int_0^K\frac{du}{4\lambda(x_0-\gamma)+(\lambda-x_0+\gamma)^2\,\mathrm{sn}^2u}-\frac{K}{\sqrt{\lambda}\,(\lambda-x_0+\gamma)},$$

where
$$4\lambda^2=12x_0^2-3,\quad 4\lambda k^2=2\lambda-3x_0,$$

and
$$x_0=\cosh\frac{\eta}{3}.$$

Hence

$$V=\frac{q}{\pi\sqrt{2rr'}}\sum_{k=0}^{k=3m \text{ or } 3m-2}\left\{2\sqrt{\lambda}\frac{\lambda+x_0-\gamma_k}{\lambda-x_0+\gamma_k}\int_0^K\frac{du}{4\lambda(x_0-\gamma_k)+(\lambda-x_0+\gamma_k)^2\mathrm{sn}^2u}\right.$$

$$-2\sqrt{\lambda}\frac{\lambda+x_0-\gamma_k'}{\lambda-x_0+\gamma_k'}\int_0^K\frac{du}{4\lambda(x_0-\gamma_k')+(\lambda-x_0+\gamma_k')^2\mathrm{sn}^2u}$$

$$\left.-\frac{K}{\sqrt{\lambda}}\left(\frac{1}{\lambda-x_0+\gamma_k}-\frac{1}{\lambda-x_0+\gamma_k'}\right)\right\},$$

where $$\gamma_k=\cos\left(\frac{\theta-\theta'}{3}+\frac{2k\pi}{3m\pm1}\right),$$

and $$\gamma_k'=\cos\left(\frac{\theta+\theta'}{3}+\frac{2k\pi}{3m\pm1}\right).$$

This includes as a particular case a conductor bounded by two planes intersecting at right angles, the dielectric angle being three right angles, under the influence of an electrified point.

5. *The Image of a Line of Uniform Density placed between Two Planes Intersecting at any Angle, the Line being Parallel to the Intersection of the Planes.*

Let ρ be the density per unit length of the line; then the potential due to it by § 1 is given by

$$V=\frac{\rho}{a\sqrt{2rr'}}\int_{-\infty}^{\infty}dz\int_{\eta}^{\infty}\frac{d\zeta}{\sqrt{\cosh\zeta-\cosh\eta}}$$

$$\times\left\{\frac{\sinh\frac{\pi\zeta}{a}}{\cosh\frac{\pi\zeta}{a}-\cos\frac{\pi}{a}(\theta-\theta')}-\frac{\sinh\frac{\pi\zeta}{a}}{\cosh\frac{\pi\zeta}{a}-\cos\frac{\pi}{a}(\theta+\theta')}\right\},$$

where $$\cosh\eta=\frac{r^2+r'^2+z^2}{2rr'}=\cosh\eta_0+\frac{z^2}{2rr'}.$$

Changing the order of integration,

$$V=\frac{\rho}{a\sqrt{rr'}}\int_{\eta_0}^{\infty}d\zeta\int_{-\sqrt{2rr'(\cosh\zeta-\cosh\eta_0)}}^{\sqrt{2rr'(\cosh\zeta-\cosh\eta_0)}}\frac{d\zeta}{\sqrt{\cosh\zeta-\cosh\eta_0-\frac{z^2}{2rr'}}}$$

$$\times\left\{\frac{\sinh\frac{\pi\zeta}{a}}{\cosh\frac{\pi\zeta}{a}-\cos\frac{\pi}{a}(\theta-\theta')}-\frac{\sinh\frac{\pi\zeta}{a}}{\cosh\frac{\pi\zeta}{a}-\cos\frac{\pi}{a}(\theta+\theta')}\right\},$$

$$= \frac{\rho\pi}{a}\int_{\eta_0}^{\infty} d\zeta \left\{ \frac{\sinh\frac{\pi\zeta}{a}}{\cosh\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta-\theta')} - \frac{\sinh\frac{\pi\zeta}{a}}{\cosh\frac{\pi\zeta}{a} - \cos\frac{\pi}{a}(\theta+\theta')} \right\};$$

Hence
$$V = \rho \log \frac{\cosh\frac{\pi\eta_0}{a} - \cos\frac{\pi}{a}(\theta+\theta')}{\cosh\frac{\pi\eta_0}{a} - \cos\frac{\pi}{a}(\theta-\theta')},$$

or
$$V = \rho \log \frac{r^{2\pi/a} + r'^{2\pi/a} - 2r^{\pi/a}r'^{\pi/a}\cos\frac{\pi}{a}(\theta+\theta')}{r^{2\pi/a} + r'^{2\pi/a} - 2r^{\pi/a}r'^{\pi/a}\cos\frac{\pi}{a}(\theta-\theta')}.$$

This shows that the result obtained by Greenhill, *Quarterly Journal of Mathematics*, Vol. XV., is true when a is not restricted to be a sub-multiple of π. It could be deduced from the form for V in § 1, involving Bessel functions, by the use of the equality

$$\int_0^{\infty} J_{n\pi/a}(\kappa r)\, J_{n\pi/a}(\kappa r')\, \frac{d\kappa}{\kappa} = \frac{a}{2n\pi}\left(\frac{r'}{r}\right)^{n\pi/a},$$

where
$$r > r' \quad \text{and} \quad n > 0,$$

which is easy to establish.

6. *The Integral* $\displaystyle\int_{\eta}^{\infty} \frac{e^{-(n\pi\zeta/a)}\, d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}}$.

It is easy to show that this integral is equal to $\sqrt{2}\, Q_{(n\pi/a)-\frac{1}{2}}(\cosh\eta)$, where $Q_{(n\pi/a)-\frac{1}{2}}$ is the zonal harmonic of the second kind and order $\frac{n\pi}{a} - \frac{1}{2}$, that is, a toroidal function of the second kind. The corresponding expression for $P_{(n\pi/a)-\frac{1}{2}}$ is

$$\frac{1}{\pi\sqrt{2}}\int_{-\eta}^{\eta} \frac{e^{n\pi\zeta/a}\, d\zeta}{\sqrt{\cosh\eta - \cosh\zeta}}.$$

The expressions for $P_{(n\pi/a)-\frac{1}{2}}(\cos\theta)$ and $Q_{(n\pi/a)-\frac{1}{2}}(\cos\theta)$ can be easily written down and are complex. It will be observed that these forms for the toroidal functions correspond exactly with the forms for Mehler's functions $P_{n\pi/a-\frac{1}{2}}$, &c., Heine, Vol. II., § 62. From § 1, it follows that a zonal harmonic of the second kind can be expressed as

an integral of the product of two Bessel functions of order $\frac{1}{2}$ higher, the relation being

$$Q_{n-\frac{1}{2}}(\cosh\eta) = \pi\sqrt{rr'}\int_0^\infty J_n(\kappa r)\,J_n(\kappa r')\,d\kappa,$$

where $$\cosh\eta = \frac{r^2+r'^2}{2rr'},$$

and n is any positive real quantity. This result may be applied to show that the addition theorem for Bessels of order $\frac{2n+1}{2}$ is the differential form of the expansion of $(\cosh\eta - \cos\theta)^{-1}$ in zonal harmonics.

7. *Potential due to a Conductor formed by Two Intersecting Spheres.*

The potential at any point, due to a conductor, charged to potential V_0, whose bounding surface consists of two segments of spheres, standing on the same circle and intersecting at any angle α, can be obtained from the above by making q equal to $-V_0 r'$, and inverting with respect to the point r', θ', the radius of inversion being r'. Take as coordinates in the inverted system η, ξ, where e^η is the ratio (greater than unity) of the distances of any point from the points in which the plane through the point and the centre of the circle of intersection of the segments perpendicular to its plane meets it, and ξ is the internal angle between these distances. Then

$$\xi = \theta - \theta',$$

the η of the new system is the η of the former, and, if R is the distance of any point in the plane $z = 0$ of the old system from r', θ',

$$R/\sqrt{2rr'} = \sqrt{\cosh\eta - \cos\xi}.$$

Hence the potential V at any point in the new system is given by

$$V = V_0 - \frac{V_0\sqrt{\cosh\eta-\cos\xi}}{\alpha}\int_\eta^\infty \frac{d\zeta}{\sqrt{\cosh\zeta-\cosh\eta}}$$

$$\times\left\{\frac{\sinh\frac{\pi\zeta}{\alpha}}{\cosh\frac{\pi\zeta}{\alpha}-\cos\frac{\pi\xi}{\alpha}} - \frac{\sinh\frac{\pi\zeta}{\alpha}}{\cosh\frac{\pi\zeta}{\alpha}-\cos\frac{\pi}{\alpha}(\xi+2\beta)}\right\},$$

where the bounding surfaces of the conductor are

$$\xi = -\beta, \quad \xi = \alpha - \beta,$$

α being the dielectric angle. When π/α is an integer p, V is given by

$$V = V_0 - V_0 \sqrt{\cosh\eta - \cos\xi}$$

$$\times \sum_{k=0}^{k=p-1} \left\{ \frac{1}{\sqrt{\cosh\eta - \cos\left(\xi + \frac{2k\pi}{p}\right)}} - \frac{1}{\sqrt{\cosh\eta - \cos\left(\xi + 2\beta + \frac{2k\pi}{p}\right)}} \right\},$$

$$\text{or} \quad V = V_0 \sqrt{\cosh\eta - \cos\xi} \sum_{k=0}^{k=p-1} \frac{1}{\sqrt{\cosh\eta - \cos\left(\xi + 2\beta + \frac{2k\pi}{p}\right)}}$$

$$- V_0 \sqrt{\cosh\eta - \cos\xi} \sum_{k=1}^{k=p-1} \frac{1}{\sqrt{\cosh\eta - \cos\left(\xi + \frac{2k\pi}{p}\right)}},$$

which is the potential due to a series of electrified points arranged along the line of centres of the spheres as in Maxwell, Vol. I., Art. 166.

When $$\alpha = \frac{2\pi}{2m+1},$$

using § 3, V is given by

$$V = V_0 - \frac{2V_0}{\pi}$$

$$\times \sum_{k=0}^{k=2m} \left\{ \frac{\sqrt{\cosh\eta - \cos\xi}}{\sqrt{\cosh\eta - \cos\left(\xi + \frac{4k\pi}{2m+1}\right)}} \tan^{-1} \sqrt{\frac{\cosh\frac{\eta}{2} + \cos\left(\frac{\xi}{2} + \frac{2k\pi}{2m+1}\right)}{\cosh\frac{\eta}{2} - \cos\left(\frac{\xi}{2} + \frac{2k\pi}{2m+1}\right)}} \right.$$

$$\left. - \frac{\sqrt{\cosh\eta - \cos\xi}}{\sqrt{\cosh\eta - \cos\left(\xi + 2\beta + \frac{4k\pi}{2m+1}\right)}} \tan^{-1} \sqrt{\frac{\cosh\frac{\eta}{2} + \cos\left(\frac{\xi}{2} + \beta + \frac{2k\pi}{2m+1}\right)}{\cosh\frac{\eta}{2} - \cos\left(\frac{\xi}{2} + \beta + \frac{2k\pi}{2m+1}\right)}} \right\}.$$

Making $m = 0$, we have the case of the spherical bowl charged to

potential V_0, giving

$$V = V_0 - \frac{2V_0}{\pi}\left\{\tan^{-1}\sqrt{\frac{\cosh\frac{\eta}{2}+\cos\frac{\xi}{2}}{\cosh\frac{\eta}{2}-\cos\frac{\xi}{2}}} - \sqrt{\frac{\cosh\eta-\cos\xi}{\cosh\eta-\cos(\xi+2\beta)}}\tan^{-1}\sqrt{\frac{\cosh\frac{\eta}{2}+\cos\left(\frac{\xi}{2}+\beta\right)}{\cosh\frac{\eta}{2}-\cos\left(\frac{\xi}{2}+\beta\right)}}\right\},$$

that is,

$$V = \frac{2V_0}{\pi}\left\{\tan^{-1}\sqrt{\frac{\cosh\frac{\eta}{2}-\cos\frac{\xi}{2}}{\cosh\frac{\eta}{2}+\cos\frac{\xi}{2}}} + \sqrt{\frac{\cosh\eta-\cos\xi}{\cosh\eta-\cos(\xi+2\beta)}}\tan^{-1}\sqrt{\frac{\cosh\frac{\eta}{2}+\cos\left(\frac{\xi}{2}+\beta\right)}{\cosh\frac{\eta}{2}-\cos\left(\frac{\xi}{2}+\beta\right)}}\right\},$$

the equation to the bowl being

$$\xi+\beta = 0.$$

The case of the circular disc is obtained by putting $\beta=\pi$, which gives

$$V = \frac{4V_0}{\pi}\tan^{-1}\sqrt{\frac{\cosh\frac{\eta}{2}-\cos\frac{\xi}{2}}{\cosh\frac{\eta}{2}+\cos\frac{\xi}{2}}}.$$

By means of § 4, the potential when

$$\alpha = 3\pi/(3m\pm1)$$

could be expressed in terms of elliptic integrals of the first and third kinds, which would include as a particular case the conductor formed by two orthogonal spheres, the dielectric angle being three right angles, but, as it will be shown afterwards that the capacity can always be expressed in a finite form, it is unnecessary to do so. The expression for the density at any point of the conductor could be written down by differentiating V with respect to ξ, and multiplying by the necessary coefficient $\frac{2}{2\pi r'}(\cosh\eta-\cos\xi)$.

8. *The Capacity of the Conductor.*

The capacity of the conductor formed by the intersection of two spheres at any angle α can be determined from the above. It can be easily proved that the capacity of any conductor is equal to the potential at any point, of the electricity induced on the inverse conducting surface with respect to this point, due to a charge $-r'$ at the point, where r' is the radius of inversion.

It follows that the capacity of the conductor under consideration is given by

$$C = \frac{r'}{\pi\sqrt{2}}\int_0^\infty \frac{d\zeta}{\sqrt{\cosh\zeta - 1}}\left\{\frac{\frac{\pi}{\alpha}\sinh\frac{\pi\zeta}{\alpha}}{\cosh\frac{\pi\zeta}{\alpha}-\cos\frac{2\pi\beta}{\alpha}} - \frac{\frac{\pi}{\alpha}\sinh\frac{\pi\zeta}{\alpha}}{\cosh\frac{\pi\zeta}{\alpha}-1} + \frac{\sinh\zeta}{\cosh\zeta-1}\right\}.$$

To evaluate this integral, let

$$\pi/\alpha = p/q,$$

where p and q are integers; then

$$C = \frac{r'}{2\pi}\int_0^\infty \frac{d\zeta}{\sinh\frac{q\zeta}{2}}\left\{\frac{p\sinh p\zeta}{\cosh p\zeta-\cos 2p\gamma} - \frac{p\sinh p\zeta}{\cosh p\zeta-1} + \frac{q\sinh q\zeta}{\cosh q\zeta-1}\right\},$$

where $\gamma = \beta/q$. Putting $e^{-\frac{1}{2}\zeta} = x$, this becomes

$$C = \frac{2r'}{\pi}\int_0^1 \frac{x^{q-1}dx}{1-x^{2q}}\left\{\frac{p\,(1-x^{4p})}{x^{4p}-2x^{2p}\cos 2p\gamma+1} - \frac{p\,(1+x^{2p})}{1-x^{2p}} + \frac{q\,(1+x^{2q})}{1-x^{2q}}\right\}dx.$$

Now,

$$\frac{px^{q-1}(1-x^{4p})}{(1-x^{2q})(x^{4p}-2x^{2p}\cos 2p\gamma+1)}$$

$$= \sum_{k=1}^{k=q-1}(-)^k\frac{p}{q}\,\frac{\sin\frac{2kp\pi}{q}\sin\frac{k\pi}{q}}{\left(\cos 2p\gamma-\cos\frac{2kp\pi}{q}\right)\left(x^2-2x\cos\frac{k\pi}{q}+1\right)}$$

$$+\sum_{k=0}^{k=2p-1}\frac{1}{2\sin\left(q\gamma+\frac{kq\pi}{p}\right)}\,\frac{\sin\left(\gamma+\frac{k\pi}{p}\right)}{x^2-2x\cos\left(\gamma+\frac{k\pi}{p}\right)+1},$$

and
$$\frac{q\,(1+x^{2q})\,x^{q-1}}{(1-x^{2q})^2}-\frac{p\,(1+x^{2p})\,x^{q-1}}{(1-x^{2q})(1-x^{2p})}$$

$$=-\frac{\partial}{\partial x}\frac{x^q}{x^{2q}-1}-\frac{p\,(1+x^{2p})\,x^{q-1}}{(1-x^{2q})(1-x^{2p})}$$

$$=-\frac{\partial}{\partial x}\sum_{k=1}^{k=q-}\frac{(-1)^k}{q}\left\{\frac{x\cos\frac{k\pi}{q}-1}{x^2-2x\cos\frac{k\pi}{q}-1}\right\}$$

$$-\sum_{k=1}^{k=q-1}(-)^k\frac{p}{q}\cot\frac{kp\pi}{q}\,\frac{\sin\frac{k\pi}{q}}{x^2-2x\cos\frac{k\pi}{q}+1}$$

$$-\sum_{k=1}^{k=p-1}\operatorname{cosec}\frac{kq\pi}{p}\,\frac{\sin\frac{k\pi}{p}}{x^2-2x\cos\frac{k\pi}{p}+1}.$$

Hence

$$\frac{\pi C}{2r'}=\sum_{k=1}^{k=q-}(-1)^k\frac{\sin\frac{2kp\pi}{q}}{\cos 2p\gamma-\cos\frac{2kp\pi}{q}}\,\frac{p\pi}{2q}\left(1-\frac{k}{q}\right)$$

$$+\sum^{k=2p-1}\frac{\frac{\pi}{2}+\tan^{-1}\cot\left(\gamma+\frac{k\pi}{p}\right)}{4\sin q\left(\gamma+\frac{k\pi}{p}\right)}-\sum_{k=1}^{k=q-1}\frac{(-1)^k}{2q}$$

$$-\sum_{k=1}^{k=q-1}(-1)^k\cot\frac{kp\pi}{q}\,\frac{p\pi}{2q}\left(1-\frac{k}{q}\right)-\sum_{k=0}^{k=p-}\frac{\pi\left(1-\frac{k}{p}\right)}{2\sin\frac{kq\pi}{p}}.$$

There are two cases, viz., q an odd integer and q an even integer.

(1) $q=2m+1$ an odd integer. Then

$$\frac{\pi C}{2r'}=\sum_{k=1}^{k=m}(-1)^k\frac{p\pi}{2q}\left\{\frac{\sin\frac{2kp\pi}{q}}{\cos 2p\gamma-\cos\frac{2kp\pi}{q}}-\cot\frac{kp\pi}{q}\right\}$$

$$+\sum_{k=1}^{k=p-1}\frac{\pi}{4}\left\{\operatorname{cosec}q\left(\gamma+\frac{k\pi}{p}\right)-\operatorname{cosec}\frac{kq\pi}{\gamma}\right\}+\frac{\pi}{4}\operatorname{cosec}q\gamma,$$

$$\text{or } \frac{C}{r'} = \sum_{k=1}^{k=m} (-1)^k \frac{\pi}{a} \left\{ \frac{\sin \frac{2k\pi^2}{a}}{\cos \frac{2\pi\beta}{a} - \cos \frac{2k\pi^2}{a}} - \cot \frac{k\pi^2}{a} \right\}$$

$$+ \sum_{k=1}^{k=p-1} \tfrac{1}{2} \left\{ \operatorname{cosec} (\beta + ka) - \operatorname{cosec} ka \right\} + \tfrac{1}{2} \operatorname{cosec} \beta.$$

(2) $q = 2m$ an even integer. Then

$$\frac{C}{r'} = \sum_{k=1}^{k=q-1} (-1)^k \frac{\pi}{a} \left(1 - \frac{k}{q}\right) \left\{ \frac{\sin \frac{2k\pi^2}{a}}{\cos \frac{2\pi\beta}{a} - \cos \frac{2k\pi^2}{a}} - \cot \frac{k\pi}{a} \right\}$$

$$+ \sum_{k=0}^{k=p-1} \frac{1 - \frac{4}{\pi} \tan^{-1} \tan \tfrac{1}{2} \left(\gamma + \frac{k\pi}{p}\right)}{2 \sin (\beta + ka)} + \frac{1}{\pi q} - \sum_{k=1}^{k=p-1} \frac{\left(1 - \frac{k}{p}\right)}{\sin ka}.$$

In the above r' is the diameter of the circle of intersection, $\xi = -\beta$ one of the bounding surfaces, $\xi = a - \beta$ the other.

9. *Particular Cases.*

1. When $q = 1$, this is the case when the problem is solvable by the method of point images and the capacity is given by

$$C = \sum_{k=1}^{k=v-1} \frac{r'}{2} \left\{ \operatorname{cosec} (\beta + ka) - \operatorname{cosec} ka \right\} + \frac{r'}{2} \operatorname{cosec} \beta,$$

which gives for the sphere $[p = 1]$

$$C = \frac{r'}{2} \operatorname{cosec} \beta = \text{the radius.}$$

2. When $q = 2$, the capacity is given by

$$C = \sum_{k=0}^{k=p-1} r' \frac{\pi - 4 \tan^{-1} \tan \tfrac{1}{2} \left(\frac{\beta}{2} + \frac{k\pi}{p}\right)}{2\pi \sin (\beta + ka)} + \frac{r'}{2\pi} - \sum_{k=1}^{k=p-1} \frac{r' (1 - k/p)}{\sin ka},$$

which gives for the spherical bowl $(p = 1)$

$$C = \frac{r'}{2\pi} + \frac{(\pi - \beta)\, r'}{2\pi \sin \beta},$$

agreeing with Lord Kelvin's result.

3. When $q = 3$, the capacity is given by

$$C = \frac{pr'}{3}\left\{\cot\frac{p\pi}{3} - \frac{\sin\frac{2p\pi}{3}}{\cos\frac{2p\beta}{3} - \cos\frac{2p\pi}{3}}\right\}$$
$$+ \sum_{k=1}^{k=p-} \frac{r'}{2}\{\operatorname{cosec}(\beta + k\alpha) - \operatorname{cosec} k\alpha\} + \frac{r'}{2}\operatorname{cosec}\beta.$$

A particular case of this result is the conductor formed by two spheres cutting at right angles, the dielectric angle α being $3\pi/2$. In this case $p = 2$, and the capacity is given by

$$C = \frac{4r'}{3\sqrt{3}}\frac{1-\cos\frac{4\beta}{3}}{1+2\cos\frac{4\beta}{3}} + \frac{r'}{2}(1+\operatorname{cosec}\beta-\sec\beta).$$

From this the capacity of a hemisphere can be found by putting $\beta = \pi$, which gives

$$C = r'\left(1-\frac{1}{\sqrt{3}}\right) = r'\cdot 42265, \text{ nearly,}$$

where r' is the diameter, that is, the capacity is ·8453 of the radius, approximately.

10. *Solvable Cases which can be obtained from the preceding.*

The case of the conductor formed by two intersecting spheres under the influence of an electrified point can be obtained by an inversion from § 8, and therefore the distribution on such a conductor under the influence of any fixed distribution.

If a sphere is described with its centre on the line of intersection of the two planes in § 1, the functional image of an electrified point for the space enclosed by the planes and the sphere can be obtained by adding to the expression there given the functional image of an electrified point placed at the inverse with respect to the sphere of the first point; whence, by inversion, can be obtained the potential due to a conductor formed by three intersecting spheres, two of which intersect at any angle whilst the third intersects them at right angles (the latter two dielectric angles being $\pi/2$). The capacity of this latter conductor could then be expressed by the method of § 8 in finite terms, and the distribution on it due to an electrified point could be obtained by another inversion. This last includes some interesting particular cases, *e.g.*, a complete sphere with a circular

which passes through its centre) projecting from it.

Note on some Properties of a Generalized Brocard Circle. By JOHN GRIFFITHS, M.A. Received January 8th, 1895. Read January 10th, 1895.

Notation.

The notation in the following pages is identical with that employed in a former note on the generalized Brocard circles of a triangle ABC. See *Proc. Lond. Math. Soc.*, Vol. XXV., Nos. 479, 480. A brief explanation of it will be sufficient.

1. The abbreviation $GBC(UU'VV'A'OWK\alpha')$ denotes the generalized Brocard circle of the first system drawn through the nine points U, U', ... α', which are all discussed—with the exception of the last—in the note in question. [In fact, α' is a new point of the circle, and the principal theorem in this paper consists in showing that it is derived from A' in the same manner as V, V', and O are from U, U', and W.]

2. The isogonal and trilinear coordinates of a point P are connected by the relations

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = \frac{\Sigma a\alpha}{\Sigma a\beta\gamma},$$

which give $$\Sigma a\,(x - yz) = 0.$$

Hence a linear equation $\lambda x + \mu y + \nu z = \delta$

denotes a circle.

3. The equation of $GBC(UU' \ldots \alpha')$ may be written in the two forms

$$x \operatorname{cosec} A \sin B \sin C \sin(2\theta - A) \operatorname{cosec}^2\theta + y \sin B (\cot\theta + \cot B) + z \sin C (\cot\theta + \cot C) = 2 \operatorname{cosec} A \sin B \sin C \cot\omega;$$

$$x \sin B \sin C \left\{\operatorname{cosec}^2 A - (\cot\omega - \cot\Omega)^2\right\} + y \sin B (\cot\omega - \cot C) + z \sin C (\cot\omega - \cot B) = 2 \operatorname{cosec} A \sin B \sin C \cot\omega;$$

where $$\cot\Omega = \Sigma \cot A,$$

and $$\cot\omega = \cot\theta + \cot B + \cot C.$$

Section I.

Common Property of the Points U, U′, A′, and W.

Let any one of the four be denoted by P, and the mid-point of BC by M; also let the circle BCP be met again by AP (produced, if necessary) in P', and by $P'M$ in P''; then P'' is also a point on GBC (U, U', ... a').

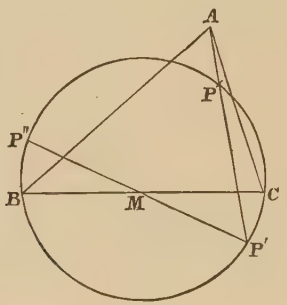


Fig. 1.

There is no difficulty in verifying this construction by the employment of isogonal coordinates. Taking (xyz), $(x'y'z')$, $(x''y''z'')$ to denote P, P', P'' respectively, we have

$$x' = x, \quad y' = -\frac{x}{z}, \quad z' = -\frac{x}{y},$$

and
$$x'' = x, \quad y'' = 2\frac{\sin C}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A \sin B} + \frac{x}{y}\frac{\sin C}{\sin B},$$

...

$$z'' = 2\frac{\sin B}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A \sin C} + \frac{x}{z}\frac{\sin B}{\sin C}.$$

(See *Proc. Lond. Math. Soc.*, June, 1894.)

Hence, if P and P'' both lie on the circumference of a G.B. circle of the first system, it follows that

$$x \sin B \sin C \left\{\operatorname{cosec}^2 A - (\cot\omega - \cot\Omega)^2\right\} + y \sin B (\cot\omega - \cot C) + z \sin C (\cot\omega - \cot B)$$
$$= 2 \operatorname{cosec} A \sin B \sin C \cot\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $x \sin B \sin C \{\operatorname{cosec}^2 A - (\cot\omega - \cot\Omega)^2\}$

$$+\left(2\frac{\sin B \sin C}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A} + \frac{x}{y}\sin C\right)(\cot\omega - \cot C)$$

$$+\left(2\frac{\sin B \sin C}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A} + \frac{x}{z}\sin B\right)(\cot\omega - \cot B)$$

$$= 2\operatorname{cosec} A \sin B \sin C \cot\omega,$$

$$\frac{2\cot\theta}{x\sin A} + \frac{\cot\omega - \cot C}{y\sin B} + \frac{\cot\omega - \cot B}{z\sin C} = \cot^2\omega + 2\cot A\cot\omega - 1$$

.....(2).

Now, since x, y, z are proportional to the trilinear coordinates of P, and $\Sigma x \sin A = \Sigma yz \sin A$, it is easy to see that (2) represents a quartic curve passing through the two circular points at infinity, and, consequently, that P is an intersection of the G.B. circle with this curve.

It thus appears that, excluding the pair of circular points at infinity, the construction gives six positions of P for each of which the companion point P'' is also on the circle. But, obviously, P'' coincides with P in two special cases, viz., when P is the intersection of the G.B. circle with the median AM. Hence there remain four positions of P to be considered, and it is found that they coincide with U, U', A', and W. The companion points are V, V', a', and O.

Section II.

Verification of the above Construction.

1. Let P coincide with U, whose coordinates are

$$x = \frac{\sin C}{\sin B};$$

$$y\sin B = (\cot\omega - \cot C)^{-1}, \quad z = \sin C(\cot\omega - \cot B).$$

U will therefore lie on the quartic represented by the equation

$$\frac{2(\cot\omega - \cot B - \cot C)}{x\sin A} + \frac{\cot\omega - \cot C}{y\sin B} + \frac{\cot\omega - \cot B}{z\sin C}$$

$$= \cot^2\omega + 2\cot A\cot\omega - 1,$$

if $2(\cot\omega - \cot B - \cot C)(\cot A + \cot C) + (\cot\omega - \cot C)^2 + \operatorname{cosec}^2 C$

$$= \cot^2\omega + 2\cot A\cot\omega - 1,$$

i.e., if $$\operatorname{cosec}^2 C = (\cot A + \cot C)(\cot B + \cot C),$$

or $$\Sigma \cot B \cot C = 1,$$

an identity.

In this case P'' is given by

$$x'' = \frac{\sin C}{\sin B} = x, \quad y'' = 2\frac{\sin C}{\sin A} - x\frac{\sin^2 B + \sin^2 C}{\sin A \sin B} + \frac{x}{y}\frac{\sin C}{\sin B}.$$

Here the fundamental relation

$$\Sigma x \sin A = \Sigma yz \sin A$$

takes the form $$\frac{x}{y} = z - \frac{\sin(B-C)}{\sin B},$$

since $$x = \frac{\sin C}{\sin B};$$

we therefore have

$$y'' = 2\frac{\sin C}{\sin A} - \frac{\sin C}{\sin B}\frac{\sin^2 B + \sin^2 C}{\sin A \sin B} + \left\{ z - \frac{\sin(B-C)}{\sin B} \right\} \frac{\sin C}{\sin B} = xz,$$

$$z'' = \frac{1}{z} + \frac{\sin(B-C)}{\sin B}.$$

In other words, P'' becomes V. See *Proceedings*, quoted *supra*.

2. Let P take the position U' given by

$$x = \frac{\sin B}{\sin C}, \quad y^{-1} \sin B = (\cot \omega - \cot C)^{-1}, \quad z^{-1} = \sin C (\cot \omega - \cot B).$$

It is then found that P will lie on the quartic if

$$\Sigma \cot B \cot C = 1,$$

as before. In this case P'' coincides with V', whose coordinates are

$$x'' = \frac{\sin B}{\sin C}, \quad y'' = \frac{1}{y} + \frac{\sin(C-B)}{\sin C}, \quad z'' = xy.$$

3. Let (x, y, z) coincide with A'. Here

$$x = \frac{\sin 2\omega}{\sin(2\omega + A)}, \quad y = x \sin C (\cot \omega - \cot C), \quad z = x \sin B (\cot \omega - \cot B).$$

This point will therefore lie on the quartic

$$\frac{2(\cot\omega-\cot B-\cot C)}{x\sin A}+\frac{\cot\omega-\cot C}{y\sin B}+\frac{\cot\omega-\cot B}{z\sin C}$$

$$=\cot^2\omega+2\cot A\cot\omega-1=\frac{\sin(2\omega+A)}{\sin A\sin^2\omega},$$

if $$\frac{2(\cot\omega-\cot B-\cot C)}{\sin A}+\frac{2}{\sin B\sin C}=\frac{2\cot\omega}{\sin A},$$

or $$\cot B+\cot C=\frac{\sin A}{\sin B\sin C}.$$

The corresponding position of P'', which I call a', is defined by

$$x''=\frac{\sin 2\omega}{\sin(2\omega+A)},$$

$$y''=\frac{2\sin B\sin C\cos 2\omega-\sin A\sin 2\omega}{\sin B\sin(2\omega+A)}+\frac{1}{\sin B(\cot\omega-\cot C)},$$

$$z''=\frac{2\sin B\sin C\cos 2\omega-\sin A\sin 2\omega}{\sin C\sin(2\omega+A)}+\frac{1}{\sin C(\cot\omega-\cot B)}.$$

4. Let P take the position W, whose coordinates are

$$x=\frac{\sin 2\theta}{\sin(2\theta-A)},\quad y=\frac{\sin\theta}{\sin(\theta+B)},\quad z=\frac{\sin\theta}{\sin(\theta+C)},$$

where $$\cot\theta=\cot\omega-\cot B-\cot C.$$

By writing these in the form

$$x\sin A=2\cot\theta\left\{\operatorname{cosec}^2A-(\cot\omega-\cot\Omega)^2\right\}^{-1},$$

$$y\sin B=(\cot\omega-\cot C)^{-1},\quad z\sin C=(\cot\omega-\cot B)^{-1},$$

it is seen that W will lie on the quartic if

$$\operatorname{cosec}^2A-(\cot\omega-\cot\Omega)^2+(\cot\omega-\cot C)^2+(\cot\omega-\cot B)^2$$

$$=\cot^2\omega+2\cot A\cot\omega-1,$$

or $$\Sigma\cot A=\cot\Omega,\quad \Sigma\operatorname{cosec}^2A=\operatorname{cosec}^2\Omega.$$

The corresponding position of P'' is defined by

$$x''=\frac{\sin 2\theta}{\sin(2\theta-A)},\quad y''=-\frac{2\sin\theta\cos(\theta+C)}{\sin(2\theta-A)},$$

and $$z'' = -\frac{2\sin\theta\cos(\theta+B)}{\sin(2\theta-A)}.$$

In other words, P'' coincides with O.

It may be here remarked that the four pairs of points UV, $U'V'$, $A'a'$, and WO form an involution on the G.B. circle, since the chords UV, &c., all meet in a point on BC. More generally, the join of the two points of intersection of a given circle with a circle through B and C passes through a given point on BC, viz., the intersection of this side with the radical axis of the given circle and the circumcircle ABC. This follows at once from the fact that the radical axes of three circles taken in pairs are concurrent.

Hence, since (1) W, O are concyclic with B and C, (2) W, U with C and A, and (3) W, U' with A and B, the intersections of WO, BC; WU, CA; and WU', AB lie on the radical axis of $GBC\,(UU'WO)$ and the circumcircle ABC. There is nodifficulty in arriving at the relations which connect the isogonal coordinates of the two points of intersection of a generalized Brocard circle with a circle drawn through two vertices of the triangle of reference. For instance, if (xyz), $(x'y'z')$ denote two points on $GBC\,(UU'WO)$ which are concyclic with B and C, we have

$$x' = x,$$

$$y' = \frac{\sin C}{\sin A}\,\frac{\cot B-\cot C}{\cot\omega-\cot C} + x\,\frac{\sin^2 C\,(\cot\omega-\cot B)-\sin^2 B\,(\cot\omega-\cot C)}{\sin A\sin B\,(\cot\omega-\cot C)}$$

$$+ z\,\frac{\sin C}{\sin B}\,\frac{\cot\omega-\cot B}{\cot\omega-\cot C},$$

$$z' = \frac{\sin B}{\sin A}\,\frac{\cot C-\cot B}{\cot\omega-\cot B} + x\,\frac{\sin^2 B\,(\cot\omega-\cot C)-\sin^2 C\,(\cot\omega-\cot B)}{\sin A\sin C\,(\cot\omega-\cot B)}$$

$$+ y\,\frac{\sin B}{\sin C}\,\frac{\cot\omega-\cot C}{\cot\omega-\cot B}.$$

These transformations give

$$y'\sin B\,(\cot\omega-\cot C)+z'\sin C\,(\cot\omega-\cot B)$$

$$= y\sin B\,(\cot\omega-\cot C)+z\sin C\,(\cot\omega-\cot B).$$

(See equation of generalized Brocard circle, *supra*.)

Section III.

Illustration of the Construction by a Diagram.

The figure shows that a G.B. circle can be constructed when its O-point is known.

M and μ are taken to be the mid-points of the side BC and the perpendicular AD, respectively; O is a given point on the line OMA_1'

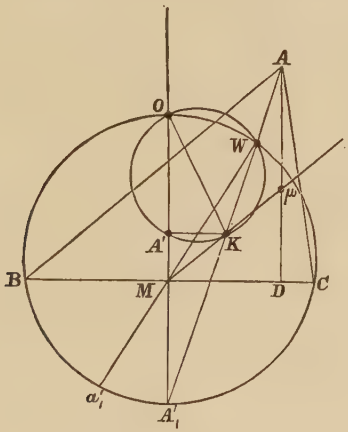


Fig. 2.

drawn parallel to AD; the circle BOC meets OMA_1' again in A_1' and AA_1' in W, AWA_1' meets $M\mu$ in K, and KA' is parallel to BC. The circle $A'KWO$, described upon OK as diameter, is a G.B. circle of the first system.

Hence we have the following properties, viz.,

(1) B, C, W, O are concyclic.

(2) The sides AB, AC subtend equal angles at W.

(3) The join of K and W passes through the vertex A of the triangle ABC.

Section IV.

The Intersection of the Chord Wa' and the Median AM lies on the Radical Axis of GBC (OWa') and the Circumcircle ABC.

This theorem is immediately suggested by the form of the expressions for the coordinates of a', viz.,

$$X = \alpha + x, \quad Y = \beta + y, \quad Z = \gamma + z,$$

where (XYZ), (xyz) denote a' and W, and

$$\alpha = \frac{\sin 2\omega}{\sin(2\omega+A)} - \frac{\sin 2\theta}{\sin(2\theta-A)},$$

$$\beta \sin B = (2\sin B \sin C \cos 2\omega - \sin A \sin 2\omega) \div \sin(2\omega+A) = \gamma \sin C.$$

(See Section II.)

Hence a', W, and the point, r, say, whose trilinear coordinates are proportional to α, β, γ are collinear. Now, since

$$\beta \sin B = \gamma \sin C,$$

r must lie on the median AM. It also lies on the radical axis in question. For, if

$$\lambda\xi + \mu\eta + \nu\zeta = \delta$$

be the equation of $GBC\,(OWa')$, we have

$$\lambda X + \mu Y + \nu Z = \delta = \lambda x + \mu y + \nu z, \quad \text{or} \quad \Sigma\lambda\,(X-x) = 0,$$

and consequently $$\Sigma\lambda\alpha = 0.$$

In other words, r is a point on the radical axis.

The theorem is thus proved.

Section V.

The four positions of P' corresponding to U, U', A', and W (see Fig. 1) are concyclic. In fact, they are on the circumference of another G.B. circle of the first system, viz., the inverse with respect to M of $GBC\,(UU'A'W)$, the constant of inversion being

$$\rho^2 = -\tfrac{1}{4}(BC)^2.$$

For $$MP' \,.\, MP'' = MB \,.\, MC = -\tfrac{1}{4}a^2.$$

(See a former note, *Proc. Lond. Math. Soc.*, June, 1894.) [In Fig. 2, the points A_1' and a_1', the projections through M of O and W on the circle $BOWC$ become the A' and a' points of the new or inverse G.B. circle.]

Section VI.

An Application of the Theory of Inversion with respect to the Point M.

Theorem.—If the coordinates of any point P on a G.B. circle are known, those of a certain other point on the circle, say Q, are also known.

Since the equation of a circle of the first system may be written as follows, viz.,

$$x \sin B \sin C \left\{ \operatorname{cosec}^2 A - (\xi - \cot \Omega)^2 \right\} + y \sin B \, (\xi - \cot C) + z \sin C \, (\xi - \cot B) = 2 \frac{\sin B \sin C}{\sin A} \xi,$$

where $$\xi = \cot \omega,$$

we may assume that the coordinates x, y, z of a point P on the circumference are functions of ξ, and certain constants dependent on the form of the triangle ABC. Hence we may take

$$x = f(\xi), \quad y = \phi(\xi), \quad \text{and} \quad z = \psi(\xi).$$

This being so, the coordinates X, Y, Z of Q are

$$X = f(\cot B + \cot C - \xi),$$

$$Y = 2 \frac{\sin C}{\sin A} - \frac{\sin^2 B + \sin^2 C}{\sin A \sin B} f(\cot B + \cot C - \xi) - \frac{\sin C}{\sin B} \psi(\cot B + \cot C - \xi),$$

$$Z = 2 \frac{\sin B}{\sin A} - \frac{\sin^2 B + \sin^2 C}{\sin A \sin C} f(\cot B + \cot C - \xi) - \frac{\sin B}{\sin C} \phi(\cot B + \cot C - \xi).$$

Ex.—Let P be U, given by

$$x = \frac{\sin C}{\sin B} = f(\xi),$$
$$y = \operatorname{cosec} B \, (\xi - \cot C)^{-1} = \phi(\xi),$$
$$z = \sin C \, (\xi - \cot B) = \psi(\xi).$$

Hence we have for Q

$$X = \frac{\sin C}{\sin B},$$

$$Y = 2 \frac{\sin C}{\sin A} - \frac{\sin^2 B + \sin^2 C}{\sin A \sin B} \frac{\sin C}{\sin B} - \frac{\sin C}{\sin B} \sin C \, (\cot C - \xi),$$

$$Z = \frac{\sin B}{\sin A} - \frac{\sin^2 B + \sin^2 C}{\sin A \sin C} \frac{\sin C}{\sin B} - \frac{\sin B}{\sin C} \operatorname{cosec} B (\cot B - \xi)^{-} ;$$

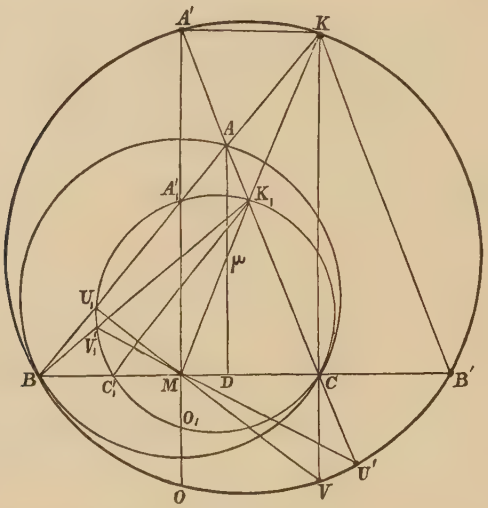
or $$X = x, \quad Y = xz, \quad Z = \frac{1}{z} + \frac{\sin (B - C)}{\sin B}.$$

In other words, Q is V.

[The coordinates of A', a' can, in like manner, be derived from those of O and W respectively.]

Section VII.

Construction of the G.B. Circles of the First System corresponding to $\omega = B$ and $\omega = C$, in illustration of the Method of Inversion.



[$MA'.MO_1 = MB.MC = MU_1.MV = MU'.MV'_1$, &c. KB', $K_1C'_1$ are parallel to CA, AB, respectively.]

For the circle corresponding to $\omega = C$, we have the following properties, viz.,

(1) The points U, W, V' and a' coincide with B.

(2) A' lies on the side CA, and K on BA.

(3) The circle touches the circumcircle ABC at B.

(4) The triangle $A'B'B$ is inversely similar to ABC, and belongs to the system of similar in-triangles whose centre of similitude is the point on the Brocard circle of ABC given by the coordinates

$$x = \frac{c}{b}, \quad y = \frac{c}{a}, \quad z = 2\cos C.$$

Hence the circle $A'B'B$ has double contact with the conic which touches the sides of ABC and has the centre of similitude in question for a focus.

The circle corresponding to $\omega = B$ may be constructed in like manner. The pair are inverse to each other with respect to M.

It may also be remarked that there is another pair of G.B. circles of the first system which touch the circle ABC, and are inverse to one another with regard to the mid-point of BC.

In general four circles in each of the three systems can be drawn to touch a given circle.

Thursday, February 14th, 1895.

Mr. A. B. KEMPE, F.R.S., Vice-President, in the Chair.

The Chairman announced the decease, since the January meeting, of Professor Cayley and Sir James Cockle, and stated that the Society had been represented at the funeral of the former by the President, himself, and Professors Elliott and Henrici.

Messrs. Walker, Glaisher, and Elliott paid tributes to the memory of the deceased members. A resolution was passed unanimously that the President should be requested to convey, in such form as he should think fit, votes of condolence from the Society to Mrs. Cayley and Lady Cockle.

The following papers were read:—

On certain Differential Operators, and their use to form a Complete System of Seminvariants of any Degree, or any Weight: Prof. Elliott.

On the Electrification of a Circular Disc in any Field of Force Symmetrical with respect to its Plane: Mr. H. M. Macdonald.

Notes on the Theory of Groups of Finite Order, iii. and iv.: Prof. W. Burnside.

The following presents were received:—

"Proceedings of the Royal Society," Vol. LVII., Nos. 340–341.

"Vierteljahrsschrift der Naturforschenden Gesellschaft zu Zürich," Jahrgang 39, Heft 3–4; Zurich, 1894.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XVIII., St. 12; Bd. XIX., St. 1; Leipzig, 1894.

"Proceedings of the Royal Irish Academy," Series 3, Vol. III., No. 3; December, 1894.

"Revue Semestrielle des Publications Mathématiques," Tome III., Partie 1ère; Amsterdam, 1895.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. IX., No. 1 ; 1894–5.

"Proceedings of the Physical Society of London," Vol. XIII., Pts. 2, 3 ; Jan.–Feb., 1895.

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"Nieuw Archief voor Wiskunde," Reeks 2, Deel I. 1 ; Amsterdam, 1894.

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Van den Berg, F. J.—"Over Co-ördinaten Stelsels voor Cirkels in het Platte vlak en door Bollen in der Ruimte," pamphlet, 8vo ; Amsterdam, 1894.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. I., No. 4.

"Nyt Tidsskrift for Matematik," A. Femte Aargang, Nos. 6, 7–8 ; B. Femte Aargang, Nos. 3, 4 ; Copenhagen.

"Bulletin des Sciences Mathématiques," Tome XIX., Jan.–Fev., 1895 ; Paris.

Bierens de Haan, D.—"Bouwstoffen voor de Geschiedenis der Wis- en Natuurkundige Wetenschappen in de Nederlanden," Deel 2, No. 1, 1^{eno} Sectie ; Amsterdam, 1893.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," 1894, No. 4.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 11–12, Sem. 2 ; Vol. IV., Fasc. 1, Sem. 1 ; Roma, 1894–95.

"Educational Times," February, 1895.

"Journal of the College of Science, Imperial University, Japan," Vol. VII., Pts. 2, 3.

McClintock, E.—"Theorems in the Calculus of Enlargement," 4to pamphlet.

"Annali di Matematica," Tomo XXIII., Fasc. 1 ; Milano, 1895.

"Journal für die reine und angewandte Mathematik," Bd. CXIV., Heft 3 ; Berlin, 1895.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Vol. VIII., Fasc. 11–12 ; 1894.

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"American Journal of Mathematics," Vol. XVII., No. 1 ; Baltimore, 1895.

On certain Differential Operators, and their use to form a Complete System of Seminvariants of any Degree, or any Weight.

By Prof. Elliott, F.R.S. Received and read February 14th, 1895.

1. There is a one-to-one correspondence between seminvariants in the unending series of letters $a, b, c, d, e, \ldots$, and products of these letters which have been called by MacMahon power enders, *i.e.*, products which when arranged from the beginning in alphabetical order end in a higher power of a letter than the first. Thus abc^4, c^3, ad^2, b^2ef^3, ... are power ending products.

This fact has been exhibited in various lights by Major MacMahon and the late Professor Cayley—alas! that we must now say the late —in a number of papers in the *American Journal of Mathematics*. The main object of this short paper is to exhibit the fact in another light by showing that a complete system of seminvariants may be deduced from a complete system of power ending products, one from one, by differential operations only. Two ways of doing this will be arrived at.

2. We regard a, nb, $\frac{n(n-1)}{1.2}c$, ... as the coefficients in a binary quantic of infinite order n, or, as will do equally well, of a binary quantic of order n which exceeds w, the weight of gradients of the type with which we are at any time dealing. It is well known that, if any seminvariant, obtained for such a quantic, involves the first $j+1$ letters only, then it is equally a seminvariant of the j-ic whose coefficients are $a, jb, \frac{j(j-1)}{1.2}c, \ldots$.

Let
$$\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \ldots \text{ to } \infty,$$
$$O_1 = b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots \text{ to } \infty,$$
$$O_2 = c\partial_b + 2d\partial_c + 3e\partial_d + \ldots \text{ to } \infty,$$

where, for instance, ∂_b means $\frac{d}{db}$. We have, if the operators stop at weight n,
$$nO_1 - O_2 \equiv O,$$

where O is the second annihilator of invariants of the binary n-ic.

We at once see that

$$\Omega O_1 - O_1\Omega = a\partial_a + b\partial_b + c\partial_c + \dots \text{ to } \infty$$
$$= i,$$

when the operation is on a homogeneous function of degree i,

$$\Omega O_1^2 - O_1^2\Omega = (\Omega O_1 - O_1\Omega)\, O_1 + O_1\,(\Omega O_1 - O_1\Omega)$$
$$= iO_1 + O_1 i = 2iO_1,$$
$$\Omega O_1^3 - O_1^3\Omega = (\Omega O_1 - O_1\Omega)\, O_1^2 + O_1\,(\Omega O_1 - O_1\Omega)\, O_1 + O_1^2\,(\Omega O_1 - O_1\Omega)$$
$$= 3iO_1^2,$$

and generally $\Omega O_1^k - O_1^k\Omega = kiO_1^{k-1}$.

Thus, if u be any homogeneous function of order i,

$$(\Omega O_1 - O_1\Omega)\, u = iu,$$
$$(\Omega O_1^2 - O_1^2\Omega)\, \Omega u = 2iO_1\Omega u,$$
$$(\Omega O_1^3 - O_1^3\Omega)\, \Omega^2 u = 3iO_1^2\Omega^2 u,$$
$$(\Omega O_1^4 - O_1^4\Omega)\, \Omega^3 u = 4iO_1^3\Omega^3 u,$$
$$\&c. \qquad \&c.,$$

and if u be rational and integral and of weight w, *i.e.*, be a gradient of type w, i, these equalities from the $(w+2)^{\text{th}}$ onwards are mere identities of zeroes, for $\Omega^{w+1}u = 0$.

Hence, by addition of multiples of the above, so chosen that terms on the right except the first cancel against second terms on the left,

$$\left\{ \frac{1}{i}\,\Omega O_1 - \frac{1}{1.2.i^2}\,\Omega O_1^2\Omega + \frac{1}{1.2.3.i^3}\,\Omega O_1^3\Omega^2 - \dots \right\} u = u,$$

in which the series on the left certainly terminates in consequence of

$$\Omega^{\,+1}u = 0,$$

so that no doubt arising from questions of convergency can enter.

The result may be written

$$\left\{ 1 - \frac{1}{i}\,\frac{\Omega O_1}{1} + \frac{1}{i^2}\,\frac{\Omega O_1^2\Omega}{1.2} - \frac{1}{i^3}\,\frac{\Omega O_1^3\Omega^2}{1.2.3} + \dots \right\} u = 0,$$

or, again, $$u = \Omega O_1 \left\{ \frac{1}{i} - \frac{1}{i^2}\,\frac{O_1\Omega}{2} + \frac{1}{i^3}\,\frac{O_1^2\Omega^2}{2.3} - \dots \right\} u,$$

in which form it shows that any gradient u is of the form Ωv. Moreover, a possible v corresponding to any u is of the form $O_1 W$.

This result might have been obtained by putting $n = \infty$ in a paper of my own (*Proceedings*, Vol. XXIII., pp. 298, &c.) on the exactness of Cayley's number of seminvariants of a given type. It was there proved that, when the number $n+1$ of $a, b, c, d, \ldots$ is finite, any gradient in them can be derived by operation with Ω on another gradient in them if $in-2w$ is positive, as is, of course, the case when n is infinite. We now see that in all cases a gradient can be derived by operation with Ω from another gradient, if the latter gradient be allowed to involve a longer series of the letters $a, b, c, d, e, \ldots$ than the former does.

3. Again, returning to the definitions in § 2,

$$\begin{aligned}
\Omega O_2 - O_2\Omega &= 2\,(b\partial_b + 2c\partial_c + 3d\partial_d + \ldots \text{ to } \infty) \\
&= 2w, \\
\Omega O_2^2 - O_2^2\Omega &= (\Omega O_2 - O_2\Omega)\,O_2 + O_2\,(\Omega O_2 - O_2\Omega) \\
&= 2\,(w+1)\,O_2 + O_2 \,.\, 2w, \\
&\qquad \text{since, if } w \text{ is the weight of } u,\ w+1 \text{ is that of } O_2 u, \\
&= 2\,(2w+1)\,O_2, \\
\Omega O_2^3 - O_2^3\Omega &= (\Omega O_2 - O_2\Omega)\,O_2^2 + O_2\,(\Omega O_2 - O_2\Omega)\,O_2 + O_2^2\,(\Omega O_2 - O_2\Omega) \\
&= 2\,(w+2)\,O_2^2 + O_2 \,.\, 2\,(w+1)\,O_2 + O_2^2 \,.\, 2w \\
&= 3\,(2w+2)\,O_2^2,
\end{aligned}$$

and generally

$$\begin{aligned}
\Omega O_2^k - O_2^k\Omega &= \{2\,(w+k-1) + 2\,(w+k-2) + \ldots + 2w\}\,O_2^{k-1} \\
&= k\,(2w+k-1)\,O_2^{k-1}.
\end{aligned}$$

Thus, u being, as before, a gradient of weight w,

$$\begin{aligned}
(\Omega O_2 - O_2\Omega)\,u &= 2wu, \\
(\Omega O_2^2 - O_2^2\Omega)\,\Omega u &= 2\,(2\overline{w-1}+1)\,O_2\Omega u, \\
&\qquad \text{for } \Omega u \text{ is of weight } w-1, \\
&= 2\,(2w-1)\,O_2\Omega u, \\
(\Omega O_2^3 - O_2^3\Omega)\,\Omega^2 u &= 3\,(2\overline{w-2}+2)\,O_2^2\Omega^2 u \\
&= 3\,(2w-2)\,O_2^2\Omega^2 u, \\
&\&c. \qquad \&c.,
\end{aligned}$$

and, generally,

$$(\Omega O_2^k - O_2^k \Omega)\,\Omega^{k-1} u = k\,(2w-k+1+k-1)\;O_2^{k-1}\Omega^{k-1}u$$

$$= k\,(2w-k+1)\;O_2^{k-1}\Omega^{k-1}u.$$

Here for the value $w+2$ of k, and for higher values, both sides vanish. Moreover, the value $w+1$ of k makes the factor $2w-k+1$ equal to w, which does not vanish when w is positive, as it must be except for gradients a^i involving a only. Also all lower values of k make this positive, and so non-vanishing. Thus, eliminating right-hand sides except the first,

$$\left\{1 - \frac{\Omega O_2}{1.2w} + \frac{\Omega O_2^2 \Omega}{1.2.2w\,(2w-1)} - \frac{\Omega O_2^3 \Omega^2}{1.2.3.2w\,(2w-1)(2w-2)} + \ldots \right.$$

$$\left. + (-1)^{w+1} \frac{\Omega O_2^{w+1} \Omega^w}{1.2.3 \ldots (w+1)\,.\,2w\,(2w-1)\,\ldots\,w} \right\} u = 0,$$

which holds for all gradients u except those of weight zero.

Hence

$$u = \Omega O_2 \left\{ \frac{1}{1.2w} - \frac{O_2 \Omega}{1.2.2w\,(2w-1)} + \frac{O_2^2 \Omega^2}{1.2.3.2w\,(2w-1)(2w-2)} - \ldots \right\} u,$$

in which the series certainly terminates in $w+1$ terms at most, no difficulty arising in consequence of vanishing of denominators or questions of convergency.

We have thus again the same result as before, excluding the case of weight zero, for which no information is this time given, that a gradient u is necessarily of the form Ωv. We have also the new information that a possible v for any u is of the form $O_2 W$. This last is not, of course, the case when u is of zero weight, *i.e.*, is a^i, so that

$$v = \frac{1}{i}\, a^{i-1} b,$$

there being no seminvariant of unit weight to add; for every term in an $O_2 W$ must contain either c or d or &c.

4. Now, take any gradient whatever of type w, i, and call it u. Ωu is then a gradient of type $w-1, i$, and may, as follows from either of the last two articles, be any gradient whatever of that type. Put Ωu

for u in the two results proved. They become

$$(1)\ \Omega\left\{1-\frac{1}{i}\frac{O_1\Omega}{1}+\frac{1}{i^2}\frac{O_1^2\Omega^2}{1.2}-\frac{1}{i^3}\frac{O_1^3\Omega^3}{1.2.3}+\ldots\right\}u=0,$$

$$(2)\ \Omega\left\{1-\frac{1}{2w-2}\frac{O_2\Omega}{1}+\frac{1}{(2w-2)(2w-3)}\frac{O_2^2\Omega^2}{1.2}\right.$$
$$-\frac{1}{(2w-2)(2w-3)(2w-4)}\frac{O_2^3\Omega^3}{1.2.3}+\ldots$$
$$\left.\ldots+(-1)^w\frac{1}{(2w-2)(2w-3)\ldots(w-1)}\frac{O_2^w\Omega^w}{1.2.3\ldots w}\right\}u=0,$$

the first being general, and the second applying when w is 2 or a greater number. Write these

$$(1)\quad \Omega\left\{1-\vartheta_1\right\}u=0,$$
$$(2)\quad \Omega\left\{1-\vartheta_2\right\}u=0,$$

where ϑ_1, ϑ_2 are differential operators whose full expressions are given. We have thus two conclusions:—

(1) u being any gradient whatever, $(1-\vartheta_1)\,u$ either vanishes or is a seminvariant.

(2) u being any gradient of weight 2 or more, $(1-\vartheta_2)\,u$ either vanishes or is a seminvariant.

Also, regarding the forms of ϑ_1, ϑ_2, which are

$$O_1\left\{\ldots\right\}u,$$
$$O_2\left\{\ldots\right\}u,$$

we draw the following conclusions:—

(1) If $(1-\vartheta_1)\,u$ is not a seminvariant but vanishes, u is of the form $O_1 v$.

(2) If $(1-\vartheta_2)\,u$ is not a seminvariant but vanishes, u is of the form $O_2 v$.

5. Now, we know that the number of asyzygetic seminvariants of type w,i is

$$(w\,;\ i,\infty)-(w-1\,;\ i,\infty),$$

where $(w\,;\ i,\infty)$ is the number of products of $a, b, c, d, e, \ldots$ of weight w and degree i. Now, this difference is also the number of power ending products of type w,i of $a, b, c, d, e, \ldots$. For every non-

power ender of type w,i can be obtained from a product of type $w-1, i$ by replacing its last letter, once only, by the next most advanced letter, and no two of type $w-1, i$ thus altered give the same one of type w,i, but every one of the first type gives one of the second.

What will be proved is that there is a one-to-one correspondence between the seminvariants and the power enders; in fact, that every power ender when operated on by either $1-\vartheta_1$ or $1-\vartheta_2$ gives a seminvariant, and that no two give the same; or, in other words, that the most general seminvariant of type w,i is obtained by operating either with $1-\vartheta_1$ or with $1-\vartheta_2$ on the most general linear function of the $(w; i, \infty)-(w-1; i, \infty)$ power enders.

This will follow if we can show that no power ending product or linear function of power ending products can be of the form $O_1 v$, or equally if we can show that none is of the form $O_2 v$. If none is of the form $O_1 v$, then, by (1) of the preceding article, every one when operated on by $1-\vartheta_1$ gives a seminvariant and not a zero. Moreover, all the seminvariants thus given are linearly independent if no linear function of the power ending products is of the form $O_1 v$; for, if $P, P', P'', \ldots$ be the power ending products, and $S, S', S'', \ldots$ the derived seminvariants, so that

$$(1-\vartheta_1) P = S, \quad (1-\vartheta_1) P' = S', \quad (1-\vartheta_1) P'' = S'', \ \ldots,$$

and if
$$\lambda S + \lambda' S' + \lambda'' S'' + \ldots = 0,$$

then
$$(1-\vartheta_1)(\lambda P + \lambda' P' + \lambda'' P'' + \ldots) = 0,$$

and this would require that $\lambda P + \lambda' P' + \lambda'' P'' + \ldots$ be of the form $O_1 v$.

And quite similarly with regard to ϑ_2 and O_2.

6. We have then only to prove, either that no linear function of power ending products is of the form $O_1 v$, or that none is of the form $O_2 v$. Either fact will suffice; but both are proved with equal ease as follows.

Take v any gradient whatever. Call the most advanced letter which occurs in it a_r, so that

$$v = A + a_r B + a_r^2 C + a_r^3 D + \ldots,$$

where $A, B, C, D, \ldots$ are free from a_r, and $B, C, D, \ldots$ do not all vanish. Then

$$\begin{aligned} O_1 v &= \left(\ldots + a_{r+1} \frac{d}{da_r}\right) v \\ &= a_{r+1} (B + 2a_r C + 3a_r^2 D + \ldots) + \text{terms free from } a_{r+1}. \end{aligned}$$

Thus, B, C, D, ... not all vanishing, and being free from a_r and a_{r+1}, $O_1 v$ must contain terms which end in the single power a_{r+1}, and so are not power enders. Thus

$$\lambda P + \lambda' P' + \lambda'' P'' + \ldots$$

cannot be of the form $O_1 v$.

Neither in like manner can it be of the form $O_2 v$.

It is then completely established, and in two ways, that all the linearly independent seminvariants of type $w.i$ are obtained from all the different power ending products of that type, one from each, by differential operation. Operation either with $1-\vartheta_1$ or $1-\vartheta_2$ suffices, the former applying to all cases whatever, the latter to all cases where the weight is 2 or more.

In fact, the latter gives all seminvariants just as much as the former. For there are no power ending products, nor seminvariants, of unit weight, and there is no failure for the zero weight of a^i.

Notes on the Theory of Groups of Finite Order. *By* W. Burnside.

Read February 14th, 1895. Received February 18th, 1895.

[*June* 21*st*, 1895.

Shortly after the following notes had been communicated to the Society, I became acquainted with a paper by Herr G. Frobenius, "Ueber auflösbare Gruppen," *Berliner Sitzungsberichte*, May, 1893. In this paper Herr Frobenius has completely anticipated the result of my Note III. I have, however, with the consent of the referees, allowed the note to stand in its original form; as, had I replaced it by a reference to Herr Frobenius' work, the remainder of my paper would not have been self-contained, and in places would have been barely intelligible. In the paper mentioned Herr Frobenius makes two other statements that bear on the present communication. He says that on some other occasion he will prove that simple groups whose orders are of the forms

$$p_1^m p_2 \quad \text{and} \quad p_1 p_2 \ldots p_{n-2} p_{n-1}^2 p_n^m,$$

where $p_1 < p_2 \ldots < p_n$, and the symbols all represent primes, do not exist. I cannot find that he has yet done so.

Secondly, he states his *belief* that the only non-soluble groups (*i.e.*, groups whose factors of composition do not consist entirely of primes), whose orders are the products of five primes, are two groups

of order 120, and three simple groups of orders $2^3 . 3 . 7$, $2^2 . 3 . 5 . 11$, $2^2 . 3 . 7 . 13$. The first part of this statement is obviously incorrect, for, if p is any prime, the group defined by

$$A^2 = 1, \quad B^3 = 1, \quad C^5 = 1, \quad D^p = 1,$$
$$ABC = 1, \quad AD = DA, \quad BD = DB, \quad CD = DC,$$

has $2^2 . 3 . 5 . p$ for its order, and 60 and p for its factors of composition.

I take this opportunity of expressing my regret that in my former "Notes on the Theory of Groups of Finite Order" (Vol. xxv., pp. 9–18) I was led, by my ignorance of Herr Frobenius' investigations on the subject, to giving as new a proof of Sylow's theorem which was in fact six years old. His two papers "Neuer Beweis des Sylowschen Satzes," *Crelle*, Vol. c., pp. 179–181, and "Ueber die Congruenz nach einem aus zwei endlichen Gruppen gebildeten Doppelmodul," *Crelle*, Vol. ci., pp. 273–299, contain a proof of the two parts of the theorem with which mine is, in essence, identical.]

The following notes are numbered consecutively to those just referred to. They are concerned chiefly with the proof of certain tests that may be applied in particular cases to determine whether it is possible for a simple group of given finite order to exist. Incidentally some other results of interest in connexion with the composition of finite groups are obtained.

If $p_1, p_2, p_3, \ldots$ are distinct primes in ascending order, the main results proved are the following:—

(i) There are no simple groups whose orders are of the forms

$$p_1 p_2 \ldots p_{n-2} p_{n-1} p_n^m, \quad p_1 p_2 \cdots p_{n-2} p_{n-1}^2 p_n, \quad p_1^m p_2, \quad p_1 p_2^m p_3.$$

(ii) Groups whose orders are of the forms

$$p_1^2 p_2 \ldots p_{n-1} p_n^m, \quad p_1^2 p_2 \ldots p_{n-2} p_{n-1}^2 p_n$$

cannot be simple unless they contain a sub-group of tetrahedral type, in which case $p_1 = 2$ and $p_2 = 3$.

(iii) A group whose order is

$$p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n},$$

in which the sub-groups of orders

$$p_1^{m_1}, \quad p_2^{m_2}, \quad \ldots \quad p_{n-1}^{m_{n-1}}$$

are all cyclical, cannot be simple.

(iv) The only simple groups whose orders consist of the product of five primes are three known groups of orders $2^3 . 3 . 7$, $2^2 . 3 . 5 . 11$, and $2^2 . 3 . 7 . 13$.

III. *On Groups in whose Order there is no repeated Prime Factor.*

A group of order $p_1 p_2 (p_2 > p_1)$, where p_1 and p_2 are primes, necessarily contains a single self-conjugate sub-group of order p_2. If there is also a self-conjugate sub-group of order p_1, the group is cyclical and contains $(p_1-1)(p_2-1)$ operations of order $p_1 p_2$; while otherwise the group must contain p_2 conjugate sub-groups of order p_1. These results, which depend on very simple considerations, are proved in Herr Netto's *Substitutionentheorie*, pp. 134, 135. They may be stated in the following form.

A group whose order is $p_1 p_2$ contains $(p_1-1)\,p_2$ operations whose orders are equal to, or multiples of, p_1, and p_2-1 operations whose orders are equal to p_2.

Consider now a group G whose order is

$$N = p_1 p_2 \dots p_n,$$

where the factors are different primes in ascending order of magnitude. I propose to show that the number of operations in G whose orders are p_r, or the product of p_r by primes greater than itself, is

$$m_r = (p_r-1)\,p_{r+1} \dots p_n.$$

Suppose that this theorem has been proved for groups whose order consists of the product of $n-1$ different prime factors; and consider, first, the case in which G contains no self-conjugate sub-group of prime order.

Any cyclical sub-group of order p_r $(r < n)$ must then be contained self-conjugately in a certain sub-group H of order

$$N_H = p_a p_b \dots p_d p_r p_\alpha p_\beta \dots p_\delta, \quad (p_a < p_b < \dots < p_r < p_\alpha < \dots),$$

and in no sub-group of higher order; so that the number of conjugate sub-groups of order p_r is $\frac{N}{N_H}$.

Since its order cannot consist of the product of more than $n-1$ different prime factors, H must contain

$$(p_r-1)\,p_\alpha p_\beta \dots p_\delta$$

different operations whose orders are p_r or products of p_r by greater primes. Hence the set of $\frac{N}{N_H}$ groups conjugate with H contain

$$\frac{N}{N_H}(p_r-1)\,p_\alpha p_\beta \dots p_\delta = \frac{N}{p_a p_b \dots p_d}\left(1-\frac{1}{p_r}\right)$$

such different operations. For, if two of the conjugate set of groups contained a common operation of order $p_r p_\alpha$, they would also contain a common operation of order p_r, and this would be contrary to the supposition that H is the most extensive sub-group that contains a

cyclical sub-group of order p_r self-conjugately. Moreover, G can contain no operations, whose orders are p_r or multiples of p_r by a greater prime, other than those enumerated, since every such operation must occur in H or in a sub-group conjugate to H.

Hence $$m_r = \frac{N}{p_a p_b \dots p_d}\left(1-\frac{1}{p_r}\right).$$

Now, since N is the number of operations in G,

$$N = 1 + \sum_{r=1}^{n} \frac{N}{p_a p_b \dots p_d}\left(1-\frac{1}{p_r}\right)$$

or $$1-\frac{1}{p_1 p_2 \dots p_n} = \sum_{r=1}^{n} \frac{1}{p_a p_b \dots p_d}\left(1-\frac{1}{p_r}\right).$$

Since $p_a, p_b, \dots p_d$ are all less than p_r, the first term on the right-hand side is necessarily $1-\frac{1}{p_1}$. Suppose, now, that the first term on the right-hand side for which $p_a p_b \dots p_d$ is not the same as $p_1 p_2 \dots p_{r-1}$ is the r^{th} term. Transposing the first $r-1$ terms, the equation becomes

$$\frac{1}{p_1 p_2 \dots p_{r-1}}\left(1-\frac{1}{p_r p_{r+1} \dots p_n}\right) = \sum_{r}^{n} \frac{1}{p_a p_b \dots p_d}\left(1-\frac{1}{p_r}\right),$$

or $$1-\frac{1}{p_r p_{r+1} \dots p_n} = p_1 p_2 \dots p_{r-1} \sum_{r}^{n} \frac{1}{p_a p_b \dots p_d}\left(1-\frac{1}{p_r}\right).$$

But all the terms on the right-hand side are positive, and, on the supposition made, the least possible value of the first term is

$$p_1\left(1-\frac{1}{p_r}\right),$$

which is necessarily greater than the left-hand side.

Hence the supposition is impossible, and therefore

$$p_a p_b \dots p_d = p_1 p_2 \dots p_{r-1}$$

for all values of r up to n.

It follows that in the case under consideration m_r has the value given.

Consider, next, the case in which G contains self-conjugate sub-groups of prime order, less than p_n, and let p_r be the smallest prime for which such a sub-group occurs. The expression

$$m_s = (p_s-1)\, p_{s+1} \dots p_n$$

will then, again, by the above reasoning, give the number of operations whose orders are p_s or multiples of p_s by higher primes for all values of p_s from p_1 to p_{r-1}; but the values of m_r must be independently investigated.

For this purpose, let P be one of the operations of the sub-group

of order p_r, and let all the operations of the group be arranged in the following familiar scheme, viz.:

$$\begin{array}{llllll} 1, & P, & P^2, & \dots & P^{p_r-1}, \\ S_1, & S_1P, & S_1P^2, & \dots & S_1P^{p_r-1}, \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S_i, & S_iP, & S_iP^2, & \dots & S_iP^{p_r-1}, \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S_{\nu-1}, & S_{\nu-1}P, & S_{\nu-1}P^2, & \dots & S_{\nu-1}P^{p_r-1}, \end{array}$$

where $$\nu = p_1 p_2 \dots p_{r-1} p_{r+1} \dots p_n.$$

Since the first line in this scheme is a self-conjugate sub-group of G, it follows that the sets of operations in the different lines combine together, line by line, in the same way as the operations of a group of order ν. Hence there must be some factor μ of ν such that each of the operations of the line

$$S_i,\ S_iP,\ S_iP^2,\ \dots\ S_iP^{p_r-1},$$

when raised to the power μ, gives an operation of the first line. Suppose, then, that

$$(S_iP^j)^\mu = P^k,$$
$$(S_iP^j)^{\mu p_r} = 1,$$

and, since p_r is relatively prime to μ, S_iP^j must be of order $\mu' p_r$, where μ' is equal to or a factor of μ. There must therefore be an operation S_i' of the group, of order μ', such that

$$S_iP^j = S_i'P^{j'}.$$

Hence one and only one of the set of operations

$$S_i,\ S_iP,\ \dots\ S_iP^{p_r-1}$$

is of an order prime to p_r.

Now, it has been seen that G contains

$$\sum_1^{r-1} (p_s-1)\, p_{s+1} \dots p_n = p_1 p_2 \dots p_n - p_r p_{r+1} \dots p_n$$

operations whose orders contain as a factor at least one prime less than p_r, and the reasoning just given shows that of the remainder $\frac{p_r-1}{p_r}$ have p_r as a factor of their order.

Hence
$$\begin{aligned} m_r &= \frac{p_r-1}{p_r}(N - p_1 p_2 \dots p_n + p_r p_{r+1} \dots p_n) \\ &= (p_r-1)\, p_{r+1} \dots p_n. \end{aligned}$$

In exactly the same way it may be shown that, if G contains a self-conjugate sub-group of order p_s, where $p_r < p_s < p_n$, then

$$m_s = (p_s-1)\, p_{s+1} \dots p_n.$$

Hence, finally, whether the sub-group of order p_r is self-conjugate or not, the expression $(p_r-1)\,p_{r+1}\ldots p_n$ gives the number of operations of the group whose orders are divisible by p_r, and by no smaller prime, on the supposition that a similar expression holds when the order is the product of $n-1$ factors. But it has been shown to be true for a group whose order consists of the product of two prime factors, and it is therefore true universally. Since

$$m_n = p_n - 1,$$

the operations of order p_n form a single sub-group, which is therefore necessarily self-conjugate in G.

Hence a group whose order contains no repeated prime factors cannot be simple; and from this it follows at once, by a consideration of the composition-series (*Reihe der Zusammensetzung*) of the group, that it must contain a self-conjugate sub-group whose order is the product of $n-1$ prime factors, and therefore that an algebraical equation the order of whose group contains no repeated prime factors is necessarily solvable by radicals.

An interesting special case of these groups is that in which all the operations of the group are of prime order. When this is so the order of the greatest sub-group H that contains a cyclical group of order p_r self-conjugately can have no factor p_s greater than p_r, since there would in that case be operations of order $p_r p_s$ in G. On the other hand, since the number of operations of order p_r in G is

$$(p_r-1)\,p_{r+1}\ldots p_n,$$

the order of H must contain every prime factor of N which is not greater than p_r, or

$$N_H = p_1 p_2 \ldots p_r.$$

In order that this may be possible, the congruence

$$p_r \equiv 1 \pmod{p_{r-s}}$$

must be satisfied for all values of r from 2 to n, and for all values of s from 1 to $r-1$, since otherwise an operation of order p_{r-s} could not transform an operation of p_r into a power of itself; and, conversely, when these conditions are satisfied a group of order N whose operations are all of prime order can always be constructed.

The composition of such a group may be determined in the following way.

Let P_n, P_{n-1} be *any* two operations of G of order p_n, p_{n-1}. Then, since the cyclical sub-group of order p_n is self-conjugate, P_{n-1} necessarily transforms P_n into a power of itself, and therefore P_n and P_{n-1} generate a sub-group of order $p_{n-1}p_n$. This sub-group contains

$(p_{n-1}-1)\,p_n$ operations of order p_n, that is, all the operations of this order contained in G, and is therefore self-conjugate in G. If, now, P_{n-2} is *any* operation of G of order p_{n-2}, it may be similarly shown that P_n, P_{n-1}, P_{n-2} generate a sub-group of order $p_{n-2}p_{n-1}p_n$ which contains all the operations of G of orders p_{n-2}, p_{n-1} and p_n, and no others, which, again, is therefore self-conjugate in G; and this process may be continued till the maximum self-conjugate sub-group of order $p_2p_3 \dots p_n$ is reached. Also, since the number of operations of G of order p_1, viz.,

$$(p_1-1)\,p_2p_3 \dots p_n,$$

is greater than any factor of N, there can be no self-conjugate sub-group containing p_1 as a factor of its order. Hence in this case the composition series of G must consist of the factors

$$p_1,\ p_2,\ \cdots\ p_n,$$

in the order written. In this case also the composition-series is clearly identical with the "chief" composition series (*Hauptreihe der Zusammensetzung*).

A simple example of such a group is that generated by the three operations S_2, S_3, S_7 which satisfy

$$S_2^2 = 1, \qquad S_3^3 = 1, \qquad S_7^7 = 1,$$
$$S_2S_3S_2 = S_3^2, \quad S_2S_7S_2 = S_7^6, \quad S_3^2S_7S_3 = S_7^2.$$

Returning now to the general case, the preceding reasoning regarding the composition of the group may be repeated almost word for word. Thus P_n and P_{n-1}, any two operations of order p_n and p_{n-1}, again generate a sub-group of order $p_{n-1}p_n$ which contains *all* the operations of G of orders p_{n-1}, $p_{n-1}p_n$, and p_n, and which is therefore self-conjugate. So, again, P_n, P_{n-1}, and P_{n-2} generate a self-conjugate sub-group of order $p_{n-2}p_{n-1}p_n$; and the process may be continued, at each step taking in the next smaller factor of N, till the group G itself is arrived at. Hence in every case the composition-series *may* consist of the prime factor of the order of the group taken in ascending order of magnitude, while, when they are taken in ascending order, the corresponding groups of the series are all self-conjugate in the main group.

When the factors of N are assigned primes, the number of possible different groups is generally very limited. As an example, the smallest number consisting of the product of four different primes may be taken, namely,

$$N = 2 \,.\, 3 \,.\, 5 \,.\, 7 = 210.$$

Corresponding to this order there are only two distinct groups. Thus, if S_5 and S_7 are any two operations of orders 5 and 7, they must be permutable, since 5 is not a factor of $7-1$. The sub-group of order 35 is therefore cyclical. If, now, S_3 is any operation of order 3, it must transform the operation $S_5 S_7$ into some power, say the n^{th}, of itself. Hence

$$S_3^{-1} S_5 S_3 = S_5^n,$$

$$S_3^{-1} S_7 S_3 = S_7^n,$$

or $$n^3 \equiv 1 \pmod 5, \qquad n^3 \equiv 1 \pmod 7,$$

and therefore $$n = 1.$$

The sub-group of order 105 is therefore cyclical, and can be generated from a single operation S_{105} of order 105. Finally, if S_2 is any operation of order 2, it must transform S_{105} either into itself or its inverse. Hence the group is defined by either

$$\text{(i.)} \quad S^2 = 1, \quad T^{105} = 1, \quad ST = TS,$$

or $$\text{(ii.)} \quad S^2 = 1, \quad T^{105} = 1, \quad ST = T^{-1}S.$$

The method of the present note may be obviously extended at once to groups in whose order the highest prime factor, only, occurs in a higher power than the first.

Thus, if $$N = p_1 p_2^m \quad (p_2 > p_1),$$

the sub-group of order p_2^m is necessarily self-conjugate, and the group therefore contains $(p_1-1)\, p_2^m$ operations in which p_1 enters as a factor of the order, and $p_2^m - 1$ operations in which it does not so enter.

If, now, the symbol m_r is used in the same sense as before, and it be assumed that

$$m_r = (p_r - 1)\, p_{r+1} \ldots p_n^m$$

holds for a given m, when the number of *different* prime factors in N does not exceed $n-1$, precisely the same proof will apply to show that, for the same m, the formula still holds when N contains n different prime factors; and that therefore, since when n is 2 the formula holds for any value of m, it holds universally.

Since the group contains exactly

$$p_n^m - 1$$

operations whose orders are equal to or powers of p_n, a sub-group of

order p_n^m must be contained in it self-conjugately, and therefore the group cannot be simple. A consideration of the composition-series of the group again shows that, since there is no simple group of order $p_1 p_2 \dots p_{n-1}$, this self-conjugate sub-group of order p_n^m cannot be a maximum self-conjugate sub-group, and that the order of such a maximum self-conjugate sub-group is necessarily of the form $\frac{N}{p_r}$, where p_r is one of the unrepeated factors of N. (There may, of course, also be maximum self-conjugate sub-groups of order $\frac{N}{p_n}$.)

It follows, again, that an equation the order of whose group is of the form considered is solvable by radicals. It also follows, exactly as in the preceding case, that the main group contains a series of self-conjugate groups of orders p_n^m, $p_{n-1} p_n^m$, $p_{n-2} p_{n-1} p_n^m$, $\dots$ $p_2 p_3 \dots p_{n-1} p_n^m$; so that, when N and the type of the sub-group of order p_n^m are given, all possible types of group may be formed by successively constructing this series of self-conjugate sub-groups in all possible ways.

IV. *On Groups of Order* $N = p_1^{m_1} p_2^{m_2} \dots p_n^{m_n}$, *in which the Sub-Groups of Orders* $p_1^{m_1}, p_2^{m_2}, \dots p_{n-1}^{m_{n-1}}$ *are all Cyclical.*

The main result obtained in the preceding note is a particular case of that obtained in the present one. By dealing, however, with the special case first, it appeared possible to bring out the nature of the method employed more simply, and to avoid detailed explanations which would have been otherwise necessary.

An application of the method of the previous note leads to the following theorem.

If a group contains a cyclical operation of order p_1^n, where p_1 is the smallest prime factor, and p_1^n the highest power of p contained in the order N, then the group contains exactly $\frac{p_1^n - 1}{p_1^n} N$ distinct operations whose orders are equal to or multiples of p_1.

Suppose that the theorem has been proved for all values of the index less than n, and in the group G of order

$$N = p_1^n qr,$$

where q and r are not necessarily primes, let the cyclical sub-group of

order p_1^n be contained self-conjugately in a sub-group H of order $p_1^n q$, and in no sub-group of higher order. Since p_1 is the smallest prime contained in the order of this sub-group, the operation P of order p_1^n, which generates the cyclical sub-group, must be permutable with every operation of H, and by constructing a scheme similar to that used before, it follows (i) that H contains $(p_1^n - 1)\, q$ operations whose orders are divisible by p_1, and (ii) that of these $(p_1^n - p_1^{n-1})\, q$ have p^n for a factor of their orders. Hence the set of r sub-groups conjugate with H contain $(p_1^n - p_1^{n-1})\, qr$ operations whose orders contain p^n as a factor, and these must be all distinct, as otherwise there would not be r distinct sub-groups of order p_1^n. By putting $n = 1$, this proves the theorem when the index is unity.

Now the sub-group of order p_1^m, which arises from the operations $P^{p_1^{n-m}}$, is certainly permutable within H, but it may be permutable in a more extensive sub-group. Suppose, then, that it is permutable within a sub-group H_m of order $p_1^n qr'$ (where $r = r'r''$).

The group h_m, which is merihedrically isomorphous with H_m in regard to the self-conjugate sub-group of order p_1^m, has $p_1^{n-m} qr'$ for its order, and its sub-groups of order p_1^{n-m} are clearly cyclical, and therefore it contains $(p_1^{n-m} - 1)\, qr'$ operations whose orders are multiples of p_1 by the supposition that has been made (the index here being less than n). To each of these operations there corresponds in H_m a set of p_1^m operations whose orders are divisible by a higher power of p_1 than p_1^m; for, if it were not so, to two suitably chosen operations of h_m of orders $p_1^s t$ and t ($s \not> n-m$ and t a factor of qr') would correspond the same set of operations of H_m, which is impossible. To each of the remaining qr' operations of the merihedrically isomorphous group correspond $p_1^m - 1$ operations of H_m whose orders are divisible by p_1, and a single operation whose order is prime to p_1.

Moreover, of these $(p_1^m - 1)\, qr'$ operations of H_m whose orders are divisible by a power of p_1 not higher than p_1^m, $(p_1^m - p_1^{m-1})\, qr'$ have p_1^m for a factor of their orders. The r'' conjugate groups of which H_m is one thus contain

$$(p_1^m - p_1^{m-1})\, qrr' = \frac{p_1^m - p_1^{m-1}}{p_1^n} N$$

distinct operations whose orders contain p_1^m, and no higher power of p_1, as a factor.

Hence G contains

$$\sum_{m=1}^{m=n} \frac{p_1^m - p_1^{m-1}}{p_1^n} N,$$

or
$$\frac{p_1^n - 1}{p_1^n} N,$$

operations whose orders are multiples of p_1. Thus the theorem is true for n if it is true for $n-1$; and, being true when $n = 1$, it holds generally.

If, now,
$$N = p_1^{m_1} p_2^{m_2} \dots p_n^{m_n},$$

where $p_1, p_2 \dots$, are in ascending order, and if the sub-groups of orders $p_1^{m_1}$ and $p_2^{m_2}$ are both cyclical, it may be shown in a similar manner that the number of operations of the group which contain p_2 as a factor of their order and do not contain p_1 is
$$\frac{p_2^{m_2} - 1}{p_1^{m_1} p_2^{m_2}} N.$$

For, if H, of order $p_1^{r_1} p_2^{m_2} p_3^{r_3} \dots p_n^{r_n}$, is the sub-group that contains $p_2^{m_2}$ self-conjugately, the isomorphous group of order $p_1^{r_1} p_3^{r_3} \dots p_n^{r_n}$ contains exactly $p_3^{r_3} \dots p_n^{r_n}$ operations whose orders do not have p_1 for a factor by the preceding theorem. Hence H contains $(p_2^{m_2} - 1) p_3^{r_3} \dots p_n^{r_n}$ operations whose orders are divisible by p_2 and not by p_1; and of these $(p_2^{m_2} - p_2^{m_2 - 1}) p_3^{r_3} \dots p_n^{r_n}$ have $p_2^{m_2}$ for a factor of their orders. The corresponding operations in the sub-groups conjugate to H are all distinct, and the main group therefore contains
$$\frac{p_2^{m_2} - p_2^{m_2 - 1}}{p_1^{r_1} p_2^{m_2}} N$$

operations whose orders are divisible by $p_2^{m_2}$ and are not divisible by p_1. Now, if r_1 is less than m_1, this number is greater than
$$\frac{N}{p_1^{m_1}},$$

the total number of operations whose orders are not divisible by p_1, which is impossible. Hence the order of H must be of the form
$$p_1^{m_1} p_2^{m_2} p_3^{r_3} \dots .$$

An operation of order $p_2^{m_2 - 1}$ is certainly self-conjugate in H, and may be self-conjugate in a more extensive group. In either case the preceding reasoning may be repeated to show that the number of operations in the group whose orders are divisible by $p_2^{m_2 - 1}$ and not by p_1 or $p_2^{m_2}$ is
$$\frac{p_2^{m_2 - 1} - p_2^{m_2 - 2}}{p_1^{m_1} p_2^{m_2}} N,$$

and hence, generally, that the number of operations whose orders are divisible by p_2 and not by p_1 is

$$\frac{p_2^{m_2}-1}{p_1^{m_1}p_2^{m_2}}N.$$

The process may be repeated step by step, so that the number of operations whose orders are divisible by p_s and by no smaller prime is

$$\frac{p_s^{m_s}-1}{p_1^{m_1}p_2^{m_2}\dots p_s^{m_s}}N \quad \text{or} \quad (p_s^{m_s}-1)\,p_{s+1}^{m_{s+1}}\dots p_n^{m_n}.$$

The sub-group of order $p_n^{m_n}$ is therefore self-conjugate, and the group is not simple. Moreover, all the conclusions arrived at in the preceding note, for the case in which the indices are all unity, hold again here both as regards the composition of the group and the mode of constructing the possible types when N is given.

V. *On Groups of Order* $N = p_1^2 p_2 \dots p_n$, *where* $p_1, p_2, \dots p_n$ *are distinct Primes in Ascending Order of Magnitude.*

When the sub-groups of order p_1^2 contained in the group are cyclical, the groups here considered are a particular case of those dealt with in the preceding note; and it has been seen that they cannot be simple, nor can they contain a sub-group which is simple. I assume, then, that the sub-groups of order p_1^2 are not cyclical, and therefore that any one of these is generated by two permutable operations of order p_1, and contains p_1+1 cyclical sub-groups of order p_1. Suppose that the greatest sub-group H which contains a sub-group of order p_1^2 self-conjugately is of order

$$n = p_1^2 p_\alpha p_\beta \dots p_\delta.$$

Then, since p_1 is the smallest prime in the order of H, no operation of H can transform one of the operations of order p_1 into a power of itself. Moreover, since H contains only p_1+1 sub-groups of order p_1, no operation of H can transform one of these sub-groups into another, since p_α is necessarily greater than p_1+1. The one exception to this last statement is when $p_1 = 2$ and $p_2 = 3$. Putting aside, then, the case in which $p_2 = 3$ for later consideration, every operation of order p_1 contained in H must be permutable with every operation of H. If, now, the main group contains s different conjugate sets of cyclical sub-groups of order p_1, then, since there is only one set of con-

jugate sub-groups of order p_1^2, the sub-group of order p_1^2 contained in H must itself contain representatives of each of the s sets. Let $P_1, P_2, \ldots P_s$ be operations contained in H representative of each of the s sets. Then P_r is certainly contained self-conjugately in H, and may be contained self-conjugately in a more extensive sub-group of order nm_r. If so, it generates one of a set of $\frac{N}{nm_r}$ conjugate cyclical sub-groups, each of which is contained in m_r sub-groups of order p_1^2. Now, the $\frac{N}{n}$ conjugate sub-groups of order p_1^2 each contain p_1+1 cyclical sub-groups of order p_1. Hence

$$(p_1+1)\frac{N}{n} = \sum_{r=1}^{r=s} m_r \cdot \frac{N}{nm_r}.$$

There are, therefore, p_1+1 distinct conjugate sets of sub-groups of order p_1 contained in the main group. If now $p_1^2 m$ is the order of the sub-group H' that contains P_1 self-conjugately, where for brevity m is written for $p_\alpha p_\beta \ldots p_\delta m_1$, the group may be expressed in the form

$$\begin{array}{llll}
1, & P_1, & \ldots & P_1^{p_1-1}, \\
S_1, & S_1P_1, & \ldots & S_1P^{p_1-1}, \\
\ldots & \ldots \quad \ldots & \ldots & \ldots \\
S_i, & S_iP_1, & \ldots & S_iP_1^{p_1-1}, \\
\ldots & \ldots \quad \ldots & \ldots & \ldots \\
S_{p_1m-1}, & S_{p_1m-1}P_1, & \ldots & S_{p_1m-1}P_1^{p_1-1},
\end{array}$$

and the different lines of this scheme combine together, as do the operations of the merihedrically isomorphous group of order $p_1 m$. Such a group contains, by Note III, $(p_1-1)\,m$ operations whose orders are equal to or multiples of p_1. If now the $(i+1)^{\text{th}}$ line of the above scheme corresponds to an operation of order $p_1 m'$ (m' a factor of m) of the isomorphous group, then $(S_iP_1^r)^{m'}$ belongs to some line of the scheme other than the first, and therefore in this case no operation of the $(i+1)^{\text{th}}$ line can be written in the form $\Sigma_i P_1^r$, where Σ_i is an operation whose order is not divisible by p_1.

If, on the other hand, the $(i+1)^{\text{th}}$ line corresponds to an operation of order m', of the isomorphous group, its operations can be written in the form

$$\Sigma_j P_1^r \quad (r = 0, 1, \ldots p_1-1),$$

where Σ_j is again an operation whose order is not divisible by p_1.

Hence the sub-group H' contains $(p_1-1)m$ distinct operations which can be written in the form

$$\Sigma_j P_1^r \quad (r = 1, 2, \ldots p_1-1).$$

The $\frac{N}{p_1^2 m}$ conjugate sub-groups therefore contain

$$\frac{p_1-1}{p_1^2} N$$

such operations, and these clearly exhaust all the operations permutable with any operations of the set of conjugate sub-groups to which P_1 belongs.

Now, it has been seen that there are $p+1$ sets of such conjugate sub-groups, and hence the main group contains

$$\frac{p_1^2-1}{p_1^2} N$$

operations whose orders are equal to or multiples of p_1.

If, now, a sub-group of order p_2 is contained self-conjugately in a group K of order $p_1^a p_2 p_3 \ldots p_s$, where a may be 0, 1 or 2, the merihedrically isomorphous group k of order $p_1^a p_3 \ldots p_s$ contains, by what has just been proved, $(p_1^a-1) p_3 \ldots p_s$ operations whose orders have p_1 as a factor, and to these correspond operations of K whose order contains p_1 as a factor. To each of the remaining $p_3 \ldots p_s$ operations of k correspond (p_2-1) operations of K, whose orders do not contain p_1, and do contain p_2, as a factor. Hence K contains $(p_2-1) p_3 \ldots p_s$ such operations, and the main group contains $\frac{N}{p_1^a p_2} (p_2-1)$ of such operations which are all distinct. This number must be less than $\frac{N}{p_1^2}$, which is the total number of operations whose orders do not contain p_1 as a factor. Hence the number of operations containing p_2, and not p_1, as a factor of their order is

$$(p_2-1) p_3 \ldots p_n.$$

In exactly the same way, it may be shown, taking the prime factors successively in ascending order of magnitude, that the number of operations containing p_s, and no lower prime, as a factor of their orders is

$$(p_s-1) p_{s+1} \ldots p_n.$$

From this it follows again that the group is necessarily composite, and that it contains self-conjugate sub-groups of orders p_n, $p_{n-1}p_n$, ..., $p_2 p_3 \dots p_n$. Moreover, since simple groups of order p_1^2 do not exist, the sub-group of order $p_2 p_3 \dots p_n$ cannot be a maximum self-conjugate sub-group, and hence there must be a self-conjugate sub-group of order $p_1 p_2 \dots p_n$.

Exactly, then, as in the previous cases when N is given, the successive self-conjugate sub-groups of orders p_n, $p_{n-1}p_n$, &c., may be constructed in all possible ways, and every type of group corresponding to the given N may be determined.

If now $p_2 = 3$, so that $p_1 = 2$, and

$$N = 2^2 3 p_3 \dots p_n ,$$

two cases may occur.

First, the group may contain no sub-group of tetrahedral type, *i.e.*, no sub-group of order 12 which consists of a self-conjugate sub-group formed of identity and 3 operations of order 2, together with 8 more operations of order 3. In this case, the preceding reasoning may be repeated, word for word, to show that there are three different conjugate sets of sub-groups of order 2, and hence that the main group contains

$$\frac{2^2-1}{2^2} N$$

operations of even order.

There is, then, no difficulty in proceeding as before to show that the number of operations whose orders are divisible by p_s, and by no lower prime, is

$$(p_s - 1) p_s \dots p_n ,$$

and thus in establishing completely for this case the previous results.

Suppose, now, secondly, that the group does contain a sub-group of tetrahedral type. Within this sub-group the three operations of order 2 form a single conjugate set, and therefore the operations of order 2 in the main group also form a single conjugate set.

Hence in this case the main group contains only

$$\frac{2-1}{2^2} N = 3 p_3 \dots p_n$$

operations of even order.

The number of operations in the main group whose orders are divisible by 3, and not by 2, will be

$$2^2\,(3-1)\,p_3 \ldots p_n,$$

$$2\,(3-1)\,p_3 \ldots p_n,$$

or $$(3-1)\,p_3 \ldots p_n,$$

according as 2^0, 2^1, or 2^2 is a factor of the order of the sub-group within which an operation of order 3 is contained self-conjugately. The number of operations whose orders are divisible neither by 2 nor 3 will therefore be

$$p_3 \ldots p_n,$$

$$5p_3 \ldots p_n,$$

or $$7p_3 \ldots p_n.$$

In the first case, the group cannot be simple, for the number of operations whose orders are divisible by p_s, and by no lower prime, will, as before, be

$$(p_s-1)\,p_{s+1} \ldots p_n.$$

In the latter two cases, however, this number may be

$$x_s\,(p_s-1)\,p_{s+1} \ldots p_n,$$

where x_s is a factor of $2^2 3\,p_3 \ldots p_{s-1}$; and where the x's are connected by the equation

$$\sum_{s=3}^{s=n} x_s\,(p_s-1)\,p_{s+1} \ldots p_n = \lambda p_3 \ldots p_n - 1, \quad \lambda = 5 \text{ or } 7.$$

There is also, by Sylow's theorem, the additional relation

$$x_n \equiv 1 \pmod{p_n}.$$

When $n=3$ it is easy to see that the only solution of this equation is

$$\lambda = 5, \quad x_3 = 6, \quad p_3 = 5,$$

which corresponds to a known result.

When $n=4$, the only solutions are given by

$$\lambda = 7, \quad x_3 = 6, \quad x_n = 12, \quad p_3 = 5, \quad p_4 = 11,$$

$$\lambda = 7, \quad x_3 = 6, \quad x_n = 14, \quad p_3 = 7, \quad p_4 = 13.$$

The orders of the corresponding groups would be $2^2 \cdot 3 \cdot 5 \cdot 11$ and $2^2 \cdot 3 \cdot 7 \cdot 13$. It is known that there is a simple group corresponding to each of these orders. A general discussion of the equation would seem to be far from simple.

A consideration of the preceding investigation will show at once that the expression

$$\frac{p_1^2-1}{p_1^2}N$$

for the number of operations in the group whose orders are divisible by p_1 holds for any group in whose order the lowest prime factor appears to the second power, so long as the group does not contain a tetrahedral sub-group. Suppose, now, that the order is

$$N = p_1^{\gamma} p_2^{m_2} p_3^{m_3} \dots p_n^{m_n},$$

and that the sub-groups of orders $p_2^{m_2}$, ... $p_{n-1}^{m_{n-1}}$ are all cyclical. Then, with the same limitation, that the group does not contain a tetrahedral sub-group, the reasoning of the present and the preceding note may be repeated, almost word for word, to show that the number of operations in the group whose order is divisible by p_s, and by no lower prime, is again

$$(p_s^{m_s}-1)\, p_{s+1}^{m_{s+1}} \dots p_n^{m_n};$$

so that again, in this case, the group cannot be simple, nor can it contain a simple sub-group (other than the cyclical sub-groups).

VI. *On Groups whose Orders are of the Forms* $p_1 p_2 \dots p_{n-1}^2 p_n$, $p_1^2 p_2 \dots p_{n-1}^2 p_n$, $p_1^m p_2$, $p_1^m p^2$, *and* $p_1 p_2^m p_3$.

The method that has been used in the three preceding notes may be applied without change to a rather more general case, as follows. Let

$$N = p_1^{m_1} p_2^{m_2} \dots p_n^{m_n}$$

be the order of a group, and suppose that the sub-group of orders $p_1^{m_1}$, $p_2^{m_2}$, ... $p_r^{m_r}$ are all cyclical. Then, proceeding step by step, it may be shown that, for all values of s from 1 to r, the group contains

$$(p_s^{m_s}-1)\, p_{s+1}^{m_{s+1}} \dots p_n^{m_n}$$

operations, whose orders are divisible by p_s, and by no lower prime, so that the number of operations whose orders contain no prime less than p_{r+1} is

$$p_{r+1}^{m_{r+1}} \dots p_n^{m_n}.$$

The same result will also hold if $m_1 = 2$, even if the sub-group of order p_1^2 is not cyclical, so long as p_2 is not equal to 3. These results, which do not now seem to require separate proof, will be used in discussing groups whose orders are of the forms in the heading of this note.

$$(\text{i})\quad N = p_1 p_2 \dots p_{n-1}^2 p_n .$$

If the sub-groups of order p_{n-1}^2 in a group whose order is of the above form are cyclical, Note IV shows that the group cannot be simple; and in any case Note III establishes that the number of operations of the group into whose orders no prime smaller than p_{n-1} enters is

$$p_{n-1}^2 p_n .$$

If, now, no operations of order p_{n-1} transform a sub-group of order p_n into itself, there must be more than p_{n-1}^2 conjugate sub-groups of order p_n, and this is clearly impossible. If the operations of a sub-group of order p_{n-1}^2 all transform a sub-group of order p_n into itself, the $p_{n-1}^2 p_n$ operations, into whose orders no prime less than p_{n-1} enters, form a sub-group which must be self-conjugate.

If, finally, a sub-group of order p_n is transformed into itself by an operation of order p_{n-1}, the resulting sub-group of order $p_{n-1} p_n$ must be cyclical unless $p_n \equiv 1 \pmod{p_{n-1}}$. If it is cyclical and not self-conjugate, it must form one of kp_n+1 conjugate sub-groups, which will contain

$$(kp_n+1)\, p_{n-1}\, (p_n-1)$$

distinct operations whose orders contain p_n as a factor. This, again, is impossible. If it is not cyclical, p_{n-1} is a factor of p_n-1. Now, in this case,

$$p_{n-1}^2 p_n - 1 = \alpha\, (p_{n-1}-1)\, p_n + \beta\, (p_n-1),$$

where the two terms on the right-hand side represent the number of operations of orders p_{n-1} and p_n contained in the group. Hence

$$(p_{n-1}-1)(p_{n-1}+1-\alpha)\, p_n = (\beta-1)(p_n-1),$$

and therefore, since p_{n-1} is a factor of p_n-1,

$$\alpha = 1,$$

and

$$p_n \not> p_{n-1}^2 - p_n + 1.$$

There is, therefore, a single set of p_n conjugate sub-groups of order p_{n-1}, and at the same time p_n conjugate sub-groups of order p_{n-1}^2; and this is certainly inconsistent with the inequality connecting p_n and p_{n-1}. The group can, therefore, in no case be simple.

$$(\text{ii})\quad N = p_1^2 p_2 \dots p_{n-1}^2 p_n .$$

When p_2 is not 3 it may be shown in precisely the same way as in the previous case that a group of order $p_1^2 p_2 \dots p_{n-1}^2 p_n$ cannot be simple, and cannot contain a simple sub-group.

(iii) $N = p_1^m p_2$, $N = p_1 p_2^m p_3$, $N = p_1^m p_2^2$.

Let p^m be the highest power of a prime p that divides the order of a group G, and let p^r be the order of the greatest group which is common to any two of the conjugate set of sub-groups of order p^m contained in G. Let h be such a sub-group of order p^r, and let h be contained in the two sub-groups H and H' of order p^m. Within H, h must be self-conjugate in some sub-group of order p^{r+1}, generated by h and some operation P of H, which is not contained in H;* and similarly in H', h must be self-conjugate in a group generated by P' and h. Hence h is self-conjugate in the group generated by P, P' and h. If the order of this group were a power of p, there would be two sub-groups of order p^m containing a common sub-group of order p^{r+1}, contrary to supposition, and therefore the sub-group h must be transformed into itself by some operation whose order is not a power of p.

Consider now a group of order $p_1^m p_2$. If a sub-group of order p_1^m is not self-conjugate it must be one of a set of p_2 conjugate sub-groups. If the operations of these are all distinct, the operation of order p_2 is self-conjugate. If they are not all distinct, then by the theorem just proved there must be a sub-group of order p_1^r which is transformed into itself by an operation of order p_2, and which being common to the p_2 sub-groups of order p_1^m must be self-conjugate. The group is therefore in any case composite.

Again, a group of order $p_1 p_2^m p_3$ must if simple contain p_3 or $p_1 p_3$ conjugate sub-groups of order p_2^m. Now the group contains $(p_1-1) p_2^m p_3$ operations whose orders are multiples of p_1. Hence if the operations of the sub-groups of order p_2^m are all distinct there must be p_3 such sub-groups, and the sub-group of order p_3 is self-conjugate. If there are p_3 sub-groups of order p_2^m whose operations are not all distinct, a sub-group of order p_2^r must be contained self-conjugately in one of order $p_2^{r+1} p_3$ or $p_1 p_2^{r+1} p_3$ ($p_1 p_2^{r+1}$ not being a possible value for the order, since the sub-group of order p_2^r is contained in more than one sub-group of order p_2^{r+1}). The sub-group of order p_2^r is therefore self-conjugate, being common to the p_3 conjugate sub-groups of order p_2^m. If there are $p_1 p_3$ conjugate sub-groups of order p_2^m, whose operations are not all distinct, the same reasoning

* This property of groups whose order is the power of a prime must be known, but I am unable to give a reference to any work in which it is explicitly stated. It may be proved as follows :—Suppose that there is no sub-group of order p^{r+1} in H containing h self-conjugately, and let S be a self-conjugate operation of h. Then the number of sub-groups of H conjugate to and other than h which contain S is a multiple of p, as also is the number which do not contain S, so that the total number in the conjugate set is congruent to unity, mod. p, which is impossible.

holds to show that they must have a common sub-group. Hence in no case can the group be simple.

A group of order $p_1^m p_2^2$, if simple, must contain p_2^2 conjugate sub-groups of order p_1^m. If the operations of these are all distinct, the sub-group of order p_2^2 is self-conjugate. If they are not all distinct the greatest sub-group h, of order p_1^r, common to more than one sub-group of order p_1^m, must be self-conjugate within a group of order $p_1^{r+t} p_2$ or $p_1^{r+t} p_2^2$, where t is not less than unity. Unless $p_2 \equiv 1 \pmod{p_1}$, only the latter case can occur, and therefore the group is certainly composite unless this condition is satisfied. When the condition is satisfied, I have not been able to show generally that the group cannot be simple, but for some of the smaller values of m this may be done without difficulty.

Thus, if the order of a group be $p_1^3 p_2^2$ and p_2 be not 3 (there is no simple group of order $2^3.3^2$) there must, if the group be simple, be p_1^3 conjugate sub-groups of order p_2^2. If two of these have a common sub-group, it must be self-conjugate within a group of order $p_1^s p_2^2$ which contains more than one sub-group of order p_2^2. Hence $s = 3$, and the sub-group of order p_2 in question is self-conjugate in the main group. If no two of the p_1^3 sub-groups of order p_2^2 have a common operation, the sub-group of order p_1^3 must be self-conjugate.

Again, if the order of a group be $p_1^4 p_2^2$, and p_2 be not 3 (there is no simple group of order $2^4.3^2$) there must be either p_1^3 or p_1^4 conjugate sub-groups of order p_2^2, and as in the previous case, it may be shown that if two sub-groups of order p_2^2 have a common sub-group of order p_2, it must be self-conjugate in the main group. Again, there must be p_2^2 conjugate sub-groups of order p_1^4. If two of these have a sub-group in common of order p_1^3, it must be self-conjugate in a group of order $p_1^4 p_2$, and the group cannot be simple.

If p_1^2 is the order of the highest common sub-group of two of the sub-groups of order p_1^4, it must be permutable with an operation of order p_2, and then this operation would be self-conjugate in a group of order $p_1^2 p_2^2$ at least. This is inconsistent with the existence of p_1^3 sub-groups of order p_2^2; and the supposition that the highest common sub-group of two sub-groups of order p_1^4 is of order p_1 leads to the same contradiction.

Finally, if the operations of the sub-groups of order p_1^4 are all distinct, the sub-group of order p_2^2 must be self-conjugate. Hence in no case can the group be simple.*

* The case of groups of order $p_1^4 p_2^2$ would be covered by a proof, that Herr Frobenius gives in his first-mentioned paper, of the non-existence of simple groups of order $p_1^4 p_2^m$, were this proof correct. Herr Frobenius, however, arrives at this

VII. *On the Simple Groups whose Orders consist of the Product of Five Primes.*

On an enumeration of all the different forms which the order of a group can take when it consists of the product of five primes, it will be found that the previous general results cover all cases with the exception of

$$\text{(i) } 2^2 3 p_3^2, \qquad \text{(ii) } 2^2 3^2 p_3, \qquad \text{(iii) } p_1 p_2^2 p_3^2,$$
$$\text{(iv) } p_1 p_2^2 p_3 p_4, \qquad \text{(v) } p_1^3 p_2 p_3;$$

so that the only simple groups, whose orders are the products of five primes, of forms different from the above are the groups of order $2^2 . 3 . 5 . 11$ and $2^2 . 3 . 7 . 13$. These remaining five cases may be dealt with as follows.

(i) $N = 2^2 3 p_3^2$.

Such a group can only be simple when 12 contains a factor of the form kp_3+1, and this can only be the case when p_3 is 5. It is, however, known that there is no simple group of order 300. (*Cf.* Dr. Cole, *American Journal of Mathematics*, Vol. xv.)

(ii) $N = 2^2 3^2 p_3$.

Sylow's theorem again here shows that the only admissible values of p_3 are 5, 7, 11, and 17; and a reference to Dr. Cole's paper proves that there are no corresponding simple groups.

(iii) $N = p_1 p_2^2 p_3^2$.

There must be either p_3^2, p_3, $p_1 p_3$ or $p_1 p_3^2$ conjugate sub-groups of order p_2^2. In the two latter cases the group would contain p_2+1 conjugate sets of sub-groups of order p_2, and could not be simple; while, if there were p_3 such sub-groups, the group evidently could not be simple, since its order contains the factor p_3^2. If there are p_3^2 sub-groups of order p_2^r, each is self-conjugate in a group of order $p_1 p_2^2$; and the group can only be simple when, within this group, the sub-groups of order p_2 are not all self-conjugate; while at the same time a sub-group of order p_2, which within this group forms one of a conjugate set of p_1, is permutable in the main group with an operation of order p_1. But, if this is the case, the sub-group of order p_2 must be self-conjugate within a group of order $p_1 p_2^2 p_3$ at least, since a group of order $p_1 p_2^2$ can only contain a single sub-group of order p_2^2. The

result by showing that in such a group the sub-group of order p_2^m must be self-conjugate. This is certainly not the case, as nothing is easier than to construct groups of the order in question, in which a sub-group of order p_2^m is not self-conjugate. *E.g.*,

$A^3 = 1$, $B^3 = 1$, $(AB)^2 = 1$, $C^4 = 1$, $D^3 = 1$, $AC = CA$, $BC = CB$, $AD = DA$, $BD = DB$, $CD = DC$.

existence of a sub-group of order $p_1 p_2^2 p_3$ is, however, clearly inconsistent with the group being simple.

(iv) $N = p_1 p_2^2 p_3 p_4$.

If p_1 is not a factor of the sub-group H that contains a group of order p_2^2 self-conjugately, there are p_2+1 distinct sets of sub-groups of order p_2, and it follows by the method of Note V that the group is not simple. The same holds also if, p_1 being a factor of the order of H, all the sub-groups of H of order p_2 are transformed into themselves by the operations of H whose orders are p_1. Suppose, now, that this is not the case, so that within H a set of p_1 sub-groups of order p_2 are conjugate. If one of these sub-groups is permutable with no operation of order p_1 within the main group, there will correspond to it within the main group a set of

$$p_1 (p_2-1) p_3 p_4$$

operations whose orders are divisible by p_2 and not by p_1; and, if the same is true of each set of p_1 conjugate sub-groups within H, the total number of operations whose orders are divisible by p_2 and not by p_1 will be $(p_2^2-1) p_3 p_4$,

from which it follows, as before, that the group is composite.

If a sub-group of order p_2 which within H forms one of a set of p_1 conjugate sub-groups is permutable within the main group with an operation of order p_1, it must be contained self-conjugately in a group of order $p_1 p_2^2 p_3$, $p_1 p_2^2 p_4$, or $p_1 p_2^2 p_3 p_4$,

and the first case only could correspond to a simple group. Now, the group of order $p_1 p_2^2 p_3$ would necessarily contain $p_1 p_3$ conjugate sub-groups of order p_2^2, so that

$$p_1 p_3 \equiv 1 \pmod{p_2},$$

and the method of Note V shows that it contains a sub-group of order p_3 self-conjugately. Also, since

$$p_3 \not\equiv 1 \pmod{p_2},$$

the operations of order p_3 must be permutable with all operations of order p_2 of the sub-group. But this is in contradiction with the existence of $p_1 p_3$ conjugate sub-groups of order p_2^2. Hence this case cannot occur.

(v) $N = p_1^3 p_2 p_3$.

Suppose, first, that there are p_3 conjugate sub-groups of order p_1^3. Each is contained self-conjugately in a sub-group of order $p_1^3 p_2$, so that the group, if simple, is transitive in p_3 symbols. Then, unless $p_2 = 3$, there must in a sub-group of order $p_1^3 p_2$ be either 1 or p_1^3 sub-groups of order p_2. If there are p_1^3, the sub-group can contain no operation of order $p_1 p_2$, and hence the sub-group of order p_1^3 must be Abelian in

type; for if it were not its p_1-1 self-conjugate operations would be transformed into themselves by an operation of order p_2, and the sub-group would contain operations of order $p_1 p_2$. Now, if two sub-groups of order p_1^3 and Abelian type contained a common operation it would be self-conjugate in a group containing more than one sub-group of order p_1^3, and, therefore, in the main-group. Hence the p_3 sub-groups of order p_1^3 contain $(p_1^3-1)\,p_3$ distinct operations, each keeping one symbol fixed. The operations of order p_2 must each keep 1 or p_1 symbols fixed; for if they kept p_1^2 or p_1^3 the group would contain operations of order $p_1 p_2$. If the operations of order p_2 keep 1 symbol fixed, they are $p_1^3\,(p_2-1)\,p_3$

in number, and the sub-group of order p_3 is self-conjugate. If the operations of order p_2 keep p_1 symbols fixed, they are

$$p_1^2\,(p_2-1)\,p_3$$

in number, and the group must contain

$$p_1^3 p_2 p_3-(p_1^3-1)\,p_3-p_1^2\,(p_2-1)\,p_3-1$$

operations of order p_3.

Now there must be p_2, $p_1 p_2$ or $p_1^2 p_2$ sub-groups of order p_3. The first two cases cannot give the above number of operations of order p_3, and the last leads to the conditions

$$p_1 = 2, \quad 4p_2-3p_3 = 1,$$

which cannot give a simple group.

If the sub-group of order $p_1^3 p_2$ contains a single sub-group of order p_2, it must keep one symbol only fixed, and must be permutable with every operation of the sub-group of order p_1^3. If now the sub-groups of order p_1^3 had a common operation, it would be permutable with two different operations of order p_2, and would be necessarily self-conjugate in a group of order $p_1^2 p_2 p_3$. This would make the group composite. If, on the other hand, no two sub-groups of order p_1^3 have a common operation, the operations of a sub-group of order $p_1^3 p_2$ all keep just one symbol fixed, and the sub-group of order p_3 is self-conjugate.

Suppose now, secondly, that the group contains $p_2 p_3$ sub-groups of order p_1^3. If the operations of these sub-groups are all distinct the group is clearly composite. If there were a sub-group of order $p_1^3 p_3$, the main group clearly could not be simple, and if there was one of order $p_1^3 p_2$ the group could be expressed as a transitive group in p_3 symbols. If there are no sub-groups of orders $p_1^3 p_2$ or $p_1^3 p_3$, the common operations of two sub-groups of order p_1^3 must be contained self-conjugately in a sub-group of order $p_1^2 p_2$ or $p_1^2 p_3$. This involves

$$p_2 \equiv 1 \pmod{p_1}, \quad p_3 \equiv 1 \pmod{p_1}.$$

Now the congruence $p_1 p_2 \equiv 1 \pmod{p_3}$

is inconsistent with the two previous ones, and, therefore, there can be no sub-group of order $p_1^2 p_3$. Every sub-group of order p_1, therefore, which is common to two sub-groups of order p_1^3 is permutable with an operation of order p_2. The main-group can, in this case, be expressed as a transitive group in $p_1 p_3$ symbols, and a simple enumeration of the operations of the $p_1 p_3$ sub-groups of order $p_1^2 p_2$ shows that they contain $p_1 p_3$ sub-groups of order $p_1 p_2$,

and $p_2 p_3$ sub-groups of order p_1,

each of the latter being self-conjugate in a sub-group of order p_1^3. Also each sub-group of order p_1^3 contains p_1 conjugate sub-groups of the set which are permutable with operations of order p_2, and these are the only sub-groups which can be common to two sub-groups of order p_1^3. Hence, besides these sub-groups and its self-conjugate sub-group, every sub-group of order p_1^3 must contain

$$p_1^3 - (p_1+1)(p_1-1) - 1 = p_1^3 - p_1^2$$

operations which occur in no other sub-group of order p_1^3. There remain

$$p_1^3 p_2 p_3 - (p_1^3 - p_1^2) p_2 p_3 - p_1 p_3 (p_1 p_2 - 1) - p_2 p_3 (p_1 - 1)$$

operations in the group. Now this number is negative, and therefore the case supposed cannot occur. Hence if the group is simple it must contain a sub-group of order $p_1^3 p_2$. Unless $p_2 = 3$ this sub-group must contain its operations of order p_2 self-conjugately, and therefore there must be p_3 sub-groups of order p_2. This being so, it is easy to show that, if the group is simple, there must be $p_1^2 p_2$ conjugate sub-groups of order p_3. The congruences

$$p_2 \equiv 1 \pmod{p_1}, \qquad p_3 \equiv 1 \pmod{p_1},$$
$$p_3 \equiv 1 \pmod{p_2}, \quad p_1^2 p_2 \equiv 1 \pmod{p_3},$$

must therefore be simultaneously satisfied. The last of these is obviously inconsistent with the first three, and therefore, unless $p_2 = 3$, the group cannot be simple.

Finally, if $p_1 = 2$, $p_2 = 3$, then

$$N = 24 p_3,$$

and, by Sylow's theorem, p_3 can only be 5, 7, 11, or 23. Of these the only value that corresponds to a simple group is 7 (*cf.* Dr. Cole's paper, *loc. cit.*), and there is only one corresponding simple group.

In conclusion, then, the only simple groups whose orders consist of the product of five primes are those of orders $2^3 . 3 . 7$, $2^2 . 3 . 5 . 11$, and $2^2 . 3 . 7 . 13$.

The Dynamics of a Top.

By A. G. Greenhill.

Read January 10th, 1895. Received February 4th, 1895.

A statement by Jacobi (*Gesammelte Werke*, t. II., p. 480) that the general motion of a top or gyrostat, moving under gravity about a fixed point in its axis, can be resolved into the relative motion of two bodies moving *à la Poinsot* about the fixed point under no forces, has attracted considerable attention of recent years, as testified by the valuable and interesting articles on this subject by

Halphen, *Comptes Rendus*, t. C., 1885;

Darboux, in Note XX. to Despeyrous' *Cours de Mécanique*, t. II., p. 525;

Routh, *Quarterly Journal of Mathematics*, Vol. XXIII., p. 34; and

Marcolongo, *Annali di Matematica*, Vol. XXII., 1894.

Dr. Routh commences with an investigation of these two associated concordant states of motion under no forces, and shows afterwards how they may be combined so as to give the motion of a top; but in the present paper it is proposed to reverse this procedure, and to start with the analysis of the motion of the top, and thence to derive Jacobi's two associated states of motion; it is hoped that this new procedure will help to throw light upon this interesting and important theorem in Dynamics.

1. We begin, then, with the equations of motion of the axis of the top, as given in Routh's *Rigid Dynamics*, following as closely as possible the notation of the article in the *Quarterly Journal of Mathematics*, Vol. XXIII.

The equations connecting ψ, the azimuth of the axis OC, and θ, the inclination of the axis to its highest vertical position OG, can then be written

$$\tfrac{1}{2}A_1\left(\frac{d\theta}{dt}\right)^2+\tfrac{1}{2}A_1\sin^2\theta\left(\frac{d\psi}{dt}\right)^2=Wg\,(d-h\cos\theta)\quad\ldots\ldots\ldots(1),$$

$$A_1\sin^2\theta\frac{d\psi}{dt}+C_1n_1\cos\theta=G_1\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Take a point P in OC at a distance l from O, such that

$$l = \frac{A_1}{Wh};$$

then P may be called the *centre of oscillation*, as in plane vibrations; and put

$$\frac{g}{l} = \frac{Wgh}{A_1} = n^2,$$

so that $2\pi/n$ seconds is the period of small plane oscillations.

The quantities employed in this paper, here and subsequently, are expressed in Dr. Routh's notation by

$$n^2 = 2f^2, \quad \frac{d}{h} = \frac{L}{f^2}, \quad E = r, \quad \frac{G_1}{A_1} = 2\frac{T}{G} = 2e, \quad \frac{C_1 n_1}{A_1} = 2\frac{T'}{G'} = 2e',$$

or

$$\frac{G_1^2}{2A_1 Wgh} = \frac{e^2}{f^2}, \qquad \frac{C_1^2 n_1^2}{2A_1 Wgh} = \frac{e'^2}{f^2}.$$

Writing equations (1) and (2)

$$\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\psi}{dt}\right)^2 = 2n^2\left(\frac{d}{h} - \cos\theta\right),$$

$$\sin^2\theta\frac{d\psi}{dt} = \frac{G_1 - C_1 n_1 \cos\theta}{A_1},$$

and, eliminating $\frac{d\psi}{dt}$,

$$\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 = 2n^2\left(\frac{d}{h} - \cos\theta\right)(1-\cos^2\theta) - \left(\frac{G_1 - C_1 n_1\cos\theta}{A_1}\right)^2$$

$$= 2n^2\Theta \quad \ldots\ldots(3),$$

suppose, where

$$\Theta = \left(\frac{d}{h} - \cos\theta\right)(1-\cos^2\theta) - \frac{(G_1 - C_1 n_1\cos\theta)^2}{2A_1 Wgh} \quad \ldots\ldots(4).$$

To solve (3) we suppose Θ to be split up into three factors, such that

$$\Theta = (\cos\theta - \cosh\theta_1)(\cos\theta - \cos\theta_2)(\cos\theta - \cos\theta_3)\ldots\ldots(5),$$

so that the inclination θ of the axis oscillates between θ_2 and θ_3,

$$\theta_2 < \theta < \theta_3.$$

2. The solution of equation (3) by elliptic functions is given by

$$\left.\begin{aligned} \wp u - e_1 &= \tfrac{1}{2}\Omega(\cos\theta - \cosh\theta_1) \\ \wp u - e_2 &= \tfrac{1}{2}\Omega(\cos\theta - \cos\theta_2) \\ \wp u - e_3 &= \tfrac{1}{2}\Omega(\cos\theta - \cos\theta_3) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

the letter Ω being employed as the *homogeneity factor* so as to agree with M. Darboux's notation (Despeyrous, t. II., p. 514); and now

$$u = qt + \omega_2 \quad \text{or} \quad qt + \omega_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

for $\cos\theta$ to oscillate between $\cos\theta_2$ and $\cos\theta_3$; and, since from (5) and (6)

$$\sin^2\theta \left(\frac{d\theta}{dt}\right)^2 = \frac{4\wp'^2 u}{\Omega^2}\left(\frac{du}{dt}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$2n^2\Theta = \frac{16n^2}{\Omega^3}(\wp u - e_1)(\wp u - e_2)(\wp u - e_3) = \frac{4n^2}{\Omega^3}\wp'^2 u \ldots\ldots\ldots (9);$$

therefore

$$q^2 = \left(\frac{du}{dt}\right)^2 = \frac{n^2}{\Omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

In Jacobi's notation, the modulus κ and its complementary modulus κ' are given by

$$\kappa^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{\cos\theta_2 - \cos\theta_3}{\cosh\theta_1 - \cos\theta_3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

$$\kappa'^2 = \frac{e_1 - e_2}{e_1 - e_3} = \frac{\cosh\theta_1 - \cos\theta_2}{\cosh\theta_1 - \cos\theta_3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

Denoting the real quarter period of Jacobi's functions by K, then the time occupied while θ grows from θ_2 to θ_3 is

$$\frac{K}{q\sqrt{(e_1 - e_3)}} = \frac{K}{n\sqrt{\{\frac{1}{2}(\cosh\theta_1 - \cos\theta_3)\}}}$$

seconds; and this is the fraction

$$\frac{1}{4\sqrt{\{\frac{1}{2}(\cosh\theta_1 - \cos\theta_3)\}}}$$

of the complete period of the top when making plane oscillations, by swinging through the angle

$$4\sin^{-1}\kappa = 4\sin^{-1}\sqrt{\left(\frac{\cos\theta_2 - \cos\theta_3}{\cosh\theta_1 - \cos\theta_3}\right)}.$$

3. If u assumes the values v_1 and v_2 when $\cos\theta$ is $+1$ and -1, then, from (6),

$$\wp u - \wp v_2 = \tfrac{1}{2}\Omega\,(1+\cos\theta) \quad \ldots\ldots(13),$$

$$\wp v_1 - \wp u = \tfrac{1}{2}\Omega\,(1-\cos\theta) \quad \ldots\ldots(14),$$

so that

$$\wp v_1 - \wp v_2 = \Omega \quad \ldots\ldots(15),$$

and, since

$$-\infty < -1 < \cos\theta_3 < \cos\theta < \cos\theta_2 < 1 < \cosh\theta_1 < \infty,$$

we therefore take

$$v_2 = p\omega_3, \quad v_1 = \omega_1 + r\omega_3 \quad \ldots\ldots(16),$$

where p and r are real fractions.

Also, putting $\cos\theta = \mp 1$ in (4) and (9),

$$\left(\frac{G_1 + C_1 n_1}{A_1}\right)^2 = -\frac{4q^2}{\Omega^2}\wp'^2 v_2, \quad \left(\frac{G_1 - C_1 n_1}{A_1}\right)^2 = -\frac{4q^2}{\Omega^2}\wp'^2 v_1 \quad \ldots(17);$$

and therefore, from (10),

$$\frac{G_1 + C_1 n_1}{\sqrt{(A_1 W g h)}} = -\frac{2i\wp' v_2}{\Omega^{\frac{3}{2}}}, \quad \frac{G_1 - C_1 n_1}{\sqrt{(A_1 W g h)}} = \frac{2i\wp' v_1}{\Omega^{\frac{3}{2}}} \quad \ldots\ldots(18).$$

Thus, if $G_1 - C_1 n_1$ is negative, we must suppose r negative, or put

$$v_1 = \omega_1 - r\omega_3 \quad \ldots\ldots(19).$$

Adding and subtracting equations (18), making use of (15),

$$\frac{G_1\sqrt{\Omega}}{\sqrt{(A_1 W g h)}} = i\frac{\wp' v_1 - \wp' v_2}{\wp v_1 - \wp v_2} \quad \ldots\ldots(20),$$

$$\frac{C_1 n_1\sqrt{\Omega}}{\sqrt{(A_1 W g h)}} = -i\frac{\wp' v_1 + \wp' v_2}{\wp v_1 - \wp v_2} \quad \ldots\ldots(21),$$

or

$$\frac{G_1^2\Omega}{4A_1 W g h} = -\wp v_1 - \wp v_2 - \wp(v_1 + v_2) \quad \ldots\ldots(22),$$

$$\frac{C_1^2 n_1^2\Omega}{4A_1 W g h} = -\wp v_1 - \wp v_2 - \wp(v_1 - v_2) \quad \ldots\ldots(23).$$

4. We shall find that (Vol. xxv., p. 281)

$$u = v_1 - v_2$$

makes $$\cos\theta = \frac{d}{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24).$$

Writing

$$\Theta = (E - \cos\theta)(1 - \cos^2\theta) - \frac{(C_1 n_1 - G_1 \cos\theta)^2}{2A_1 W g h} \quad \ldots\ldots (25),$$

then $$E = \frac{d}{h} - \frac{G_1^2 - C_1^2 n_1^2}{2A_1 W g h} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (26),$$

and this is the quantity denoted by r in Dr. Routh's article; and we find that (p. 281)

$$u = v_1 + v_2$$

makes $$\cos\theta = E \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27),$$

so that, putting

$$v_1 + v_2 = v,$$

$$v_1 - v_2 = w,$$

$$\left.\begin{aligned} \wp v - \wp u &= \tfrac{1}{2}\Omega\,(E - \cos\theta) \\ \wp v - e_1 &= \tfrac{1}{2}\Omega\,(E - \cosh\theta_1) \\ \wp v - e_2 &= \tfrac{1}{2}\Omega\,(E - \cos\theta_2) \\ \wp v - e_3 &= \tfrac{1}{2}\Omega\,(E - \cos\theta_3) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (28),$$

$$\left.\begin{aligned} \wp w - \wp u &= \tfrac{1}{2}\Omega\left(\frac{d}{h} - \cos\theta\right) \\ \wp w - e_1 &= \tfrac{1}{2}\Omega\left(\frac{d}{h} - \cosh\theta_1\right) \\ \wp w - e_2 &= \tfrac{1}{2}\Omega\left(\frac{d}{h} - \cos\theta_2\right) \\ \wp w - e_3 &= \tfrac{1}{2}\Omega\left(\frac{d}{h} - \cos\theta_3\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (29).$$

5. Writing equation (2) in the form

$$\sin\theta \frac{d\psi}{d\theta}\sqrt{\Theta} = \frac{-C_1 n_1 \cos\theta + G}{\sqrt{(2A_1 W g h)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (30),$$

$$\psi = \frac{G_1 - C_1 n_1}{\sqrt{(2A_1 W g h)}} \int \frac{\sin\theta\, d\theta}{(1 - \cos\theta)\sqrt{\Theta}} + \frac{G_1 + C_1 n_1}{\sqrt{(2A_1 W g h)}} \int \frac{\sin\theta\, d\theta}{(1 + \cos\theta)\sqrt{\Theta}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (30^*),$$

then ψ is the sum of two elliptic integrals of the third kind, with Jacobian parameters v_1 and v_2; and Legendre's theorem for the addition of these parameters shows that these two integrals depend upon a single integral, of the form

$$\frac{C_1 n_1 - G_1 E}{2\sqrt{(2A_1 Wgh)}} \int \frac{\sin\theta\, d\theta}{(E-\cos\theta)\sqrt{\Theta}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31),$$

and we find, in fact (as is readily verified by a differentiation),

$$\psi = \frac{G_1 t}{2A_1} - \tan^{-1}\frac{\sqrt{(2A_1 Wgh)}\sqrt{\Theta}}{C_1 n_1 - G_1 \cos\theta} + \frac{C_1 n_1 - G_1 E}{2\sqrt{(2A_1 Wgh)}} \int \frac{\sin\theta\, d\theta}{(E-\cos\theta)\sqrt{\Theta}} \quad \ldots\ldots\ldots (32).$$

6. To agree again with Darboux's notation, we put

$$\frac{G_1^2 \Omega}{4A_1 Wgh} = L^2, \quad \frac{C_1^2 n^2 \Omega}{4A_1 Wgh} = B^2 \quad \ldots\ldots\ldots\ldots\ldots (33),$$

so that, from (22) and (23),

$$L^2 = -\wp v_1 - \wp v_2 - \wp v \quad \ldots\ldots\ldots\ldots\ldots\ldots (34),$$

$$B^2 = -\wp v_1 - \wp v_2 - \wp w \quad \ldots\ldots\ldots\ldots\ldots\ldots (35),$$

$$L^2 - B^2 = \wp w - \wp v \quad \ldots\ldots\ldots\ldots\ldots\ldots (35^*).$$

Then, from equation (25),

$$\cosh\theta_1 + \cos\theta_2 + \cos\theta_3 = E + \frac{2L^2}{\Omega} \quad \ldots\ldots\ldots\ldots (36),$$

and, from (28), by addition,

$$3\wp v = \tfrac{3}{2}\Omega E - \tfrac{1}{2}\Omega\,(\cosh\theta_1 + \cos\theta_2 + \cos\theta_3)$$
$$= \Omega E - L^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37),$$

so that, from (28), or (13) and (14),

$$\Omega\cos\theta = \Omega E - 2\wp v + 2\wp u$$
$$= L^2 + \wp v + 2\wp u$$
$$= 2\wp u - \wp v_1 - \wp v_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (38);$$

and therefore

$$\left.\begin{aligned} \Omega\cosh\theta_1 &= L^2 + \wp v + 2e_1 \\ \Omega\cos\theta_2 &= L^2 + \wp v + 2e_2 \\ \Omega\cos\theta_3 &= L^2 + \wp v + 2e_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (39).$$

Again, from (25),

$$\cos\theta_2\cos\theta_3+\cos\theta_3\cosh\theta_1+\cosh\theta_1\cos\theta_2$$
$$=-1+\frac{G_1C_1n_1}{A_1Wgh}=-1+\frac{2LC_1n_1}{\sqrt{(\Omega A_1Wgh)}},$$

so that, multiplying by Ω^2, and employing (39),

$$\frac{2LC_1n_1\Omega^{\frac{3}{2}}}{\sqrt{(A_1Wgh)}}=\Omega^2(1+\cos\theta_2\cos\theta_3+\cos\theta_3\cosh\theta_1+\cosh\theta_1\cos\theta_2)$$
$$=\Omega^2+3L^4+6L^2\wp v+3\wp^2v-g_2 \quad\ldots\ldots(40);$$

this relation is implied in Darboux's (18), Despeyrous, II., p. 515.

From (25), again, as well as (37),

$$i\wp'v=\frac{C_1n_1-GE}{2\sqrt{(A_1Wgh)}}\Omega^{\frac{3}{2}}$$
$$=\frac{C_1n_1\Omega^{\frac{3}{2}}}{2\sqrt{(A_1Wgh)}}-L\Omega E$$
$$=\frac{C_1n_1\Omega^{\frac{3}{2}}}{2\sqrt{(A_1Wgh)}}-L^3-3L\wp v,$$

so that, multiplying by L,

$$\frac{LC_1n_1\Omega^{\frac{3}{2}}}{2\sqrt{(A_1Wgh)}}=L^4+3L^2\wp v+Li\wp'v \quad\ldots\ldots(41),$$

or
$$B\Omega=L^3+3L\wp v+i\wp'v\ldots\ldots(41^*);$$

and therefore, from (40),

$$\Omega^2+3L^4+6L^2\wp v+3\wp^2v-g_2=4L^4+12L^2\wp v+4Li\wp'v,$$

or
$$\Omega^2=L^4+6L^2\wp v+4Li\wp'v-3\wp^2v+g_2$$
$$=(L^2+3\wp v)^2+4Li\wp'v-2\wp''v\ldots\ldots(42).$$

With this value of Ω we shall find

$$\tanh\theta_1=2\frac{-L\sqrt{(e_1-\wp v)}+\sqrt{(\wp v-e_2\,.\,\wp v-e_3)}}{L^2+\wp v+2e_1} \quad\ldots\ldots(43),$$

$$\tan\theta_2=2\frac{L\sqrt{(\wp v-e_2)}+\sqrt{(e_1-\wp v\,.\,\wp v-e_3)}}{L^2+\wp v+2e_2} \quad\ldots\ldots(44),$$

$$\tan\theta_3=2\frac{L\sqrt{(\wp v-e_3)}+\sqrt{(e_1-\wp v\,.\,\wp v-e_2)}}{L^2+\wp v+2e_3} \quad\ldots\ldots(45),$$

and the complete motion of the top can be made to depend upon the constants e_1, e_2, e_3, $\wp v$, and L.

7. When v is of the form

$$v = \omega_1 + \frac{P\omega_3}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46),$$

where P and μ are integers, the solution can be effected by the associated pseudo-elliptic integral of order μ, which we can write in the form

$$I\left(\omega_1 + \frac{P\omega_3}{\mu}\right) = \tfrac{1}{2}\int \frac{\rho\,(\sigma - s) - \mu\sqrt{(-\Sigma)}}{(\sigma - s)\sqrt{S}}\,ds$$

$$= \tfrac{1}{2}i\log\left\{\frac{\sigma\,(u+v)}{\sigma\,(u-v)}\right\}^{\mu} e^{-(\rho i + 2\mu\zeta v)u} \quad \ldots\ldots\ldots\ldots (47),$$

where (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 209)

$$S = 4s\,(s+x)^2 - \left\{(y+1)\,s + xy\right\}^2$$

$$= 4\,(s-s_1)(s-s_2)(s-s_3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48),$$

$$\sigma - s = \wp v - \wp u = \tfrac{1}{2}\Omega\,(E - \cos\theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (49),$$

and where Σ denotes the value of S when $s = \sigma$.

Then

$$I\left(\omega_1 + \frac{P\omega_3}{\mu}\right) = \tfrac{1}{2}\rho\int\frac{ds}{\sqrt{S}} - \mu\,\frac{C_1 n_1 - G_1 E}{2\sqrt{(2A_1 W g h)}}\int\frac{\sin\theta\,d\theta}{(E-\cos\theta)\sqrt{\Theta}}$$

$$= \frac{\rho}{2\sqrt{\Omega}}\int\frac{\sin\theta\,d\theta}{\sqrt{(2\Theta)}} + \frac{\mu G_1 t}{2A_1} - \mu\tan^{-1}\frac{\sqrt{(2A_1 W g h\Theta)}}{C_1 n_1 - G_1\cos\theta} - \mu\psi$$

$$= \frac{\rho + 2\mu L}{2\sqrt{\Omega}}\,nt - \mu\tan^{-1}\frac{\sqrt{(2A_1 W g h\Theta)}}{C_1 n_1 - G_1\cos\theta} - \mu\psi,$$

or

$$\mu\psi - \frac{\rho + 2\mu L}{2\sqrt{\Omega}}\,nt = -\mu\tan^{-1}\frac{\sqrt{(2A_1 W g h\Theta)}}{C_1 n_1 - G_1\cos\theta} - I\left(\omega_1 + \frac{P\omega_3}{\mu}\right) \ldots (50),$$

so that $\mu\psi$, with the addition of the secular term

$$-\frac{\rho + 2\mu L}{2\sqrt{\Omega}}\,nt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51),$$

can now be expressed as an inverse circular function of θ.

The secular term can be made to disappear by taking

$$L = -\frac{\rho}{2\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52);$$

and then $\quad (\sin\theta)^{\mu}\cos\mu\psi \quad$ and $\quad (\sin\theta)^{\mu}\sin\mu\psi$

are rational functions of $\cos\theta$, which can be determined by a verification consisting of differentiation and squaring and adding.

Writing $\sigma_1, \sigma_2, \sigma_3$ for $\sigma - s_1, \sigma - s_2, \sigma - s_3$,

respectively, then equations (39), (41), (42) can be written

$$\left.\begin{aligned} \Omega \cosh\theta_1 &= L^2 - \sigma_1 + \sigma_2 + \sigma_3 \\ \Omega \cos\theta_2 &= L^2 + \sigma_1 - \sigma_2 + \sigma_3 \\ \Omega \cos\theta_3 &= L^2 + \sigma_1 - \sigma_2 - \sigma_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (53),$$

$$\frac{C_1 n_1 \Omega^2}{2\sqrt{(A_1 W g h)}} = L^3 + L(\sigma_1 + \sigma_2 + \sigma_3) + \sqrt{(-\Sigma)} \quad \ldots\ldots\ldots (54),$$

$$\Omega^2 = (L^2 + \sigma_1 + \sigma_2 + \sigma_3)^2 + 4L\sqrt{(-\Sigma)} - 4(\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2) \ldots (55).$$

There are cusps on the circle $\theta = \theta_2$ when $w = \omega_2$; and then

$$\cos\theta_2 = \frac{d}{h} = \frac{G_1}{C_1 n_1} = \frac{1 + \cosh\theta_1 \cos\theta_3}{\cosh\theta_1 + \cos\theta_3}.$$

8. Thus, for instance, with $2\mu = 4$, we can take (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 212)

$$s_1 = (1+c)^2, \quad s_2 = c^2, \quad s_3 = 0, \quad \rho = 2,$$

$$\sigma = c + c^2, \quad \sqrt{(-\Sigma)} = 2(c + c^2) \quad \ldots\ldots\ldots\ldots\ldots (56),$$

and then

$$I(\omega_1 + \tfrac{1}{2}\omega_3) = \tfrac{1}{2}\int \frac{2(c+c^2-s) - 4(c+c^2)}{(c+c^2-s)\sqrt{S}}\, ds$$

$$= \cos^{-1}\frac{\sqrt{s}}{c+c^2-s} = \sin^{-1}\frac{\sqrt{\{(1+c)^2 - s \,.\, c^2 - s\}}}{c+c^2-s} \quad \ldots\ldots (57).$$

The secular term attached to 2ψ is destroyed by taking $L = -\frac{1}{2}$, so that, putting

$$c = \tfrac{1}{2}(2a-1), \quad 1 + c = \tfrac{1}{2}(2a+1),$$

$$\Omega^2 = a^2(a^2+2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58),$$

$$\left.\begin{aligned} \cosh\theta_1 &= \frac{a+2}{\sqrt{(a^2+2)}}, & \sinh\theta_1 &= \sqrt{\left(\frac{4a+2}{a^2+2}\right)} \\ \cos\theta_2 &= \frac{a-2}{\sqrt{(a^2+2)}}, & \sin\theta_2 &= \sqrt{\left(\frac{4a-2}{a^2+2}\right)} \\ \cos\theta_3 &= -\frac{2a^2+1}{2a\sqrt{(a^2+2)}}, & \sin\theta_3 &= \frac{1}{2a}\sqrt{\left(\frac{4a^2-1}{a^2+2}\right)} \end{aligned}\right\} \quad \ldots\ldots (59),$$

$$\frac{G_1^2}{A_1 W g h} = \frac{1}{a\sqrt{(a^2+2)}}, \quad \frac{C_1^2 n_1^2}{A_1 W g h} = \frac{9a}{(a^2+2)^{\frac{3}{2}}} \quad \ldots\ldots\ldots\ldots(60),$$

$$L^2 = \tfrac{1}{4}, \quad B^2 = \frac{9a^2}{4(a^2+2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(60^*),$$

and the cone described by the axis of the top is given by

$$\sin^2\theta\, e^{2\psi i} = \frac{2\sqrt{2}\sqrt{a}}{(a^2+2)^{\frac{3}{4}}}\sqrt{(\cos\theta - \cos\theta_3)}$$

$$+ i\left\{\cos\theta + \frac{a}{\sqrt{(a^2+2)}}\right\}\sqrt{(\cosh\theta_1 - \cos\theta\,.\,\cos\theta_2 - \cos\theta)} \ldots(61).$$

When $a = 1$ or $c = \frac{1}{2}$, there are four cusps on the circle

$$\theta = \theta_2 = \cos^{-1}(-\tfrac{1}{3}\sqrt{3});$$

and the time occupied by the axis of the top in describing the four loops is $4 \times 3^{-\frac{1}{2}}$ times the period when making plane oscillations through an angle

$$4\sin^{-1}\tfrac{1}{3}.$$

9. So also with $2\mu = 6$, and the corresponding parameters

$$v = \omega_1 + \tfrac{1}{3}\omega_3, \quad \text{or} \quad \omega_1 + \tfrac{2}{3}\omega_3,$$

we take

$$s_1 = (1-c)^2, \quad s_2 = c^2, \quad s_3 = (c-c^2)^2,$$

$$\sigma = 2c(1-c)^2, \quad \text{or} \quad 2c^2 - 2c^3,$$

$$\rho = 2(2-c)(1-2c), \quad \text{or} \quad 2(1+c)(1-2c) \ldots\ldots(62),$$

$$\sqrt{(-\Sigma)} = 2c(1-c)^2(2-c)(1-2c),$$

or

$$2c^2(1-c)(1+c)(1-2c) \ldots\ldots\ldots\ldots\ldots(62^*),$$

and then the corresponding pseudo-elliptic integrals (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 218)

$$I(\omega_1 + \tfrac{1}{3}\omega_3) \quad \text{or} \quad I(\omega_1 + \tfrac{2}{3}\omega_3)$$

will serve to construct other solvable cases of top motion.

Putting

$$S = 4(s-s_1)(s-s_2)(s-s_3),$$

these integrals are

$$I(\omega_1 + \tfrac{1}{3}\omega_3)$$

$$= \tfrac{1}{2}\int \frac{2(2-c)(1-2c)\left\{2c(1-c)^2 - s\right\} - 6c(1-c)^2(2-c)(1-2c)}{\left\{2c(1-c)^2 - s\right\}\sqrt{S}}\, ds$$

$$= \sin^{-1}\frac{\left\{s - (1-c)^2(2-3c+2c^2)\right\}\sqrt{(c^2-s)}}{\left\{2c(1-c^2) - s\right\}^{\frac{3}{2}}}$$

$$= \cos^{-1}\frac{(2-c)(1-2c)\sqrt{\left\{(1-c)^2 - s\,.\,s - (c-c^2)^2\right\}}}{\left\{2c(1-c)^2 - s\right\}^{\frac{1}{2}}}$$

$$I(\omega_1+\tfrac{2}{3}\omega_3)$$

$$= \tfrac{1}{2}\int \frac{2(1+c)(1-2c)(2c^2-2c^3-s)-6c^2(1-c)(1+c)(1-2c)}{(2c^2-2c^3-s)\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(s-c^2+c^3-2c^4)\sqrt{\{(1-c)^2-s\}}}{(2c^2-2c^3-s)^{\frac{3}{2}}}$$

$$= \sin^{-1}\frac{(1+c)(1-2c)\sqrt{\{c^2-s\,.\,s-(c-c^2)^2\}}}{(2c^2-2c^3-s)^{\frac{3}{2}}}.$$

10. First, when $v = \omega_1+\frac{1}{3}\omega_3$,

and $\rho = 2(2-c)(1-2c)$,

the secular term associated with 3ψ is made to vanish by putting

$$L = -\tfrac{1}{6}\rho = -\tfrac{1}{3}(2-c)(1-2c),$$

and now, from (42) and (53),

$$81\Omega^2 = (1+c)^2\{27(1-c)^6-2(1-4c+c^2)^3\},$$

$$9\Omega\cosh\theta_1 = (1+c)(13-33c+21c^2-5c^3),$$

$$9\Omega\cos\theta_2 = -(5-16c+12c^2-16c^3+5c^4),$$

$$9\Omega\cos\theta_3 = -(1+c)(5-21c+33c^2-13c^3).$$

From (39), (43), (44), (45),

$$3\Omega\sinh\theta_1 = 2(1-c^2)(2-c)\sqrt{(1-2c)},$$

$$3\Omega\sin\theta_2 = 2(1-c+c^2)\sqrt{(1-2c\,.\,2c-c^2)},$$

$$3\Omega\sin\theta_3 = 2(1-c^2)(1-2c)\sqrt{(2c-c^2)}.$$

The equation connecting θ and ψ can now be written in the form

$$\sin^3\theta\cos 3\psi = (Q\cos\theta-R)\sqrt{(\cos\theta_2-\cos\theta)},$$

or

$$\sin^3\theta\sin 3\psi = (\cos^2\theta-C\cos\theta+D)\sqrt{(\cosh\theta_1-\cos\theta\,.\,\cos\theta-\cos\theta_3)},$$

and, we find by squaring and adding, that

$$C = -\tfrac{1}{2}(\cosh\theta_1+\cos\theta_3)$$

$$= -\frac{2(1+c)^2(2-c)(1-2c)}{9\Omega},$$

$$D = \frac{(1+c)^2(19-84c+141c^2-160c^3+141c^4-84c^5+19c^6)}{81\Omega^2},$$

$$Q = \frac{2\sqrt{2}\,(1+c)^2(2-5c+2c^2)(5-8c+5c^2)}{(9\Omega)^{\frac{3}{2}}},$$

$$R = -\frac{2\sqrt{2}\,(1+c)^2(2-5c+2c^2)(7-12c-3c^2+32c^3-3c^4-12c^5+7c^6)}{(9\Omega)^{\frac{5}{2}}};$$

and by a logarithmic differentiation, and comparison with (30),

$$L = \frac{G_1\sqrt{\Omega}}{2\sqrt{(A_1Wgh)}} = -\frac{2-5c+2c^2}{3},$$

$$B = \frac{C_1n_1\sqrt{\Omega}}{2\sqrt{(A_1Wgh)}} = \frac{(1+c)^2(2-5c+2c^2)(5-8c+5c^2)}{27\Omega}.$$

A point on the axis OC now describes a closed spherical curve with six loops or waves; and, when $c = 2-\sqrt{3}$, there are six cusps on the circle $\theta = \theta_2 = \frac{2}{3}\pi$; and the time of describing the six loops is $3^{\frac{3}{2}}$ times the period when making plane oscillations through an angle of 60°.

11. Secondly, when $v = \omega_1 + \frac{2}{3}\omega_3$,

and $\rho = 2(1+c)(1-2c)$,

the secular term associated with 3ψ disappears when

$$L = -\tfrac{1}{6}\rho = -\tfrac{1}{3}(1+c)(1-2c);$$

and now
$$81\Omega^2 = (2-c)^2\{2(2-2c-c^2)^3+27c^6\},$$
$$9\Omega\cosh\theta_1 = 10-20c+6c^2+4c^3-5c^4,$$
$$9\Omega\cos\theta_2 = -(2-c)(4-6c-6c^2-5c^3),$$
$$9\Omega\cos\theta_3 = -(2-c)(4-6c-6c^2+13c^3).$$

The equations connecting θ and ψ are now of the form

$$\sin^3\theta\cos 3\psi = (Q\cos\theta - R)\sqrt{(\cosh\theta_1-\cos\theta)},$$

or

$$\sin^3\theta\sin 3\psi = (\cos^2\theta - C\cos\theta + D)\sqrt{(\cos\theta_2-\cos\theta\,.\,\cos\theta-\cos\theta_3)},$$

and we find

$$C = -\tfrac{1}{2}(\cos\theta_2+\cos\theta_3) = \frac{2(1+c)(2-c)^2(1-2c)}{9\Omega},$$

$$D = -\frac{(2-c)^2(8-24c+48c^2-20c^3-6c^4+30c^5-19c^6)}{81\Omega^2},$$

$Q =$ ... ... $R =$ ... ...

obtainable from the preceding values by writing $1-c$ for c.

A point on the axis OC describes a closed spherical curve with three loops or waves; and, when

$$c = \sqrt[3]{4} - \sqrt[3]{2},$$

there are three cusps on the circle

$$\theta = \theta_2 = \pi - \tan^{-1} \sqrt[3]{2};$$

and the time of describing the three loops is

$$\frac{3}{\sqrt[3]{2}\,\sqrt{(3-\sqrt[3]{2})}\,\sqrt[4]{(\sqrt[3]{4}+1)}}$$

times the period of plane oscillations through an angle

$$4 \tan^{-1} (2 - \sqrt[3]{4}).$$

So also for higher values of 2μ, namely, 8, 10, 12, 14, 16, 18, ...; the even values being taken because the resolution of the cubic S is required in these dynamical applications.

Jacobi's Theorems on the Motion of a Top.

12. So far the treatment of the motion of the axis of a top, as given in the *Proc. Lond. Math. Soc.*, Vol. xxv., p. 291, has been amplified to a certain extent; but now we proceed to introduce Jacobi's theorems (*Gesammelte Werke*, Vol. II., p. 480).

Measure off a length OG along the upward vertical from O, representing to an appropriate scale the dynamical quantity G_1; and measure off OC along the axis of the top, to represent to the same scale the dynamical quantity $C_1 n_1$; draw the horizontal plane through G perpendicular to OG, and call this *the invariable plane of* G; and draw the plane through C perpendicular to OC, and call it *the invariable plane of* C (Fig. 1).

Then, if the vector OH represents to the same scale the resultant angular momentum of the system, the point H must lie in the line of intersection of the invariable planes of G and C, because the components of angular momentum about the vertical OG and about the axis OC are G_1 and $C_1 n_1$ respectively.

If this line of intersection cuts the vertical plane GOC in K, then

$$CH^2 - GH^2 = CK^2 - GK^2 = OG^2 - OC^2 = G_1^2 - C_1^2 n_1^2 \quad \ldots\ldots(63).$$

13. The point H moves in the invariable plane of G with velocity equal to the moment of the impressed couple of gravity, and parallel to the axis of this couple.

The velocity of H is therefore in the direction HK, perpendicular to the plane GOC, and equal to $Wgh \sin\theta$; and the moment of this velocity about G is

$$Wgh \sin\theta \,.\, GK = Wgh\,(OC - OG\cos\theta) \quad \ldots\ldots\ldots\ldots(64),$$

so that

$$\rho^2 \frac{d\varpi}{dt} = Wgh\,(C_1 n_1 - G_1 \cos\theta) \quad \ldots\ldots\ldots\ldots(65),$$

if ρ, ϖ denote the polar coordinates of H in the invariable plane of G.

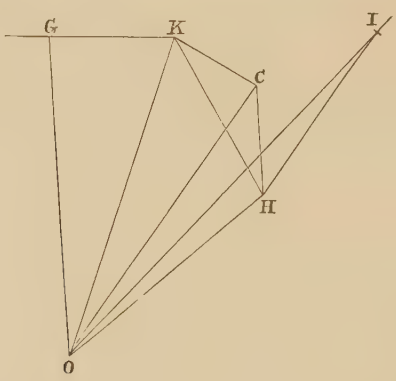

Fig. 1.

Again, in the notation of Routh's *Rigid Dynamics*, ω_1 and ω_2 now denoting components of the angular velocity,

$$\begin{aligned} OH^2 &= A_1^2(\omega_1^2 + \omega_2^2) + C_1^2 n_1^2 \\ &= A^2\left(\frac{d\theta^2}{dt^2} + \sin^2\theta\,\frac{d\psi^2}{dt^2}\right) + C_1^2 n_1^2 \\ &= 2A_1 Wg\,(d - h\cos\theta) + C_1^2 n_1^2 \quad \ldots\ldots\ldots\ldots(66), \end{aligned}$$

so that, from (26) and (28),

$$\begin{aligned} GH^2 = \rho^2 &= 2A_1 Wg\,(d - h\cos\theta) + C_1^2 n_1^2 - G_1^2 \\ &= 2A_1 Wgh\,(E - \cos\theta) \\ &= \frac{4A_1 Wgh}{\Omega}\,(\wp v - \wp' u) \quad \ldots\ldots\ldots\ldots(67). \end{aligned}$$

Therefore, from (65),

$$\frac{d\varpi}{dt} = \frac{C_1 n_1 - G_1 \cos\theta}{2A_1 (E - \cos\theta)}$$

$$= \frac{G_1}{2A_1} + \frac{C_1 n_1 - G_1 E}{2A_1} \frac{1}{E - \cos\theta},$$

$$\varpi = \frac{G_1 t}{2A_1} + \frac{C_1 n_1 - G_1 E}{2\sqrt{(2A_1 W g h)}} \int \frac{\sin\theta\, d\theta}{(E - \cos\theta)\sqrt{\Theta}}$$

$$= \frac{G_1 t}{2A_1} + \tfrac{1}{2} i \int \frac{\wp' v\, du}{\wp v - \wp u} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (68),$$

which, combined with (67), give the well known relations of a *herpolhode;* thus H describes a herpolhode in the invariable plane of G, with parameter v; this is one of Jacobi's theorems.

14. A reference to (32) shows that the angle between the vertical planes GOC and GOH, or

$$\varpi - \psi = \tan^{-1} \frac{\sqrt{(2A_1 W g h \Theta)}}{C_1 n_1 - G_1 \cos\theta}$$

$$= \sin^{-1} \frac{\sqrt{\Theta}}{\sin\theta \sqrt{(E - \cos\theta)}}$$

$$= \cos^{-1} \frac{C_1 n_1 - G_1 \cos\theta}{\sin\theta \sqrt{(2A_1 W g h)} \sqrt{(E - \cos\theta)}} \quad \ldots\ldots\ldots (69),$$

so that the herpolhode of H is algebraical when ψ is pseudo-elliptic, and when the accompanying secular term is at the same time made to vanish.

The tangent at H being perpendicular to the plane GOC, it follows that this plane is stationary, as H passes through a point of inflexion on the herpolhode; the herpolhode must therefore have points of inflexion when the path of a point C on the axis of the top is looped.

Generally, the component velocity of C perpendicular to the plane GOC is

$$C_1 n_1 \sin\theta \frac{d\psi}{dt} = \frac{C_1^2 n_1^2}{A_1} \frac{G_1 - C_1 n_1 \cos\theta}{C_1 n_1 \sin\theta}$$

$$= \frac{C_1^2 n_1^2}{A_1} \tan CGK,$$

$$A_1 \sin\theta \frac{d\psi}{dt} = C_1 n_1 \tan CGK = CK \quad \ldots\ldots\ldots\ldots (70).$$

This vanishes, and the plane GOC is stationary, when C lies in the invariable plane of G, and is therefore coincident with K; and the angle between the planes GOC and COH is then a right angle.

Fig. 1 shows immediately that the angle between the planes GOC and GOH, or

$$\varpi - \psi = \cos^{-1}\frac{GK}{GH} = \cos^{-1}\frac{C_1 n_1 - G_1\cos\theta}{\sin\theta\sqrt{(2A_1 Wgh)}\sqrt{(E-\cos\theta)}},$$

because $$GH^2 = 2A_1 Wgh\,(E-\cos\theta),$$

and $$GK\sin\theta = OC - OG\cos\theta = C_1 n_1 - G_1\cos\theta;$$

and therefore also

$$KH^2 = 2A_1 Wgh\,(E-\cos\theta) - \frac{(C_1 n_1 - G_1\cos\theta)^2}{\sin^2\theta}$$

$$= 2A_1 Wgh\,\frac{\Theta}{\sin^2\theta} = A_1^2\left(\frac{d\theta}{dt}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(71),$$

$$CH^2 = KH^2 + KC^2 = A_1^2\left\{\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\psi}{dt}\right)^2\right\} \quad \ldots(71^*).$$

15. Similarly, the angle between the planes GOC and HOC is

$$\cos^{-1}\frac{CK}{CH} = \cos^{-1}\frac{G_1 - C_1 n_1\cos\theta}{\sin\theta\sqrt{(2A_1 Wgh)}\sqrt{(D-\cos\theta)}} \quad \ldots\ldots(72),$$

on putting $$\frac{d}{h} = D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(73);$$

this property will enable us to prove the second of Jacobi's theorems, which asserts that the path of H in the invariable plane of C is another herpolhode, and that its parameter is

$$v_1 - v_2 = w$$

(*Gesammelte Werke*, Vol. II., Note B, p. 476).

Employing accented letters, ρ' and ϖ', to denote the polar coordinates of H in the invariable plane of C, then, from (66) and (24),

$$\rho'^2 = CH^2 = OH^2 - OC^2$$

$$= 2A_1 Wg\,(d - h\cos\theta)$$

$$= 2A_1 Wgh\,(D-\cos\theta)$$

$$= \frac{4A_1 Wgh}{\Omega}(\wp w - \wp u) \quad \ldots\ldots\ldots\ldots\ldots(74).$$

The angle ϖ' being measured from a straight line OA, fixed in the body at right angles to OC, and the angle between the planes AOC and GOC being denoted, as in Euler's notation, by ϕ, then the angle between the planes GOC and HOC is $\varpi'-\phi$; so that

$$\varpi'-\phi = \cos^{-1}\frac{G_1-C_1n_1\cos\theta}{\sin\theta\sqrt{(2A_1Wgh)}\sqrt{(D-\cos\theta)}}$$

$$= \sin^{-1}\frac{\sqrt{\Theta}}{\sin\theta\sqrt{(D-\cos\theta)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(75),$$

analogous to (69).

But, from Euler's relations,

$$\frac{d\phi}{dt} = n_1-\cos\theta\frac{d\psi}{dt}$$

$$= \left(1-\frac{C_1}{A_1}\right)n_1+\frac{C_1n_1-G_1\cos\theta}{A_1\sin^2\theta},$$

so that, with

$$\frac{d\cos\theta}{dt} = -\sqrt{\left(\frac{2Wgh}{A_1}\right)}\sqrt{\Theta},$$

$$\frac{d\varpi'}{dt} = \frac{d\phi}{dt}+\frac{d}{dt}\cos^{-1}\frac{G_1-C_1n_1\cos\theta}{\sin\theta\sqrt{(2A_1Wgh)}\sqrt{(D-\cos\theta)}},$$

and, after reduction, we find

$$\frac{d\varpi'}{dt} = \left(1-\frac{C_1}{A_1}\right)n_1+\frac{G_1-C_1n_1\cos\theta}{2A_1(D-\cos\theta)}$$

$$= \left(1-\tfrac{1}{2}\frac{C_1}{A_1}\right)n_1+\frac{G_1-C_1n_1D}{2A_1}\frac{1}{D-\cos\theta} \quad \ldots\ldots\ldots\ldots\ldots(76),$$

or

$$\varpi' = \left(1-\tfrac{1}{2}\frac{C_1}{A_1}\right)n_1t+\frac{G_1-C_1n_1D}{2\sqrt{(2A_1Wgh)}}\int\frac{\sin d\theta}{(D-\cos\theta)\sqrt{\Theta}}$$

$$= \left(1-\tfrac{1}{2}\frac{C_1}{A_1}\right)n_1t+\tfrac{1}{2}i\int\frac{\wp'w\,du}{\wp w-\wp u} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(77),$$

which, combined with the value of ρ'^2 in (74), proves the second part of Jacobi's theorem, that H describes in the invariable plane of C a herpolhode of parameter

$$w = v_1-v_2$$

16. By means of Euler's three angles θ, ϕ, ψ, the position of the top as a solid body is completely determined, the formulas being

$$u = qt + \omega_2 \quad \text{or} \quad qt + \omega_3,$$

$$\tan^2 \tfrac{1}{2}\theta = \frac{1-\cos\theta}{1+\cos\theta} = \frac{\wp \tfrac{1}{2}(v+w) - \wp u}{\wp u - \wp \tfrac{1}{2}(v-w)} \quad \ldots\ldots\ldots\ldots\ldots (78),$$

$$\phi = \varpi' - \sin^{-1} \frac{\sqrt{\Theta}}{\sin\theta \sqrt{(D-\cos\theta)}}$$

$$= \left(1 - \tfrac{1}{2}\frac{C_1}{A_1}\right) n_1 t - \sin^{-1} \frac{\sqrt{\Theta}}{\sin\theta\sqrt{(D-\cos\theta)}} + \tfrac{1}{2} i \log \frac{\sigma(u+w)}{\sigma(u-w)} e^{-2u\zeta w}$$

$\ldots\ldots\ldots\ldots (79),$

$$\psi = \varpi - \sin^{-1} \frac{\sqrt{\Theta}}{\sin\theta\sqrt{(E-\cos\theta)}}$$

$$= \frac{G_1 t}{2A_1} - \sin^{-1} \frac{\sqrt{\Theta}}{\sin\theta\sqrt{(E-\cos\theta)}} + \tfrac{1}{2} i \log \frac{\sigma(u+v)}{\sigma(u-v)} e^{-2u\zeta v} \ldots\ldots\ldots (80).$$

17. Since the axis OI of instantaneous rotation lies in the plane HOC, the direction of motion of C is perpendicular to this plane; and therefore the path of C cuts the vertical plane GOC at an angle

$$\tan^{-1} \frac{G_1 - C_1 n_1 \cos\theta}{\sqrt{(2A_1 W g h \Theta)}} = \cos^{-1} \frac{\sqrt{\Theta}}{\sin\theta\sqrt{(D-\cos\theta)}} \quad \ldots\ldots\ldots (81),$$

or it cuts the horizontal circle through C at an angle $\varpi' - \phi$; and this is a right angle when the plane GOC is stationary.

As H passes through a point of inflexion of the herpolhode in the invariable plane of C, the plane HOC is stationary; and C at the same time passes through a point of inflexion on its spherical path.

18. When the momental ellipsoid at O becomes a sphere, or

$$C_1 = A_1,$$

the axis OI of instantaneous angular velocity ω coincides with OH, and

$$OH = A_1 \omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (82).$$

But in the general case, when the momental ellipsoid at O is a spheroid, take a fixed point F in OC, such that

$$\frac{OF}{OC} = \frac{A_1}{C_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (83),$$

and call the plane through F perpendicular to OF the *invariable plane of* F (Fig. 1).

Now, if HI, drawn parallel to OC, cuts the invariable plane of F in I, the vector OI will represent $A_1\omega$, or A_1 times the resultant angular velocity; and I describes a herpolhode in the invariable plane of F equal and parallel to the herpolhode described by H in the invariable plane of C.

It can readily be proved now that the angle between the vertical planes GOC and GOI is

$$\cos^{-1}\frac{(A_1\cos^2\theta+C_1\sin^2\theta)\,n_1-G_1\cos\theta}{\sin\theta\sqrt{\{2A_1Wgh\,(E-\cos\theta)\}}} \quad \ldots\ldots\ldots\ldots(84),$$

reducing to (69) when $A_1=C_1$.

Darboux's Mechanical Representation of the Motion of the Axis of a Top.

19. M. Darboux has shown, in Notes xviii. and xix. of Despeyrous' *Cours de Mécanique*, how the generating lines of an articulated deformable hyperboloid can be employed to imitate the motion of the axis of a top.

We begin with the consideration of the properties of the confocal system of quadrics, given by

$$\frac{x^2}{\alpha^2+\lambda}+\frac{y^2}{\beta^2+\lambda}+\frac{z^2}{\lambda}=1 \quad \ldots\ldots\ldots\ldots(85),$$

$$\frac{x^2}{\alpha^2+\mu}+\frac{y^2}{\beta^2+\mu}+\frac{z^2}{\mu}=1 \quad \ldots\ldots\ldots\ldots(86),$$

$$\frac{x^2}{\alpha^2+\nu}+\frac{y^2}{\beta^2+\nu}+\frac{z^2}{\nu}=1 \quad \ldots\ldots\ldots\ldots(87),$$

having the focal ellipse

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{0}=1 \ldots\ldots\ldots\ldots(88),$$

and the focal hyperbola

$$\frac{x^2}{\alpha^2-\beta^2}+\frac{y^2}{0}+\frac{z^2}{-\beta^2}=1 \ldots\ldots\ldots\ldots(89).$$

We can now put, employing m as a homogeneity factor,

$$\left.\begin{array}{lll} \alpha^2+\lambda=m^2(e_1-\wp v_2), & \beta^2+\lambda=m^2(e_2-\wp v_2), & \lambda=m^2(e_3-\wp v_2) \\ \alpha^2+\mu=m^2(e_1-\wp u), & \beta^2+\mu=m^2(e_2-\wp u), & \mu=m^2(e_3-\wp u) \\ \alpha^2+\nu=m^2(e_1-\wp v_1), & \beta^2+\nu=m^2(e_2-\wp v_1), & \nu=m^2(e_3-\wp v_1) \end{array}\right\} \ldots\ldots\ldots\ldots(90),$$

$$\left.\begin{aligned} x^2 &= \frac{\alpha^2+\lambda\,.\,\alpha^2+\mu\,.\,\alpha^2+\nu}{\alpha^2-\beta^2\,.\,\alpha^2} = m^2\,\frac{e_1-\wp v_2\,.\,e_1-\wp u\,.\,e_1-\wp v_1}{e_1-e_2\,.\,e_1-e_3} \\ y^2 &= \frac{\beta^2+\lambda\,.\,\beta^2+\mu\,.\,\beta^2+\nu}{\beta^2-\alpha^2\,.\,\beta^2} = m^2\,\frac{e_2-\wp v_2\,.\,e_2-\wp u\,.\,e_2-\wp v_1}{e_2-e_3\,.\,e_2-e_1} \\ z^2 &= \frac{\lambda\mu\nu}{\alpha^2\beta^2} = m^2\,\frac{e_3-\wp v_2\,.\,e_3-\wp u\,.\,e_3-\wp v_1}{e_3-e_1\,.\,e_3-e_2} \end{aligned}\right\} \quad \ldots(91),$$

where $v_2 = p\omega_3$, for the ellipsoid,

$u = \omega_3 + qt$, for the hyperboloid of one sheet,

$v_1 = \omega_1 + r\omega_3$, for the hyperboloid of two sheets;

and now
$$\frac{\beta^2}{\alpha^2} = \frac{e_2-e_3}{e_1-e_3} = \kappa^2 \quad \ldots\ldots(92),$$

so that the modulus of the elliptic functions is the ratio of the axes of the focal ellipse.

Then (Salmon, *Solid Geometry*, Chap. VIII.)

$$\begin{aligned} x^2+y^2+z^2 &= \alpha^2+\lambda+\beta^2+\mu+\nu \\ &= m^2\,(-\wp v_2-\wp u-\wp v_1 \quad \ldots\ldots(93), \end{aligned}$$

and the squares of the semi-axes of the central section made by a plane parallel to the tangent plane of the hyperboloid (86) are

$$\mu-\lambda \quad \text{and} \quad \mu-\nu;$$

so that, if θ denotes the angle between the generating lines of the hyperboloid of one sheet (86),

$$\tan^2\tfrac{1}{2}\theta = -\frac{\mu-\lambda}{\mu-\nu} \quad \ldots\ldots(94),$$

$$\cos\theta = \frac{\lambda-2u+\nu}{\lambda-\nu} = \frac{2\wp u-\wp v_1-\wp v_2}{\wp v_1-\wp v_2} \quad \ldots\ldots(95),$$

and we notice that $\lambda = \mu$ or $\wp u = \wp v_2$

makes $\cos\theta = -1$,

while $\mu = \nu$ or $\wp u = \wp v_1$

makes $\cos\theta = 1$,

as before, in the top; so that we can carry on with the previous notation of § 3.

Also, from (23) and (66),

$$OH^2 = \frac{4A_1Wgh}{\Omega}\{\wp(v_1-v_2)-\wp u\}$$

$$+\frac{4A_1Wgh}{\Omega}\{\wp v_1-\wp v_2-\wp(v_1-v_2)\}$$

$$= \frac{4A_1Wgh}{\Omega}(-\wp v_1-\wp u-\wp v_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(96),$$

so that, with

$$m^2 = \frac{4A_1Wgh}{\Omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots(97),$$

we may take the point H at (x, y, z) on the hyperboloid of one sheet, which is then moved so that one generating line through H is vertical, and then the other generating line will keep parallel to the axis of the top.

20. To hold this hyperboloid in position, M. Darboux employs a second hyperboloid of half the size, two generating lines being coincident with those passing through H, and the opposite pair being the lines OG and OC, passing through O (Fig. 2).

The generator OG being held vertical, any point H in the parallel opposite generator HJ will describe a horizontal plane; and now, if H is guided along a herpolhode, always moving perpendicular to the plane GOC, that is, normally to the hyperboloid, the generator OC will imitate the motion of the axis of a top.

21. The instantaneous axis of rotation will be represented by the vector OI to a point I fixed in the generator through H, parallel to OC; and it has already been shown in § 18 that I describes a herpolhode in the invariable plane of F.

The point I can be joined to a certain fixed point G' on OG by a generating line IG' of fixed length, and I is therefore constrained to lie on a sphere, with centre G'; hence Darboux's theorem, that the motion of the top can be imitated by rolling the herpolhode of I in the invariable plane of F on a fixed sphere, with centre in OG, the angular velocity being proportional to OI (Despeyrous, II., p. 538).

22. To construct these hyperboloids in Henrici's manner, consider them when flattened in the plane of the focal ellipse, corresponding to

$$\mu = 0, \quad u = \omega_3.$$

The coordinates of H are now given by

$$x^2 = \frac{\alpha^2+\lambda \,.\, \alpha^2+\nu}{\alpha^2-\beta^2} = m^2 \frac{e_1-\wp v_2 \,.\, e_1-\wp v_1}{e_1-e_2},$$

$$y^2 = \frac{\beta^2+\lambda \,.\, \beta^2+\nu}{\beta^2-\alpha^2} = m^2 \frac{e_2-\wp v_2 \,.\, e_2-\wp v_1}{e_2-e_1},$$

$$OH^2 = x^2+y^2 = m^2 (-\wp v_1-\wp v_2-e_3),$$

and if S, S' denote the foci of the focal ellipse,

$$SH \,.\, S'H = m^2 (\wp v_1-\wp v_2) = m^2\Omega = 4A_1 Wgh \ldots\ldots\ldots (98).$$

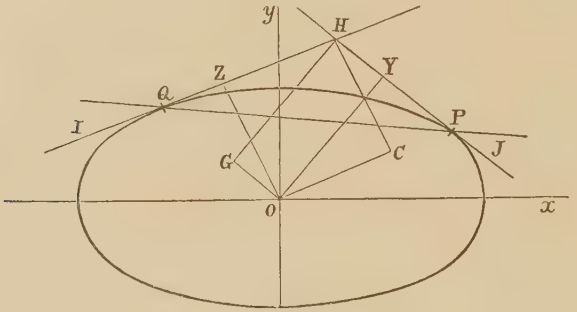

Fig. 2.

Drawing the tangents HJ and HI through H to the focal ellipse, and the perpendiculars OY and OZ upon them from the centre O; drawing also the perpendicular HG and HC upon the lines OG and OC through O parallel to the tangents HJ and HI, then we find that

$$OY^2 = GH^2 = \rho^2 = m^2 (\wp v-e_3) \ldots\ldots\ldots\ldots\ldots (99),$$

$$OZ^2 = CH^2 = \rho'^2 = m^2 (\wp w-e_3) \ldots\ldots\ldots\ldots\ldots (100);$$

and therefore

$$OG^2 = HY^2 = m^2 (-\wp v_1-\wp v_2-\wp v) = m^2 L^2 \ldots\ldots\ldots (101),$$

$$OC^2 = HZ^2 = m^2 (-\wp v_1-\wp v_2-\wp w) = m^2 B^2 \ldots\ldots\ldots (102).$$

The coordinates of P and Q, the points of contact of the tangents HJ and HI, will be given by

$$\left.\begin{aligned} \frac{x^2}{\alpha^2} &= \frac{e_1-e_3}{e_1-e_2}\frac{\wp v-e_2}{\wp v-e_3}, \text{ and } \frac{e_1-e_3}{e_1-e_2}\frac{\wp w-e_2}{\wp w-e_3} \\ \frac{y^2}{\beta^2} &= \frac{e_2-e_3}{e_1-e_2}\frac{e_1-\wp v}{\wp v-e_3}, \text{ and } \frac{e_2-e_3}{e_1-e_2}\frac{\wp_1-\wp w}{\wp w-e_3} \end{aligned}\right\} \ldots\ldots\ldots (103).$$

Any other two pairs of tangents to the focal ellipse will mark the position of the requisite number of rods, to serve as generating

lines connecting the opposite pairs HI, HJ and $H'I'$, $H'J'$; and now the design of the larger hyperboloid is complete; the smaller hyperboloid of half the scale having HI, HJ and OC, OG as opposite pairs of generators.

23. When flattened in the plane of the focal ellipse, H is at its maximum distance from O, and the angle GOC is θ_3, the maximum value of θ.

As the articulated model is gradually deformed, e_3 must be replaced by the variable $\wp u$, and

$$OY^2 = GH^2 = \rho^2 = m^2(\wp v - \wp u) \quad \ldots\ldots\ldots\ldots\ldots\ldots (104),$$

$$OZ^2 = CH^2 = \rho'^2 = m^2(\wp w - \wp u) \quad \ldots\ldots\ldots\ldots\ldots\ldots (105),$$

but OG, OC, HY, HZ remain constant.

When the model is flattened in the plane of the focal hyperbola,

$$u = \omega_2, \quad \wp u = e_2,$$

and OH has its minimum value; and the angle between OG and OC becomes θ_2, the minimum value of θ.

24. When $G_1 = 0$ or $L = 0$, the point H must move to Y, a point on the pedal of the focal ellipse with respect to the centre; and then

$$\wp' a = \wp' b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (106).$$

So, also, when $C_1 n_1 = 0$ or $B = 0$, as in the spherical pendulum, then

$$\wp' a = -\wp' b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (107),$$

and the point H must move to Z, on the pedal of the focal ellipse; we thus obtain a geometrical interpretation of the equation

$$\wp' u = e \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (108),$$

discussed by Halphen in his *Fonctions elliptiques*, t. I., p. 110.

Equation (41) shows that, in the spherical pendulum,

$$L^3 + 3L\wp v + i\wp' v = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (109),$$

or
$$L = \left\{\sqrt{(\wp^3 - \tfrac{1}{4}\wp'^2)} - \tfrac{1}{2}i\wp'\right\}^{\frac{1}{3}} - \left\{\sqrt{(\wp^3 - \tfrac{1}{4}\wp'^2)} + \tfrac{1}{2}i\wp'\right\}^{\frac{1}{3}} \quad \ldots (110),$$

and this is the condition that

$$\frac{d}{du}\,\frac{\sigma(u+v)}{\sigma u\,\sigma v}\,e^{(iL-\zeta v)u}$$

should be a solution of Lamé's equation for $n = 2$.

This relation can also be written

$$-\frac{2}{L}+\frac{1}{L+\sqrt{\left(\frac{\sigma_2\sigma_3}{-\sigma_1}\right)}}+\frac{1}{L+\sqrt{\left(\frac{\sigma_3\sigma_1}{-\sigma_2}\right)}}+\frac{1}{L+\sqrt{\left(\frac{\sigma_1\sigma_2}{-\sigma_3}\right)}}=0,$$

or (§ 27)
$$\frac{2}{h}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \quad \text{.....(111)}$$

in Darboux's notation (Halphen, *F.E.*, II., p. 102), or

$$2\frac{G^2}{T}=2D=A+B+C,$$

in Dr. Routh's notation.

Generally, in Darboux's notation,

$$B\Omega=abch\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{2}{h}\right)$$

$$=h\,(bc+ca+ab)-2abc,$$

or
$$h'\Omega=Qh-2R,$$

as in Darboux's equations (18), p. 515, or (6), p 531 (Despeyrous, *Cours de Mécanique*, t. II.).

25. Along the generator OG or HJ the parameter

$$v_1+v_2=v$$

is constant; while

$$v_1-v_2=w$$

is constant along OC or HI.

Starting with H at the point Y, when G_1 and $L=0$, then, for any other position of Y on the generator HJ,

$$HY=mL \quad \text{.....(112)},$$

and, from (38) and (42),

$$\Omega\cos\theta=L^2+\wp v+2\wp u,$$

$$\Omega^2=(L^2+3\wp v)^2+4L\wp' v-2\wp'' v,$$

and the elimination of Ω^2 gives the relation connecting $\cos^2\theta$ with L or HY/m.

The herpolhodes for different positions of H on HK must receive an appropriate constant angular velocity round OG to realize the true motion; and the corresponding rolling quadrics are confocal, in accordance with Sylvester's theorem.

So also for the relation connecting HZ and the angle between the generating lines for different positions of H on the generator HI.

26. We conclude, in accordance with the order of procedure in this paper, with the investigation of the properties of the quadric surfaces which will trace out the herpolhodes described by H in the invariable planes of G and of C, when rolled upon these planes, their centre being fixed at O.

If a quadric surface, coaxial with the deformable hyperboloid, is to roll on the invariable plane of G, so that the points of contact form the locus of H in this plane, then, denoting the distance OG by δ, and by P_1, P_2, P_3 the points in which the generating line HJ, perpendicular to the invariable plane of G, meets the principal planes, it follows, by well-known theorems of Solid Geometry, that the squares of the semi-axes of the rolling quadric are

$$\delta \,.\, HP_1, \quad \delta \,.\, HP_2, \quad \delta \,.\, HP_3,$$

the line HJ being the normal at H to the rolling quadric; and these semi-axes are constant, since δ and the lengths HP_1, HP_2, HP_3 remain constant while the hyperboloid is deformed.

27. Write the equations of the polhode on this rolling quadric, with Dr. Routh's notation, in the form

$$Ax^2 + By^2 + Cz^2 = D\delta^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (113),$$

$$A^2x^2 + B^2y^2 + C^2z^2 = D^2\delta^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (114),$$

where
$$D = G^2/T \quad \ldots\ldots\ldots\ldots\ldots\ldots (115);$$

or, in M. Darboux's notation,

$$\frac{p^2}{a} + \frac{q^2}{b} + \frac{r^2}{c} = h \quad \ldots\ldots\ldots\ldots\ldots\ldots (116),$$

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (117),$$

where, to identify the notations, we put

$$x = mp, \quad y = mq, \quad z = mr;$$

and then
$$D\delta^2 = m^2T, \quad D\delta = mG.$$

Then the squares of the semi-axes of the rolling quadric are

$$\frac{D}{A}\delta^2 = m^2ah, \quad \frac{D}{B}\delta^2 = m^2bh, \quad \frac{D}{C}\delta^2 = m^2ch \quad \ldots\ldots\ldots (118),$$

while
$$\delta^2 = m^2h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (119),$$

so that
$$\frac{D}{A} = \frac{a}{h}, \quad \frac{D}{B} = \frac{b}{h}, \quad \frac{D}{C} = \frac{c}{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots (120).$$

Darboux's a, b, c, and h, or the reciprocals of Routh's A, B, C, and D, are thus proportional to

$$HP_1, \quad HP_2, \quad HP_3, \quad \text{and} \quad HY.$$

Now, when the hyperboloid is flattened in the plane of the focal ellipse

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \quad z = 0,$$

corresponding to $u = \omega_3$, then (Fig. 2)

$$HP_3 = HP,$$

and

$$\frac{D}{C}\delta^2 = m^2 ch = HY \,.\, HP,$$

or

$$\frac{D}{C} = \frac{c}{h} = \frac{HP}{HY} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (121).$$

But, from a property of the ellipse,

$$PY^2 = \frac{\alpha^2 - \delta^2 \,.\, \delta^2 - \beta^2}{\delta^2} = -m^2 \frac{\wp v - e_1 \,.\, \wp v - e_2}{\wp v - e_3} \quad \ldots\ldots\ldots (122),$$

so that $$\left(\frac{D}{C} - 1\right)^2 = \left(\frac{c}{h} - 1\right)^2 = \frac{PY^2}{HY^2} = -\frac{m^2\sigma_1\sigma_2}{\delta^2\sigma_3} = -\frac{\sigma_1\sigma_2}{h^2\sigma_3},$$

or

$$c - h = \sqrt{\left(-\frac{\sigma_1\sigma_2}{\sigma_3}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (123),$$

with

$$m^2 = \frac{\delta^2}{h^2} = \frac{4A_1 W g h}{\Omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots (124),$$

$$h^2 = L^2, \quad h = \pm L \quad \ldots\ldots\ldots\ldots\ldots\ldots (125),$$

according as L is positive or negative.

Similarly,

$$a - h = \sqrt{\left(-\frac{\sigma_2\sigma_3}{\sigma_1}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (126),$$

$$b - h = \sqrt{\left(-\frac{\sigma_3\sigma_1}{\sigma_2}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (127),$$

or

$$(b-h)(c-h) = -\sigma_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (128),$$

$$(c-h)(a-h) = -\sigma_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (129),$$

$$(a-h)(b-h) = -\sigma_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (130).$$

28. Denote by accented letters the corresponding quantities for the coaxial quadric which rolls on the invariable plane of C, and of which HI, the other generating line through H of the deformable hyperboloid, is the normal at H.

Then the locus of H on this quadric is the same polhode as before, but now determined by the equations

$$A'x^2 + B'y^2 + C'z^2 = D'\delta'^2 \quad \text{.....(131)},$$

$$A'^2x^2 + B'^2y^2 + C'^2z^2 = D'^2\delta'^2 \quad \text{.....(132)},$$

or

$$\frac{p^2}{a'} + \frac{q^2}{b'} + \frac{r^2}{c'} = h' \quad \text{.....(133)},$$

$$\frac{p^2}{a'^2} + \frac{q^2}{b'^2} + \frac{r^2}{c'^2} = 1 \quad \text{.....(134)},$$

with

$$x = mp, \quad y = mq, \quad z = mr,$$

$$D'\delta'^2 = m^2T', \quad D'\delta' = mG'.$$

If the generating line HI cuts the principal planes of the deformable hyperboloid in Q_1, Q_2, Q_3, then, as in § 27, the squares of the semi-axes of this rolling quadric are

$$\frac{D'}{A'}\delta'^2 = m^2a'h' = \delta' . HQ_1 \quad \text{.....(135)},$$

$$\frac{D'}{B'}\delta'^2 = m^2b'h' = \delta' . HQ_2 \quad \text{.....(136)},$$

$$\frac{D'}{C'}\delta'^2 = m^2c'h' = \delta . HQ_3 \quad \text{.....(137)},$$

so that Darboux's a', b', c', and h',

or the reciprocals of Routh's

$$A', \quad B', \quad C', \quad \text{and} \quad D' = G'^2/T',$$

are proportional to HQ_1, HQ_2, HQ_3, and HZ,

where OZ is the perpendicular from O on the generating line HI.

Denoting $\wp w - e_s$ by τ_s,

then, as for the first rolling quadric, we find

$$a'-h' = \sqrt{\left(-\frac{\tau_2\tau_3}{\tau_1}\right)}, \quad b'-h' = \sqrt{\left(-\frac{\tau_3\tau_1}{\tau_2}\right)}, \quad c'-h' = \sqrt{\left(-\frac{\tau_1\tau_2}{\tau_3}\right)} \quad \text{.....(138)},$$

and
$$\left.\begin{aligned}(b'-h')(c'-h') &= -\tau_1\\ (c'-h')(a'-h') &= -\tau_2\\ (a'-h')(b'-h') &= -\tau_3\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (138^*).$$

Thus, for instance, with the hyperboloid flattened in the plane of the focal ellipse, the ratio of the squares of the corresponding axes of the rolling quadrics

$$\frac{\frac{D'}{C'}\delta'^2}{\frac{D}{C}\delta^2} = \frac{c'h'}{ch} = \frac{QH.HZ}{PH.HY} = \frac{QH}{PH}\,\frac{h'}{h},$$

or
$$\frac{c'}{c} = \frac{QH}{PH} = \frac{OY}{OZ} = \sqrt{\frac{\sigma_3}{\tau_3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (139),$$

because the triangles OPH, OQH are of equal area.

29. Also
$$\sigma + h^2 = \tau + h'^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (140);$$

these and the other various relations connecting the quantities A, B, C, D, δ, and A', B', C', D', δ', or a, b, c, h, and a', b', c', h', are discussed in the articles of M. Darboux and Dr. Routh, making use of the algebraical relations; and from their equations some additional results can be deduced, for instance,

$$\lambda = -\left(\sqrt{\frac{\tau_1}{\sigma_1}} + \sqrt{\frac{\tau_2}{\sigma_2}} + \sqrt{\frac{\tau_3}{\sigma_3}}\right) \quad \ldots\ldots\ldots\ldots (141),$$

$$\frac{a^2}{a'^2} = \frac{a}{a'} = \frac{\tau_1}{\sigma_1}, \text{ \&c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (142),$$

$$h(b+c) - bc = h'(b'+c') - b'c' \quad \ldots\ldots\ldots\ldots (143),$$

or
$$T\left(\frac{1}{B} + \frac{1}{C}\right) - \frac{G^2}{BC} = T'\left(\frac{1}{B'} + \frac{1}{C'}\right) - \frac{G'^2}{B'C'} \quad \ldots\ldots (144),$$

$$(h-a)(b-c) = (h'-a')(b'-c') \quad \ldots\ldots\ldots\ldots (145),$$

$$\frac{a}{b} - \frac{a}{c} = -\frac{a'}{b'} + \frac{a'}{c'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (146),$$

or
$$\frac{B-C}{A} = -\frac{B'-C'}{A'} \quad \ldots\ldots\ldots\ldots\ldots\ldots (147),$$

$$2Ph - Q = 2P'h' - Q' \quad \ldots\ldots\ldots\ldots\ldots\ldots (148),$$

$$\Omega hh' = Qh^2 - 2Rh = Q'h'^2 - 2R'h' \quad \ldots\ldots\ldots\ldots (149),$$

$$\Omega^2 = \Omega'^2 = Q^2 - 4R(P-h) = Q'^2 - 4R'(P'-h') \quad \ldots\ldots (150),$$

$$(PQ-R)\,h^3-(Q^2+PR)\,h^2+2QRh-R^2$$

$$= \text{a similar expression with accented letters} \quad \ldots\ldots\ldots\ldots(151),$$

$$h'+h = \frac{\left[\sqrt{(-\sigma_1)}+\sqrt{(-\tau_1)}\right](\sqrt{\sigma_2}+\sqrt{\tau_2})(\sqrt{\sigma_3}+\sqrt{\tau_3})}{\sigma-\tau} \quad \ldots(152),$$

$$h'-h = -\frac{\left[\sqrt{(-\sigma_1)}-\sqrt{(-\tau_1)}\right](\sqrt{\sigma_2}-\sqrt{\tau_2})(\sqrt{\sigma_3}-\sqrt{\tau_3})}{\sigma-\tau} \quad \ldots(153),$$

and so forth.

30. But it will be instructive to bring out the geometrical interpretation of these relations; and, first of all, we examine the geometrical properties of the herpolhode.

We notice that $\dfrac{x^2}{a^2+\mu}$, $\dfrac{y^2}{\beta^2+\mu}$, $\dfrac{z^2}{\mu}$

are constant during the deformation of the hyperboloid by variation of μ; and that we can put

$$\left.\begin{aligned} lAx^2 &= (B-C)(a^2+\mu), & l'A'x^2 &= (B'-C')(a^2+\mu) \\ lBy^2 &= (C-A)(\beta^2+\mu), & l'By^2 &= (C'-A')(\beta^2+\mu) \\ lCz^2 &= (A-B)\,\mu, & l'C'z^2 &= (A'-B')\,\mu \end{aligned}\right\} \quad \ldots(154),$$

so that, in consequence of

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{\beta^2+\mu}+\frac{z^2}{\mu}=1,$$

we find

$$l = \frac{B-C}{A}+\frac{C-A}{B}+\frac{A-B}{C} = -\frac{(B-C)(C-A)(A-B)}{ABC},$$

$$l' = \frac{B'-C'}{A'}+\frac{C'-A'}{B'}+\frac{A'-B'}{C'} = -\frac{(B'-C')(C'-A')(A'-B')}{A'B'C'},$$

and

$$\left.\begin{aligned} \frac{x^2}{a^2+\mu} &= -\frac{BC}{(C-A)(A-B)} = -\frac{B'C'}{(C'-A')(A'-B')} \\ \frac{y^2}{\beta^2+\mu} &= -\frac{CA}{(A-B)(B-C)} = -\frac{C'A'}{(A'-B')(B'-C')} \\ \frac{z^2}{\mu} &= -\frac{AB}{(B-C)(C-A)} = -\frac{A'B'}{(B'-C')(C'-A')} \end{aligned}\right\} \quad \ldots(155).$$

Therefore $$\left(\frac{B-C}{A}\right)^2 = \left(\frac{B'-C'}{A'}\right)^2, \text{ \&c.};$$

and taking the square roots with opposite signs, because like signs lead merely to the result

$$A = A', \quad B = B', \quad C = C',$$

we find, as before, in (147),

$$\frac{B-C}{A} = -\frac{B'-C'}{A'}, \text{ \&c.,}$$

and $$l = -l' \quad \ldots\ldots\ldots\

is conjugate to the plane GOH, or parallel to the tangent line at F in the plane EOF parallel to the invariable plane of G, OF being the radius of the quadric (113) which is parallel to GH (Fig. 3).

This theorem can also be proved, in Poinsot's manner, from purely geometrical conditions; for, as the ellipsoid turns about OH in rolling on the plane GHK, the line OF is the ultimate intersection of the plane OEF with its consecutive position in the body; so that as OH moves to OH' in the body, the plane OHH' is conjugate to OF, and HH' is thus ultimately parallel to OE.

The three radii OE, OF, OH of the quadric (113) thus form a conjugate system, and the plane OGK is perpendicular to HK; and therefore, by the theorems of Solid Geometry for conjugate diameters (Salmon, *Solid Geometry*, § 97),

$$OE^2+OF^2+OH^2=\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)D\delta^2 \quad \text{.....(161),}$$

$$OE^2.OF^2.\sin^2 EOF+OK^2.OE^2+OF^2.OG^2 = \left(\frac{1}{BC}+\frac{1}{CA}+\frac{1}{AB}\right)D^2\delta^4 \quad \text{.....(162),}$$

$$OG^2.OE^2.OF^2.\sin^2 EOF=\frac{D^3\delta^6}{ABC} \quad \text{.....(163).}$$

32. Putting $GH=\rho$, $GK=p$, $OG=\delta$,

then these equations give

$$OE^2+OF^2=\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)D\delta^2-\delta^2-\rho^2,$$

$$(p^2+\delta^2)\,OE^2+\delta^2.OF^2=\left(\frac{1}{BC}+\frac{1}{CA}+\frac{1}{AB}\right)D^2\delta^4-\frac{D^3\delta^4}{ABC},$$

so that $$p^2.OE^2=\left(\rho^2+\frac{A-D.B-D.C-D}{ABC}\delta^2\right)\delta^2 \quad \text{.....(164),}$$

$$p^2.OF^2=\left\{\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)D\delta^2-\delta^2-\rho^2\right\}p^2 - \frac{A-D.B-D.C-D}{ABC}\delta^4-\delta^2\rho^2 \quad \text{.....(165).}$$

From (163), $$OE^2 . OF^2 . \frac{p^2}{\rho^2} = \frac{D^3\delta^4}{ABC},$$

or $$p^4 . OE^2 . OF^2 = \frac{D^3\delta^4}{ABC} p^2\rho^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (166);$$

and therefore

$$\frac{D^3\delta^4}{ABC} p^2\rho^2 = \left[\left\{ \left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) D\delta^2 - \delta^2 - \rho^2 \right\} p^2 - \frac{A-D.B-D.C-D}{ABC} \delta^4 - \delta^2\rho^2 \right] \left(\rho^2 + \frac{A-D.B-D.C-D}{ABC} \delta^2 \right) \delta^2,$$

$$p^2 = \frac{\left(\rho^2 + \frac{A-D.B-D.C-D}{ABC} \right)^2 \delta^2}{\left\{ \left(\frac{A}{D} + \frac{B}{D} + \frac{C}{D} - 1 \right) \delta^2 - \rho^2 \right\} \left(\rho^2 + \frac{A-D.B-D.C-D}{ABC} \delta^2 \right) - \frac{D^3\delta^2\rho^2}{ABC}} \quad \ldots\ldots\ldots\ldots (167),$$

and this is the relation connecting p and ρ in the herpolhode.

Thence

$$\left(\frac{\rho^2 d\varpi}{d\rho^2} \right)^2 = \tfrac{1}{4} \tan^2 GHK = \tfrac{1}{4} \frac{p^2}{p^2 - \rho^2}$$

$$= \frac{\left(\rho^2 + \frac{A-D.B-D.C-D}{ABC} \delta^2 \right)^2 \delta^2}{R} \quad \ldots\ldots\ldots (168),$$

where

$$R = -4 \left(\rho^2 + \frac{B-D.C-D}{BC} \delta^2 \right) \left(\rho^2 + \frac{C-D.A-D}{CA} \delta^2 \right) \times \left(\rho^2 + \frac{A-D.B-D}{AB} \delta^2 \right),$$

and

$$\frac{d\varpi}{d\rho^2} = \frac{\delta}{\sqrt{R}} + \frac{A-D.B-D.C-D}{ABC} \frac{\delta^3}{\rho^2\sqrt{R}} \quad \ldots\ldots\ldots (169),$$

the differential equation of the herpolhode, employed in the previous investigations.

33. But the relation connecting

$$OH^2 = \rho^2 + \delta^2 \quad \text{and} \quad OK^2 = p^2 + \delta^2$$

should be the same for both herpolhodes described by H, the one in the plane of G and the other in the plane of C.

Putting, then,

$$\rho^2 + \delta^2 = r^2 \quad \text{and} \quad p^2 + \delta^2 = q^2$$

for the moment, we find

$$\frac{q^2}{r^2 - q^2} = \frac{Hr^2 + K\delta^2}{\frac{1}{4}R}\,\delta^4 \quad \text{.....(170),}$$

where

$$\tfrac{1}{4}R = -\left\{r^2 - \left(\frac{D}{B} + \frac{D}{C} - \frac{D^2}{BC}\right)\delta^2\right\}\left\{r^2 - \left(\frac{D}{C} + \frac{D}{A} - \frac{D^2}{CA}\right)\delta^2\right\}$$

$$\times \left\{r^2 - \left(\frac{C}{A} - \frac{D}{B} - \frac{D^2}{AB}\right)\delta^2\right\},$$

$$H = \left(\frac{1}{BC} + \frac{1}{CA} + \frac{1}{AB}\right)D^2 - 2\,\frac{D^3}{ABC} \quad \text{.....(171),}$$

$$K = \left(1 - \frac{D}{A}\right)\left(1 - \frac{D}{B}\right)\left(1 - \frac{D}{C}\right)\left(\frac{D^2}{BC} + \frac{D^2}{CA} + \frac{D^2}{AB} - \frac{D^3}{ABC}\right)$$

$$- \frac{D^2}{BC} - \frac{D^2}{CA} - \frac{D^2}{AB} + 2\,\frac{D^3}{ABC} \quad \text{.....(172).}$$

The expression in (170) should be unaltered when

$$A, \quad B, \quad C, \quad D, \quad \text{and} \quad \delta$$

are replaced by the corresponding accented letters; and therefore

$$\left(\frac{D}{B} + \frac{D}{C} - \frac{D^2}{BC}\right)\delta^2 = \left(\frac{D'}{B'} + \frac{D'}{C'} - \frac{D'^2}{B'C'}\right)\delta'^2, \quad \text{&c.....(173),}$$

or, forming the differences

$$\frac{(B-C)(A-D)}{ABC}\,D\delta^2 = \frac{(B'-C')(A'-D')}{A'B'C'}\,D'\delta'^2 \quad \text{.....(174),}$$

each of them being in fact $-\beta^2$, from (159).

Since (147) $\qquad \dfrac{B-C}{A} = -\dfrac{B'-C'}{A'}$,

this last relation (174) becomes

$$\frac{A-D}{BC}D\delta^2 = -\frac{A'-D'}{B'C'}D'\delta'^2 \quad \text{.....(175),}$$

or (§ 27)

$$\frac{AT-G^2}{BC} = -\frac{A'T'-G'^2}{B'C'} \quad \text{.....(176),}$$

with two similar relations, and these can be written

$$\frac{A(AT-G^2)}{A'(A'T'-G'^2)} = \frac{B(BT-G^2)}{B'(B'T'-G'^2)} = \frac{C(CT-G^2)}{C'(C'T'-G'^2)} = -\frac{ABC}{A'B'C'},$$

as required for the coincidence of the polhode cones

$$A(AT-G^2)x^2 + B(BT-G^2)y^2 + C(CT-G^2)z^2 = 0,$$

$$A'(A'T'-G'^2)x^2 + B'(B'T'-G'^2)y^2 + C'(C'T'-G'^2)z^2 = 0.$$

So also the comparison of the two forms

$$H\delta^4 = H'\delta'^4 \quad \text{.....(177)}$$

and

$$K\delta^6 = K'\delta'^6 \quad \text{.....(178)}$$

will lead to relations implied in the preceding equations.

In Darboux's notation, with $\delta^2 = m^2h^2$,

$$\frac{q^2}{r^2-q^2}$$

$$= \frac{Hh^4\frac{r^2}{m^2} + Kh^6}{-\left\{\frac{r^2}{m^2}-(b+c)h+bc\right\}\left\{\frac{r^2}{m^2}-(c+a)h+ca\right\}\left\{\frac{r^2}{m^2}-(a+b)h+ab\right\}}$$

and .....(179),

$$Hh^4 = (bc+ca+ab)h^2 - 2abch = Qh^2 - 2Rh$$

$$= \Omega hh' = Q'h'^2 - 2R'h' \quad \text{.....(180),}$$

while

$$Kh^6 = (h-a)(h-b)(h-c)(Qh-R) - h^3(Qh-2R)$$

$$= (R-PQ)h^3 + (Q^2+PR)h^2 - 2QRh + R^2 \quad \text{.....(181),}$$

and this remains unaltered when the letters are accented, as in (151).

34. In Jacobi's notation, we put

$$v_2 = p\mathrm{K}'i, \quad v_1 = \mathrm{K} + r\mathrm{K}'i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(182),$$

and changing to the complementary modulus κ', the excentricity of the focal ellipse, we can put

$$\alpha^2+\lambda = \alpha^2 \frac{1}{\mathrm{sn}^2 p\mathrm{K}'}, \quad \beta^2+\lambda = \alpha^2 \frac{\mathrm{dn}^2 p\mathrm{K}'}{\mathrm{sn}^2 p\mathrm{K}'}, \quad \lambda = \alpha^2 \frac{\mathrm{cn}^2 p\mathrm{K}'}{\mathrm{sn}^2 p\mathrm{K}'} \quad \ldots(183),$$

$$\alpha^2+\nu = \kappa'^2\alpha^2 \,\mathrm{sn}^2 r\mathrm{K}', \quad \beta^2+\nu = -\kappa'^2\alpha^2 \,\mathrm{cn}^2 r\mathrm{K}', \quad \nu = -\alpha^2 \,\mathrm{dn}^2 r\mathrm{K}$$

$$\ldots\ldots\ldots\ldots \quad \ldots(184),$$

and the coordinates of H are

$$\alpha \frac{\mathrm{sn}\, r\mathrm{K}'}{\mathrm{sn}\, p\mathrm{K}'}, \quad \beta \frac{\mathrm{cn}\, p\mathrm{K}' \,\mathrm{dn}\, r\mathrm{K}'}{\kappa \,\mathrm{sn}\, r\mathrm{K}'} \quad \ldots\ldots\ldots\ldots\ldots\ldots(185).$$

We now find that the excentric angles, measured from the minor axis, of P and Q, the points of contact of the tangents drawn from H to the focal ellipse, are

$$\mathrm{am}\left\{(1-p-r)\,\mathrm{K}', \kappa'\right\} \quad \text{and} \quad \mathrm{am}\left\{(1-p+r)\,\mathrm{K}', \kappa'\right\} \quad \ldots(186),$$

while OY and OZ make angles

$$\mathrm{am}\left\{(p+r)\,\mathrm{K}', \kappa'\right\} \quad \text{and} \quad \mathrm{am}\left\{(p-r)\,\mathrm{K}', \kappa'\right\} \quad \ldots\ldots\ldots(187)$$

with the major axis, so that

$$\theta_3 = \mathrm{am}\left\{(p+r)\,\mathrm{K}', \kappa'\right\} - \mathrm{am}\left\{(p-r)\,\mathrm{K}', \kappa'\right\} \quad \ldots\ldots\ldots(188);$$

also

$$OY = \alpha \,\mathrm{dn}\left\{(p+r)\,\mathrm{K}', \kappa'\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(189),$$

$$OZ = \alpha \,\mathrm{dn}\left\{(p-r)\,\mathrm{K}', \kappa'\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(190).$$

35. As an application, take $p+r = \frac{1}{2}$ as in § 8; then

$$OY = \alpha \,\mathrm{dn}\, \tfrac{1}{2}\mathrm{K}' = \alpha\sqrt{\kappa} = \sqrt{(\alpha\beta)} \quad \ldots\ldots\ldots\ldots\ldots(191).$$

If at the same time the secular term attached to the azimuth ψ, or to the angle ϖ in the herpolhode described by H in the invariable plane of G, is made to vanish,

$$L = -\tfrac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(192),$$

and the algebraical herpolhode discussed by Halphen (*Fonctions elliptiques*, II., p. 282) is obtained.

We may write its equation, connecting the coordinates ξ, η,

$$(\xi^2+b^2)(\eta^2+b^2) = a^4 \quad \text{.....(193),}$$

or
$$\tfrac{1}{4}\rho^4 \sin^2 2\varpi + b^2\rho^2 + b^4 - a^4 = 0 \quad \text{.....(194),}$$

or
$$\rho^2 \sin^2 2\varpi + 2b^2 = 2\surd(a^4 \sin^2 2\varpi + b^4 \cos^2 2\varpi) \quad \text{.....(195),}$$

and
$$\frac{a^4-b^4}{b^2} > \rho^2 > 2(a^2-b^2),$$

and it is produced by rolling the hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{-b^2} + \frac{z^2}{-a^2} = 1 \quad \text{.....(196)}$$

upon a fixed plane at a distance b from its centre.

The squared modulus κ^2 is now equal to the anharmonic ratio of the four quantities a^2, b^2, $-b^2$, $-a^2$; so that

$$\kappa^2 = \left(\frac{a^2-b^2}{a^2+b^2}\right)^2 = \frac{\beta^2}{\alpha^2} \quad \text{.....(197),}$$

while
$$\frac{a^4-b^4}{b^2} = \alpha\beta \quad \text{.....(198),}$$

so that
$$\left.\begin{aligned} \alpha^2 &= \frac{(a^2+b^2)^2}{b^2} = b^2\left(\frac{a^2}{b^2}+1\right)^2 \\ \beta^2 &= \frac{(a^2-b^2)^2}{b^2} = b^2\left(\frac{a^2}{b^2}-1\right)^2 \end{aligned}\right\} \quad \text{.....(199),}$$

and the equation of the focal ellipse is

$$\frac{x^2}{b^2\left(\frac{a^2}{b^2}+1\right)^2} + \frac{y^2}{b^2\left(\frac{a^2}{b^2}-1\right)^2} + \frac{z^2}{0} = 1 \quad \text{.....(200).}$$

The equation of the tangent HP is

$$x\,\mathrm{cn}\,\tfrac{1}{2}\mathrm{K}' + y\,\mathrm{sn}\,\tfrac{1}{2}\mathrm{K}' = \surd(\alpha\beta) \quad \text{.....(201),}$$

or
$$\left.\begin{aligned} x\sqrt{\left(\frac{\kappa}{1+\kappa}\right)} + y\sqrt{\left(\frac{\kappa}{1+\kappa}\right)} &= \surd(\alpha\beta) \\ x\sqrt{\left(\frac{a^2-b^2}{2a^2}\right)} + y\sqrt{\left(\frac{a^2+b^2}{2a^2}\right)} &= \sqrt{\left(\frac{a^4-b^4}{b^2}\right)} \end{aligned}\right\} \quad \text{.....(202);}$$

and therefore, at the point of contact P,

$$x^2 = \frac{b^4}{2a^2}\left(\frac{a^2}{b^2}+1\right)^2, \quad y^2 = \frac{b^4}{2a^2}\left(\frac{a^2}{b^2}-1\right)^2 \quad \text{.....(203).}$$

At the point H,

$$\frac{x^2}{y^2} = -\frac{\alpha^2+\lambda \,.\, \alpha^2+\nu}{\beta^2+\lambda \,.\, \beta^2+\nu}$$

$$= \frac{e_1-\wp v_2 \,.\, e_1-\wp v_1}{e_2-\wp v_2 \,.\, \wp v_1-e_2}$$

$$= \frac{\cosh\theta_1+1 \,.\, \cosh\theta_1-1}{\cos\theta_2+1 \,.\, 1-\cos\theta_2} = \frac{\sinh^2\theta_1}{\sin^2\theta_2} \quad \ldots\ldots\ldots(204),$$

and from § 8, with the parameter a employed there (which must be distinguished from a^2 as employed here)

$$\kappa = \frac{2a-1}{2a+1},$$

$$\frac{\sinh^2\theta_1}{\sin^2\theta_2} = \frac{2a+1}{2a-1} = \frac{1}{\kappa},$$

so that

$$\frac{x^2}{y^2} = \frac{a^2+b^2}{a^2-b^2} \quad \ldots\ldots\ldots(205),$$

and therefore at H, the point of intersection of OH with the tangent HP,

$$x^2 = \frac{a^2}{2}\left(\frac{a^2}{b^2}+1\right), \quad y^2 = \frac{a^2}{2}\left(\frac{a^2}{b^2}-1\right) \quad \ldots\ldots\ldots(206).$$

Similarly, we find that, at Q,

$$x^2 = \frac{b^8}{2a^6}\left(\frac{a^2}{b^2}+1\right)^3\left(\frac{a^2}{b^2}-2\right)^2, \quad y^2 = \frac{b^8}{2a^6}\left(\frac{a^2}{b^2}-1\right)^3\left(\frac{a^2}{b^2}+2\right)^2 \quad \ldots(207).$$

Replacing the value of a in § 8 by $\frac{a^2}{2b^2}$, so as to agree with the notation of this article, we find that the cone described by the axis of the top is given by

$$\left.\begin{aligned} \sin^2\theta\cos 2\psi &= 4\sqrt{2}\,\frac{ab^2}{(a^4+8b^4)^{\frac{3}{4}}}\sqrt{\left\{\frac{a^4+2b^4}{a^2\sqrt{(a^4+8b^4)}}-\cos\theta\right\}} \\ \sin^2\theta\sin 2\psi &= \left\{\frac{a^2}{\sqrt{(a^4+8b^4)}}-\cos\theta\right\} \\ &\quad \times\sqrt{\left\{\frac{a^2-4b^2}{\sqrt{(a^4+8b^4)}}+\cos\theta\,.\,\frac{a^2+4b^2}{\sqrt{(a^4+8b^4)}}+\cos\theta\right\}} \end{aligned}\right\} \quad \ldots\ldots\ldots(208),$$

but θ is now measured from the downward vertical through O.

Thus, for instance, if

$$a^2 = 2b^2, \quad \kappa = \tfrac{1}{3};$$

the point Q is at an end of the minor axis of the focal ellipse, and the spherical curve described by C has cusps.

If $$a^2 = 3b^2, \quad \kappa = \tfrac{1}{2};$$

the curve of C has loops, and Halphen's herpolhode has points of inflexion, where

$$\rho^2 = \tfrac{16}{3}b^2,$$

and $$8b^2 > \rho^2 > 4b^2;$$

the coordinates of H are $\tfrac{1}{2}\sqrt{6}\,b, \ \tfrac{1}{2}\sqrt{3}\,b;$

of P are $\tfrac{4}{3}\sqrt{6}\,b, \ \tfrac{2}{3}\sqrt{3}\,b;$

of Q are $\tfrac{4}{9}\sqrt{6}\,b, \ \tfrac{10}{9}\sqrt{3}\,b;$

the equation of the focal ellipse being

$$\frac{x^2}{16b^2} + \frac{y^2}{4b^2} = 1 \quad \text{.....(209).}$$

These give suitable dimensions for a model, like the one constructed by Chateau of Paris, according to M. Darboux's instructions.

36. The results for the motion of the top when

$$v = \omega_1 + \tfrac{1}{3}\omega_3, \quad \text{and} \quad \omega_1 + \tfrac{2}{3}\omega_3,$$

and when, in addition, the secular term associated with 3ψ is made to disappear, as in §§ 10 and 11, so that the path of the axis OC is given algebraically, may be stated here in conclusion, expressed in the notation defined above.

With $$v = \omega_1 + \tfrac{1}{3}\omega_3,$$

we must put $$h = -L = \tfrac{1}{3}(2-c)(1-2c);$$

$$-\frac{\sigma_2\sigma_3}{\sigma_1} = (2c-c^2)^2, \quad -\frac{\sigma_3\sigma_1}{\sigma_2} = (1-c)^4, \quad -\frac{\sigma_1\sigma_2}{\sigma_3} = (1-2c)^2,$$

and thus Darboux's a, b, c (his c being replaced by $[c]$ to distinguish it) are given by

$$a = \ \tfrac{1}{3}(1+c)(2-c),$$
$$b = -\tfrac{1}{3}(1-c+c^2),$$
$$[c] = -\tfrac{1}{3}(1+c)(1-2c),$$

and for the rolling quadric

$$\frac{D}{A} = \frac{a}{h} = \frac{1+c}{1-2c}, \quad \frac{D}{B} = \frac{b}{h} = -\frac{1-c+c^2}{(2-c)(1-2c)}, \quad \frac{D}{C} = \frac{[c]}{h} = -\frac{1+c}{2-c}.$$

The herpolhode of H in the invariable plane of G is now an algebraical curve, given by (§ 9)

$$I(\omega_1+\tfrac{1}{3}\omega_3) = 3\varpi,$$

and

$$\rho^2 = m^2\{2c(1-c)^2-s\};$$

so that

$$\left(\frac{\rho}{m}\right)^3\cos 3\varpi$$

$$= (2-5c+2c^2)\sqrt{\left\{(1-c)^2(1-2c)+\frac{\rho^2}{m^2}.(1-c)^2(2c-c^2)-\frac{\rho^2}{m^2}\right\}},$$

$$\left(\frac{\rho}{m}\right)^3\cos 3\varpi$$

$$= \left\{(1-c)^2(2-5c+2c^2)+\frac{\rho^2}{m^2}\right\}\sqrt{\left\{-c(2-5c+2c^2)+\frac{\rho^2}{m^2}\right\}}.$$

With

$$v = \omega_1+\tfrac{2}{3}\omega_3,$$

we must put

$$h = -L = \tfrac{1}{3}(1+c)(1-2c),$$

$$-\frac{\sigma_2\sigma_3}{\sigma_1} = c^4, \quad -\frac{\sigma_3\sigma_1}{\sigma_2} = (1-c^2)^2, \quad -\frac{\sigma_1\sigma_2}{\sigma_3} = (1-2c)^2,$$

and

$$a = \tfrac{1}{3}(1-c+c^2), \quad b = -\tfrac{1}{3}c(1+c), \quad [c] = \tfrac{1}{3}c(1-2c).$$

For the rolling quadric

$$\frac{D}{A} = \frac{a}{h} = \frac{1-c+c^2}{(1+c)(1-2c)},$$

$$\frac{D}{B} = \frac{b}{h} = -\frac{c}{1-2c},$$

$$\frac{D}{C} = \frac{[c]}{h} = \frac{c}{1+c}.$$

The algebraical herpolhode of H in the invariable plane of G is now given by

$$I(\omega_1+\tfrac{2}{3}\omega_3) = 3\varpi,$$

and

$$\rho^2 = m^2(2c^2-2c^3-s);$$

so that

$$\left(\frac{\rho}{m}\right)^3\cos 3\varpi = \left\{c^2(1+c)(1-2c)-\frac{\rho^2}{m^2}\right\}\sqrt{\left\{(1-c^2)(1-2c)+\frac{\rho^2}{m^2}\right\}},$$

$$\left(\frac{\rho}{m}\right)^3\sin 3\varpi = (1+c)(1-2c)\sqrt{\left\{-c^2(1-2c)+\frac{\rho^2}{m^2}.c^2(1-c^2)-\frac{\rho^2}{m^2}\right\}}.$$

[37. We can utilize other results of the article on "Pseudo-Elliptic Integrals," Vol. xxv.; thus, from p. 288, with

$$\begin{aligned} v &= \omega_1 + \tfrac{1}{4}\omega_3, \\ \sigma &= c\,(1-c)^2\,(1-2c)^2\,(1-2c+2c^2), \\ \sigma_1 &= -\tfrac{1}{4}\,(1-2c)^3\,(1-2c+2c^2)(1-4c+2c^2), \\ \sigma_2 &= c\,(1-c)^3\,(1-2c+2c^2)(1-4c+2c^2), \\ \sigma_3 &= c\,(1-c)^3\,(1-2c)^3, \\ \surd(-\Sigma) &= c\,(1-c)^3\,(1-2c)^3\,(1-2c+2c^2)(1-4c+2c^2), \\ \rho &= (3-8c+6c^2)(1-4c+2c^2). \end{aligned}$$

With

$$\begin{aligned} v &= \omega_1 + \tfrac{3}{4}\omega_3, \\ \sigma &= c^2\,(1-c)(1-2c)(1-2c+2c^2), \\ \sigma_1 &= -\tfrac{1}{4}\,(1-2c)(1-2c+2c^2)(1-4c+2c^2), \\ \sigma_2 &= c^3\,(1-c)(1-2c+2c^2)(1-4c+2c^2), \\ \sigma_3 &= c^3\,(1-c)(1-2c), \\ \surd(-\Sigma) &= c^3\,(1-c)(1-2c)(1-2c+2c^2)(1-4c+2c^2), \\ \rho &= (1+2c^2)(1-4c+2c^2). \end{aligned}$$

The cone described by the axis of the top in the corresponding states of motion will now have eight loops, given by equations of the form

$$\begin{aligned} &\sin^4\theta\cos(4\psi - pt) \\ &= (P\cos^3\theta + Q\cos^2\theta + R\cos\theta + S)\surd(\cos\theta - \cos\theta_3), \\ &\sin^4\theta\sin(4\psi - pt) \\ &= (\cos^3\theta + C\cos^2\theta + D\cos\theta + E)\surd(\cosh\theta_1 - \cos\theta\cos\theta_2 - \cos\theta)\,; \end{aligned}$$

with

$$P = \surd 2\,\frac{p}{n} = \frac{\rho + 8L}{\surd(2\Omega)}.$$

38. Again, from p. 290, with

$$\begin{aligned} v &= \omega_1 + \tfrac{1}{6}\omega_3, \\ \sigma &= 8c\,(c+1)^2\,(c-1), \\ \surd(-\Sigma) &= 8c\,(c+1)^3\,(c-1)(-c^2+4c+1), \\ \rho &= (c+3)(c^2-4c-1)\,; \end{aligned}$$

and with
$$v = \omega_1 + \tfrac{3}{5}\omega_3,$$
$$\sigma = 4c\,(c+1)(c-1)^3,$$
$$\surd(-\Sigma) = 8c^2\,(c+1)(c-1)^3(-c^2+4c+1),$$
$$\rho = (3c-1)(c^2-4c-1)\,;$$

and the cone described by the axis of the top has ten loops, given by equations of the form

$$\sin^5\theta\cos(5\psi - pt)$$
$$= (P\cos^4\theta + Q\cos^3\theta + R\cos^2\theta + S\cos\theta + T)\surd(\cos\theta_2 - \cos\theta),$$

$$\sin^5\theta\sin(5\psi - pt)$$
$$= (\cos^4\theta + C\cos^3\theta + D\cos^2\theta + E\cos\theta + F)\surd(\cosh\theta_1 - \cos\theta\,.\,\cos\theta - \cos\theta_3),$$

$$P = \surd 2\,\frac{p}{n} = \frac{\rho + 10L}{\surd(2\Omega)}.$$

So also, with parameters of the form

$$v = \omega_1 + \tfrac{2}{5}\omega_3 \quad \text{or} \quad \omega_1 + \tfrac{4}{5}\omega_3,$$

when the cone described by the axis will have five loops, given by equations of the form

$$\sin^5\theta\cos(5\psi - pt)$$
$$= (P\cos^4\theta + Q\cos^3\theta + R\cos^2\theta + S\cos\theta + T)\surd(\cosh\theta_1 - \cos\theta),$$

$$\sin^5\theta\sin(5\psi - pt)$$
$$= (\cos^4\theta + C\cos^3\theta + D\cos^2\theta + E\cos\theta + F)\surd(\cos\theta_2 - \cos\theta\,.\,\cos\theta - \cos\theta_3).$$

39. It is readily proved that the angle between GH and the projection of Ox on the tangent plane GHK (Fig. 3)

$$= \tan^{-1}\frac{B-C}{A-D}\,\frac{yz}{\delta x} = \tan^{-1}\surd\left(-\frac{\wp v - e_a\,.\,\wp u - e_b\,.\,\wp u - e_c}{\wp u - e_a\,.\,\wp v - e_b\,.\,\wp v - e_c}\right)$$

from (91), (119), and (123); so that, if ρ_a, ϖ, denote the polar coordinates of the projection on the invariable plane of G of a point fixed in Ox at a distance k_a from O, then, from (68),

$$\varpi_a = \frac{G_1 t}{2A_1} + \tfrac{1}{2}\int\frac{i\wp' v\,du}{\wp v - \wp u} + \tan^{-1}\surd\left(-\frac{\wp v - e_a\,.\,\wp u - e_b\,.\,\wp u - e_c}{\wp u - e_a\,.\,\wp v - e_b\,.\,\wp v - e_c}\right)$$
$$= \frac{G_1 t}{2A_1} - \frac{\tfrac{1}{2}i\wp' v}{\wp v - e_a}\,u + \tfrac{1}{2}\int\frac{i\wp'(v-\omega_a)\,du}{\wp(v-\omega_a) - \wp u}.$$

This is of the form

$$\varpi_a = \tfrac{1}{2}\int \frac{a\{\wp(v-\omega_a)-\wp u\}+i\wp'(v-\omega_a)}{\wp(v-\omega_a)-\wp u}\,du \quad \ldots\ldots\ldots (210),$$

while $$\left(\frac{\rho_a}{k_a}\right)^2 = \sin^2 xOG = 1-\frac{A^2x^2}{D^2\delta^2}.$$

But, from (154),

$$x^2 = \frac{BC}{(C-A)(A-B)}\,m^2(\wp u-e_a),\ \ldots,$$

and, from (168),

$$\rho^2+\frac{(B-D)(C-D)}{BC}\,\delta^2 = m^2(e_a-\wp u),\ \ldots.$$

Also $$\rho^2 = m^2(\wp v-\wp u),$$

so that, putting $$u=v,\quad \rho^2=0,$$

$$\frac{(B-D)(C-D)}{BC}\,\delta^2 = m^2(e_a-\wp v),\ \ldots,$$

and
$$m^2\{\wp(v-\omega_a)-e_a\} = \frac{m^2(e_a-e_b)\,m^2(e_a-e_c)}{m^2(\wp v-e_a)}$$
$$= \frac{(C-A)(A-B)\,D^2}{A^2BC}\,\delta^2.$$

Therefore
$$\left(\frac{\rho_a}{k_a}\right)^2 = 1-\frac{A^2BCm^2(\wp u-e_a)}{(C-A)(A-B)\,D^2\delta^2}$$
$$= 1-\frac{\wp u-e_a}{\wp(v-\omega_a)-e_a}$$
$$= \frac{\wp(v-\omega_a)-\wp u}{\wp(v-\omega_a)-e_a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (211),$$

and (210), (211) prove that (ρ_a, ϖ_a) describes a herpolhode, denoted by σ_a in Poinsot's *Théorie nouvelle de la rotation des corps*, p. 127.

In the curve σ_a', described by the point A', in which Ox cuts the invariable plane of G,

$$\rho_a'^2 = GA'^2 = OA'^2-OG^2 = \frac{D^2\delta^4}{A^2x^2}-\delta^2$$
$$= \frac{(C-A)(A-B)\,D^2\delta^4}{A^2BCm^2(\wp u-e_a)}-\delta^2 = \frac{\wp(v-\omega_a)-e_a}{\wp u-e_a}\,\delta^2-\delta^2$$
$$= \frac{\wp(u-\omega_a)-e_a}{\wp v-e_a}\,\delta^2-\delta^2 = \frac{\wp(u-\omega_a)-\wp v}{\wp v-e_a}\,\delta^2.\Big]$$

The Electrical Distribution induced on a Circular Disc placed in any Field of Force. *By* H. M. Macdonald. Read February 14th, 1895. Received, in revised form, April 22nd, 1895.

The potential due to the inducing system can be expanded in a series of the form

$$\Sigma\Sigma\, A_{n\mu} J_n(\mu r) \cos(n\phi + \alpha_\mu)$$

for points on the disc, and the component of the force due to it perpendicular to the plane of the disc can be represented by a similar series. It will be sufficient to obtain the potential V due to the induced charge for the term of the series $J_n(\mu r)\cos(n\phi+\alpha_\mu)$, where n and μ are unrestricted. The solution of the problem has already been obtained in two particular cases, when $n=0$ (Gallop, *Quarterly Journal*, Vol. xxi., p. 229) and $n=1$ (Basset, *Camb. Phil. Soc. Proc.*, Vol. v., p. 425).

1. *To Determine a Function W_n such that for Points on the Disc*

$$W_n = J_n(\mu r),$$

and for Points in its Plane not on it

$$\frac{\partial W_n}{\partial z} = 0.$$

Let I denote the integral

$$\int_0^\infty e^{-\kappa z} J_{n-\frac{1}{2}}(\kappa r')\, J_n(\kappa r)\, \kappa^{\frac{1}{2}}\, d\kappa\ ;$$

then, if $r > r'$,

$$I = \frac{J_{n-\frac{1}{2}}\left(r'\frac{\partial}{\partial z}\right)}{\left(\frac{\partial}{\partial z}\right)^{n-\frac{1}{2}}} \int_0^\infty e^{-\kappa z}\, \kappa^n J_n(\kappa r)\, d\kappa,$$

that is,

$$I = \frac{2^n r^n \Pi\left(n-\frac{1}{2}\right)}{\sqrt{\pi}}\, \frac{J_{n-\frac{1}{2}}\left(r'\frac{\partial}{\partial z}\right)}{\left(\frac{\partial}{\partial z}\right)^{n-\frac{1}{2}}}\, \frac{1}{(r^2+z^2)^{n+\frac{1}{2}}},$$

since $$\int_0^\infty e^{-\kappa z}\kappa^n J_n(\kappa r)\,d\kappa = \frac{\Pi(n-\frac{1}{2})}{\sqrt{\pi}}\,\frac{2^n r^n}{(r^2+z^2)^{n+\frac{1}{2}}}$$

(Sonnine, *Math. Ann.*, Bd. XVI.).

Hence, when $z=0$,

$$I_0 = \frac{\sqrt{2}}{\pi}\,\frac{\Pi(n-\frac{1}{2})}{\Pi(n-1)}\,r^n r'^{n-\frac{1}{2}}\int_0^\pi \frac{\sin^{2n-1}\theta\,d\theta}{(r^2-r'^2\cos^2\theta)^{n+\frac{1}{2}}}, \quad r>r'$$

that is, $$I_0 = \frac{2\sqrt{2}}{\pi}\,\frac{\Pi(n-\frac{1}{2})}{\Pi(n-1)}\,\frac{r^n}{r'^{n-\frac{1}{2}}}\int_0^{r'}\frac{(r'^2-x^2)^{n-1}}{(r^2-x^2)^{n+\frac{1}{2}}}\,dx,$$

or $$I_0 = \frac{2\sqrt{2}}{\pi}\,\frac{\Pi(n-\frac{1}{2})}{\Pi(n-1)}\,\frac{1}{r^n r'^{n-\frac{1}{2}}\sqrt{r^2-r'^2}}\int_0^{r'}\frac{y^{2n-1}dy}{\sqrt{r'^2-y^2}},$$

whence $$I_0 = \sqrt{\frac{2}{\pi}}\,\frac{r'^{n-\frac{1}{2}}}{r^n\sqrt{r^2-r'^2}},$$

when $r>r'$;

also $$\left(\frac{\partial I}{\partial z}\right)_0 = 0,$$

when $r>r'$.

Again, when $r<r'$,

$$I = \frac{J_n\left(r\frac{\partial}{\partial z}\right)}{\left(\frac{\partial}{\partial z}\right)^n}\int_0^\infty e^{-\kappa z}\kappa^{n+\frac{1}{2}}J_{n-\frac{1}{2}}(\kappa r')\,d\kappa;$$

that is, $$I = \frac{2^{n+\frac{1}{2}}\Pi(n)}{\sqrt{\pi}}\,r'^{n-\frac{1}{2}}\,\frac{J_n\left(r\frac{\partial}{\partial z}\right)}{\left(\frac{\partial}{\partial z}\right)^n}\,\frac{z}{(r'^2+z^2)^{n+1}};$$

therefore $$I_0 = 0,$$

and $$\left(\frac{\partial I}{\partial z}\right)_0 = \sqrt{\frac{2}{\pi}}\,\frac{r^n}{r'^{n-\frac{1}{2}}(r'^2-r^2)^{\frac{1}{2}}}, \quad r<r'.$$

Writing

$$W_n = \int_0^\infty e^{-\kappa z}d\kappa\int_0^a J_{n-\frac{1}{2}}(\mu r')\,J_{n-\frac{1}{2}}(\kappa r')\,J_n(\kappa r)\,r'\kappa^{\frac{1}{2}}\mu^{\frac{1}{2}}dr',$$

when $z = 0$, and $r < a$,

$$W_n = \int_0^r J_{n-\frac{1}{2}}(\mu r')\, r' \mu^{\frac{1}{2}} \sqrt{\frac{2}{\pi}}\, \frac{r'^{n-\frac{1}{2}}}{r^n \sqrt{r^2 - r'^2}}\, dr',$$

from the foregoing; that is,

$$W_n = \sqrt{\frac{2\mu r}{\pi}} \int_0^{\frac{1}{2}\pi} J_{n-\frac{1}{2}}(\mu r \sin\theta) \sin^{n+\frac{1}{2}}\theta\, d\theta\ ;$$

therefore $$W_n = J_n(\mu r),$$

when $z = 0$ and $r < a$.

Again, when $z = 0$ and $r < a$,

$$\frac{\partial W_n}{\partial z} = -\int_0^\infty d\kappa \int_0^a J_{n-\frac{1}{2}}(\mu r')\, J_{n-\frac{1}{2}}(\kappa r')\, J_n(\kappa r)\, \kappa^{\frac{3}{2}} \mu^{\frac{1}{2}} r'\, dr';$$

that is,

$$\frac{\partial W_n}{\partial z} = -\mu J_n(\mu r) + \int_0^\infty d\kappa \int_a^\infty J_{n-\frac{1}{2}}(\mu r')\, J_{n-\frac{1}{2}}(\kappa r')\, J_n(\kappa r)\, \kappa^{\frac{3}{2}} \mu^{\frac{1}{2}} r'\, dr',$$

or $$\frac{\partial W_n}{\partial z} = -\mu J_n(\mu r) - \int_a^\infty \sqrt{\frac{2}{\pi}}\, \frac{r^n}{r'^{n-\frac{1}{2}} (r'^2 - r^2)^{\frac{3}{4}}} J_{n-\frac{1}{2}}(\mu r')\, \mu^{\frac{1}{2}} r'\, dr',$$

from the above. Hence

$$\frac{\partial W_n}{\partial z} = -\mu J_n(\mu r) - \sqrt{\frac{2\mu}{\pi}}\, r^n \int_a^\infty \frac{J_{n-\frac{1}{2}}(\mu r')\, dr'}{r'^{n-\frac{3}{2}} (r'^2 - r^2)^{\frac{3}{4}}},$$

when $z = 0$ and $r < a$.

Similarly, when $z = 0$ and $r > a$,

$$W_n = \sqrt{\frac{2\mu r}{\pi}} \int_0^{\sin^{-1}(a/r)} J_{n-\frac{1}{2}}(\mu r \sin\theta) \sin^{n+\frac{1}{2}}\theta\, d\theta,$$

$$\frac{\partial W_n}{\partial z} = 0.$$

Hence, if the potential at the point r, ϕ of the disc (radius a) is

$$A_{n\mu} J_n(\mu r) \cos(n\phi + \alpha_\mu),$$

the potential at any point on the positive side of the plane of the disc is given by

$$A_{n\mu} W_n \cos(n\phi + \alpha_\mu),$$

where W_n is the integral defined above, and the potential on the negative side is obtained from this by changing the sign of z.

The surface density of the distribution on either side of the disc necessary to produce this potential is given by

$$\sigma = \frac{A_{n\mu}\cos(n\phi+\alpha_\mu)}{4\pi}\left\{\mu J_n(\mu r)+\sqrt{\frac{2\mu}{\pi}}\,r^n\int_a^\infty \frac{J_{n-\frac{1}{2}}(\mu r')\,dr'}{r'^{n-\frac{3}{2}}(r'^2-r^2)^{\frac{1}{2}}}\right\};$$

for the case $n=0$, this agrees with the result found by Gallop, *Quarterly Journal*, Vol. XXI., p. 234.

2. *To find the Potential at any Point due to any Inducing System.*

It will be sufficient to consider the case where the inducing system lies wholly on the positive side of the disc; in this case, for points in the immediate neighbourhood of the plane of the disc, the potential of the inducing system can be represented by a series of the form

$$\Sigma\Sigma\, A_{n\mu}\, e^{\mu z} J_n(\mu r)\cos(n\phi+\alpha_\mu) \equiv V_1.$$

Let V be the potential due to the induced charge on the disc; then over the disc

$$V+V_1=0,$$

and it is easy to verify that

$$V = -\Sigma\Sigma\, A_{n\mu} W_n \cos(n\phi+\alpha_\mu),$$

for this satisfies the above condition, and makes $\dfrac{\partial V}{\partial z}$ continuous for all points in the plane of the disc not on it. The density of the distribution induced on the positive side of the disc is given by

$$\sigma = -\frac{1}{4\pi}\frac{\partial}{\partial z}(V+V_1);$$

that is,

$$\sigma = -\frac{1}{4\pi}\Sigma\Sigma\, A_{n\mu}\left\{2\mu J_n(\mu r)+\sqrt{\frac{2\mu}{\pi}}\,r^n\int_a^\infty \frac{J_{n-\frac{1}{2}}(\mu r')}{r'^{n-\frac{3}{2}}(r'^2-r^2)}\,dr'\right\}$$
$$\times\cos(n\phi+\alpha_\mu);$$

the density on the negative side is given by

$$\sigma = -\frac{1}{4\pi}\Sigma\Sigma\, A_{n\mu}\sqrt{\frac{2\mu}{\pi}}\,r^n\int_a^\infty \frac{J_{n-\frac{1}{2}}(\mu r')}{r'^{n-\frac{3}{2}}(r'^2-r^2)^{\frac{3}{2}}}\,dr'\cos(n\phi+\alpha_\mu).$$

Thursday, March 14th, 1895.

Major P. A. MACMAHON, R.A., F.R.S., President, in the Chair.

Mr. Franklin Pierce Matz, M.A., M.Sc., Ph.D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, U.S.A., was elected a member.

The President announced that he had written letters of condolence to Lady Cockle and Mrs. Cayley, and had received their acknowledgements of receipt of the same, which he communicated to the meeting.

Professor Hill read a paper by Mr. F. H. Jackson, entitled "Certain Π Functions," and the President (Mr. Kempe, Vice-President, in the Chair) communicated a paper on "The Perpetuant Invariants of Binary Quantics." Lt.-Col. Cunningham gave a proof that $2^{197}-1$ is divisible by 7487.

The President read a letter from Rev. T. C. Simmons announcing what the writer believed to be a "New Theorem in Probability."

The following presents were received :—

"Proceedings of the Royal Society," No. 342.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XIX., St. 2 ; Leipzig, 1895.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. IX., No. 2 ; 1894–95.

"Proceedings of the Physical Society of London," Vol. XIII., Pt. 4 ; March, 1895.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XII., No. 2 ; Coimbra, 1895.

"Bulletin des Sciences Mathématiques," Tome XIX., Mar., 1895 ; Paris.

"Bulletin de la Société Mathématique de France," Tome XXII., No. 10 ; Paris, 1895.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. I., No. 5 ; New York, 1895.

"Rendiconti del Circolo Matematico di Palermo," Tomo IX., Fasc. 1 and 2 ; 1895.

Brioschi, F.—"Notizie sulla Vita e sulle Opere di A. Cayley," 4to pamph. ; Roma, 1895.

Donisthorpe, W.—"A System of Measures," 4to ; London, 1895. From the Author.

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," 39–53 ; Oct., 1894 to Dec., 1894.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. IV., Fasc. 2, 3, 4 ; Roma, 1895.

Lie, Sophus.—"Untersuchungen über Unendliche Continuirliche Gruppen," Roy. 8vo ; Leipzig, 1895.

"Educational Times," March, 1895.

"Annales de la Faculté des Sciences de Toulouse," Tome IX., Fasc. 1 ; Paris, 1895.

"Philosophical Transactions of the Royal Society," Vol. CLXXXV., Pt. 1.

"Indian Engineering," Vol. XVII., Nos. 4–7.

The Perpetuant Invariants of Binary Quantics. *By* Major P. A. MacMahon, R.A., F.R.S. Read March 14th, 1895. Received 16th May, 1895.

It was in Vol. v. of the *American Journal of Mathematics* that Sylvester first proposed the problem of the enumeration of the perpetuants of given degree and weight.* Of a given degree Cayley's rule gives a generating function which enumerates the asyzygetic seminvariants. A knowledge of the perpetuants of lower degrees leads to the generating function for the compound seminvariants of the given degree. Since these forms are not linearly independent, it is necessary to find the generating function of the syzygies which connect them. We have, then, the means for arriving at the generating function of the perpetuants. It is merely necessary to subtract the generating function of the syzygies from that of the compound forms, and then subtract the difference from that of the asyzygetic forms. This procedure was adopted by Sylvester. For the first four degrees no syzygies arise, and the perpetuant generating functions were found to be

$$x^0,\quad \frac{x^2}{1-x^2},\quad \frac{x^3}{(1-x^2)(1-x^3)},\quad \frac{x^7}{(1-x^2)(1-x^3)(1-x^4)},$$

respectively ; the enumeration of the perpetuants being given, for a weight w, by the coefficient of x^w in the developments.

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* "On Sub-Invariants, *i.e.*, Semi-Invariants to Binary Quantics of Unlimited Order," *Amer. Math. Jour.*, Vol. v., p. 79.

Syzygies first present themselves for the degree 5. Sylvester, in the paper quoted, did not succeed in correctly enumerating them. This was accomplished by Hammond,* who established the generating function

$$\frac{x^7}{(1-x^2)(1-x^4)},$$

which immediately led to the true generating function for perpetuants of degree 5, viz.,

$$\frac{x^{15}}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}.$$

Cayley† continued the investigation on the same lines, but adding the notion, due to the author of the present paper, of the transformation of seminvariants into non-unitary symmetric functions. Considerable light was thus thrown upon the structure of the syzygies in general, and in particular upon those of degree 6. No new generating function was obtained, as the enumeration of the syzygies of degree 6 proved to be impracticable. The simplest perpetuant of degree 6 was first obtained by the author of this paper.‡ It proved to be of weight 31. The research proceeded on the lines laid down by Sylvester, Hammond, and Cayley, and principally by the use of Cayley's exceedingly useful algorithm for the multiplication of symmetric functions, the whole of the syzygies up to the weight 31 inclusive were calculated as far as was necessary for the purpose in hand. The generating function for the syzygies was not obtained. It should be mentioned also that on p. 45 of the paper the perpetuant of weight 31 is correctly identified, but that the non-exemplar perpetuants of this weight are incorrectly enumerated. The number was given as 5, whereas, as will subsequently appear, we now know the number to be 16.

In a second paper§ in the same volume, the author again considered the question, and showed that on a certain hypothesis, the truth of which he was unable to assert, the generating function for perpetuants of degree $\theta\,(>2)$ was

$$\frac{x^{2^{\theta-1}-1}}{(1-x^2)(1-x^3)\ldots(1-x^\theta)}.$$

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* *Amer. Math. Jour.*, Vol. IV. (1882), pp. 218–228, "On the Solution of the Differential Equation of Sources."

† *Amer. Math. Jour.*, Vol. VII. (1885), pp. 1–25, "A Memoir on Seminvariants."

‡ "On Perpetuants," *Amer. Math. Jour.*, Vol. VII., pp. 26–46.

§ "A Second Paper on Perpetuants," *Amer. Math. Jour.*, Vol. VII., pp. 259–263.

This prediction was subsequently verified by Stroh,* who, in § 10 of the paper quoted in the foot-note, established the generating function by an ingenious method which differed totally from that adopted by previous investigators in the same field.

Cayley† followed with interesting remarks and developments of Stroh's theory.

Stroh considers the general seminvariants of degree θ and weight w,

$$\Omega_\theta^w = (a_1\beta_1 + a_2\beta_2 + \ldots + a_\theta\beta_\theta)^w,$$

where $\beta_1, \beta_2, \ldots \beta_\theta$ are arbitrary quantities merely subject to the condition

$$\Sigma\beta = 0,$$

and $a_1, a_2, \ldots a_\theta$ are *umbræ*, such that, after expansion,

$$a_1^s = a_2^s = \ldots = a_\theta^s = (1^s) \text{ or } = a_s.$$

Assuming

$$(1+\mu\beta_1)(1+\mu\beta_2)\ldots(1+\mu\beta_\theta) = 1 + \mu^2 B_2 + \mu^3 B_3 + \ldots + \mu^\theta B_\theta,$$

the expanded function Ω_θ^w

can be exhibited as a linear function of products of powers of

$$B_2, B_3, \ldots B_\theta,$$

of weight w. Appearing as a coefficient of each B term of this function, we find a seminvariant of the binary quantic

$$u^n - \binom{n}{1} a_1 u^{n-1} + \binom{n}{2} a_2 u^{n-2} - \ldots = 0,$$

where n may be supposed to be infinite. Stroh shows that the whole of the seminvariants of degree θ and weight w thus present themselves. To exhibit certain of the seminvariants in terms of seminvariants of lower degree by means of products of degree θ, we may, since $\beta_1, \beta_2, \ldots \beta_\theta$ are merely subject to the condition

$$\Sigma\beta = 0,$$

suppose $$\beta_1 + \beta_2 + \ldots + \beta_\phi = \beta_{\phi+1} + \beta_{\phi+2} + \ldots + \beta_\theta = 0,$$

where ϕ may be any integer less than θ.

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* "Ueber die Symbolische Darstellung den Grundsyzyganten einer binären Form sechster Ordnung und eine Erweckerung der Symbolik von Clebsch," *Math. Ann.*, t. xxxvi. (1890), pp. 263–303.

† "On Symmetric Functions and Seminvariants," *Amer. Math. Jour.*, Vol. xv., pp. 1–69.

We have then $$\Omega_\theta^w = (\Omega_\phi + \Omega_{\theta-\phi})^w,$$

and Ω_θ^w is thus shown to be reducible. But Ω_θ^w is no longer the perfectly general seminvariant that it was proved to be before the introduction of the new conditions

$$\beta_1 + \beta_2 + \ldots + \beta_\phi = \beta_{\phi+1} + \beta_{\phi+2} + \ldots + \beta_\theta = 0.$$

These conditions necessitate the vanishing of a certain function of the quantities

$$B_2,\ B_3,\ \ldots\ B_\theta,$$

so that a certain number of B products, and therefore also of seminvariants, have disappeared from

$$\Omega_\theta^w.$$

These are the perpetuants of the degree θ and weight w.

This is very clearly stated by Stroh; and Cayley, with further amplification of statement, actually determines the conditions for the first six degrees.

The above is merely historical.

I am now principally concerned with the two papers of Stroh and Cayley last mentioned, which, from their recent appearance, will be fresh in the memory of mathematicians.

I propose to present Stroh's theory and Cayley's developments from a purely algebraical point of view—that is to say, without the employment of any umbral symbols—and also to actually identify each of the whole series of perpetuants of all degrees and weights.

First, consider a transformation of Stroh's general seminvariant obtained by employing umbræ with a different signification.

In the form

$$\Omega_\theta^w = (a_1\beta_1 + a_2\beta_2 + \ldots + a_\theta\beta_\theta)^w,$$

let a_s be an umbral symbol, such that after evolution a_s^σ is to be replaced by $\sigma!\ a_\sigma$.

We find $$\Omega_\theta^w = \Sigma \frac{w!}{\pi_1!\ \pi_2!\ \ldots} a_1^{\pi_1} a_2^{\pi_2} \ldots (\pi_1\pi_2\ldots)_\beta,$$

where $(\pi_1\pi_2\ldots)_\beta$ denotes the symmetric function

$$\Sigma\beta_1^{\pi_1}\beta_2^{\pi_2}\ldots.$$

Thence $$\Omega_\theta^w = w!\ \Sigma\, a_{\pi_1} a_{\pi_2} \ldots (\pi_1\pi_2\ldots)_\beta.$$

The symmetric functions on the dexter side are to be expressed in terms of the elementary functions $B_2, B_3, \ldots B_\theta$, and the dexter has then to be arranged as a linear function of products

$$B_s B_t \ldots .$$

Let $$(\pi_1 \pi_2 \ldots)_\beta = \Sigma C_{st\ldots} B_s B_t \ldots ;$$

then $$\frac{1}{w!} \Omega_\theta^w = \Sigma \left\{ \Sigma C_{st\ldots} a_{\pi_1} a_{\pi_2} \ldots \right\} B_s B_t \ldots .$$

The whole coefficient of $B_s B_t$ is

$$\Sigma C_{st\ldots} a_{\pi_1} a_{\pi_2} \ldots .$$

But $C_{st\ldots}$ is the coefficient of $B_s B_t \ldots$ in the expression of $(\pi_1 \pi_2 \ldots)_\beta$; therefore, by the well known law of reciprocity, it is also the coefficient $B_{\pi_1} B_{\pi_2} \ldots$ in the expression of $(st \ldots)_\beta$ or of $a_{\pi_1} a_{\pi_2} \ldots$ in the expression of $(st \ldots)$, where $(st \ldots)$ denotes a symmetric function of the quantities of which

$$a_1, a_2, \ldots a_{\pi_1} \ldots a_{\pi_2}, \ldots$$

are the elementary symmetric functions. Hence

$$\Sigma C_{st\ldots} a_{\pi_1} a_{\pi_2} \ldots = (st \ldots),$$

and $$\frac{1}{w!} \Omega_\theta^w = \Sigma (st \ldots) B_s B_t \ldots .$$

Since $B_1 = 0$, we have on the right a linear function of the non-unitary symmetric functions of weight w and of degree not exceeding θ.

These non-unitariants (Cayley, *loc. cit.*) of the roots of the equation

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots = 0$$

are, as is well known, seminvariants of the binary quantic

$$x^n - n a_1 x^{n-1} y + n (n-1) a_2 x^{n-2} y^2 - \ldots .$$

Thus transformed, Stroh's general seminvariant assumes a simple and elegant form, and suggests the following method of viewing the subject.

I, first of all, retain B_1, so as to consider the reducibility of symmetric functions in general, and subsequently cause B_1 to vanish, so as to restrict the investigation to seminvariants.

Taking an arbitrary quantity μ, let

$$\begin{aligned}&(1+\mu a_1)(1+\mu a_2)(1+\mu a_3)\ \ldots\ \text{ad inf.}\\ &= 1+\mu\,(1)+\mu^2\,(1^2)+\mu^3\,(1^3)+\ldots\\ &= 1+\mu a_1+\mu^2 a_2+\mu^3 a_3+\ldots,\end{aligned}$$

where a_1, a_2, a_3, ... are not the umbræ before mentioned, but quantities obeying the *ordinary* laws of algebraical quantity.

Let, also,
$$\begin{aligned}&(1+\mu\beta_1)(1+\mu\beta_2)\ \ldots\ (1+\mu\beta_\theta)\\ &= 1+\mu B_1+\mu^2 B_2+\ldots+\mu^\theta B_\theta;\end{aligned}$$

then
$$\begin{aligned}&(1+\mu a_1\beta_1)(1+\mu a_1\beta_2)\ \ldots\ (1+\mu a_1\beta_\theta)\\ &= 1+\mu a_1 B_1+\mu^2 a_1^2 B_2+\ldots+\mu^\theta a_1^\theta B_\theta,\end{aligned}$$

and
$$\begin{aligned}&\prod_s (1+\mu a_s\beta_1)(1+\mu a_s\beta_2)\ \ldots\ (1+\mu a_s\beta_\theta)\\ &= \prod_s (1+\mu a_s B_1+\mu^2 a_s^2 B_2+\ldots+\mu^\theta a_s^\theta B_\theta),\end{aligned}$$

the products extending to the quantities

$$a_1,\ a_2,\ a_3,\ \ldots$$

of unlimited number.

Multiplying out the dexter of this identity and therein representing the coefficients of μ^κ by $Z_{\kappa,\theta}$, we have

$$\begin{aligned}&1+\mu Z_{1,\theta}+\mu^2 Z_{2,\theta}+\mu^3 Z_{3,\theta}+\ldots\\ &= 1+\mu\,(1)\,B_1+\mu^2\left\{(2)\,B_2+(1^2)\,B_1^2\right\}\\ &\qquad+\mu^3\left\{(3)\,B_3+(21)\,B_2B_1+(1^3)\,B_1^3\right\}+\ldots,\end{aligned}$$

where on the right the coefficient of μ^κ involves linearly all the symmetric functions of a_1, a_2, a_3, ... of weight κ and degree not exceeding θ.

[Taking Ω_θ with changed umbræ

$$Z_{\kappa,\theta}=\frac{1}{\kappa!}\,\Omega_\theta^\kappa,$$

and the sinister is (when $B_1=0$)

$$\exp\,(\mu\Omega_\theta).]$$

Thus
$$Z_{1,\theta} = (1)\, B_1,$$
$$Z_{2,\theta} = (2)\, B_2 + (1^2)\, B_1^2,$$
$$Z_{3,\theta} = (3)\, B_3 + (21)\, B_2 B_1 + (1^3)\, B_1^3,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$Z_{\kappa,\theta} = \Sigma\, (p_1^{\pi_1} p_2^{\pi_2} \ldots)\, B_{p_1}^{\pi_1} B_{p_2}^{\pi_2} \ldots,$$

the summation being for all partitions of κ into parts not exceeding θ in magnitude.

Taking $\phi < \theta$, write
$$(1+\mu\beta_1)(1+\mu\beta_2) \ldots (1+\mu\beta_\phi)$$
$$= 1 + \mu B_1' + \mu^2 B_2' + \ldots + \mu^\phi B_\phi'$$
$$(1+\mu\beta_{\phi+1})(1+\mu\beta_{\phi+2}) \ldots (1+\mu\beta_\theta)$$
$$= 1 + \mu B_1'' + \mu^2 B_2'' + \ldots + \mu^{\theta-\phi} B_{\theta-\phi}'',$$

and thence
$$\prod_s (1+\mu a_s\beta_1)(1+\mu a_s\beta_2) \ldots (1+\mu a_s\beta_\phi)$$
$$= \prod_s (1 + \mu a_s B_1' + \mu^2 a_s^2 B_2' + \ldots + \mu^\phi a_s^\phi B_\phi')$$
$$\prod_s (1+\mu a_s\beta_{\phi+1})(1+\mu a_s\beta_{\phi+2}) \ldots (1+\mu a_s\beta_\theta)$$
$$= \prod_s (1 + \mu a_s B_1'' + \mu^2 a_s^2 B_2'' + \ldots + \mu^{\theta-\phi} a_s^{\theta-\phi} B_{\theta-\phi}''),$$

whence we derive
$$1 + \mu Z_{1,\phi} + \mu^2 Z_{2,\phi} + \mu^3 Z_{3,\phi} + \ldots$$
$$= 1 + \mu\,(1)\, B_1' + \mu^2 \left\{ (2)\, B_2' + (1^2)\, B_1'^2 \right\} + \ldots,$$

the parts of the partitions being limited not to exceed ϕ in magnitude; and
$$1 + \mu Z_{1,\theta-\phi} + \mu^2 Z_{2,\theta-\phi} + \mu^3 Z_{3,\theta-\phi} + \ldots$$
$$= 1 + \mu\,(1)\, B_1'' + \mu^2 \left\{ (2)\, B_2'' + (1^2)\, B_1''^2 \right\} + \ldots,$$

the parts being limited not to exceed $\theta - \phi$ in magnitude.

Moreover,
$$1 + \mu B_1 + \mu^2 B_2 + \ldots + \mu^\theta B_\theta$$
$$= (1 + \mu B_1' + \mu^2 B_2' + \ldots + \mu^\phi B_\phi')(1 + \mu B_1'' + \mu^2 B_2'' + \ldots + \mu^{\theta-\phi} B_{\theta-\phi}''),$$

and
$$1 + \mu Z_{1,\theta} + \mu^2 Z_{2,\theta} + \mu^3 Z_{3,\theta} + \ldots$$
$$= (1 + \mu Z_{1,\phi} + \mu^2 Z_{2,\phi} + \mu^3 Z_{3,\phi} + \ldots)(1 + \mu Z_{1,\theta-\phi} + \mu^2 Z_{2,\theta-\phi} + \mu^3 Z_{3,\theta-\phi} + \ldots).$$

Comparing coefficients of μ^κ,

$$Z_{\kappa,\theta} = Z_{\kappa,\phi} + Z_{\kappa-1,\phi} Z_{1,\theta-\phi} + \ldots + Z_{1,\phi} Z_{\kappa-1,\theta-\phi} + Z_{\kappa,\theta-\phi},$$

$Z_{\kappa,\theta}$ involves symmetric functions of weight κ and of degree $\leqslant \theta$, while any product on the right

$$Z_{\kappa-s,\phi} Z_{s,\theta-\phi}$$

involves products of two symmetric functions, the one of weight $\kappa - s$ and degree $\leqslant \phi$, and the other of weight s and degree $\leqslant \theta - \phi$.

Moreover, the quantities

$$B_1,\ B_2,\ \ldots\ B_\theta$$

are expressible in terms of the quantities

$$B'_1,\ B'_2,\ \ldots\ B'_\phi;\quad B''_1,\ B''_2,\ \ldots\ B''_{\theta-\phi},$$

by a series of relations of the form

$$B_s = B'_s + B'_{s-1} B''_1 + \ldots + B'_1 B''_{s-1} + B''_s,$$

and, by reason of these relations,

$$B_1,\ B_2,\ \ldots\ B_\theta$$

are not subject to any condition.

Hence, by comparison of the two sides of the relation

$$Z_{\kappa,\theta} = Z_{\kappa,\phi} + Z_{\kappa-1,\phi} Z_{1,\theta-\phi} + \ldots + Z_{\kappa,\theta-\phi},$$

we are able to express certain symmetric functions of weight κ and degree $\leqslant \theta$ as sums of products of pairs of symmetric functions, each pair involving one function of degree $\leqslant \phi$, and one of degree $\leqslant \theta - \varphi$.

We have, in fact, a general theorem of reducibility.

Supposing $\theta > 1$ and ϕ any one of the integers 1, 2, 3, ... $\theta - 1$, it can be demonstrated that every monomial symmetric function of degree θ is reducible by the aid of symmetric functions whose partitions are subsequent to it in dictionary order, and of products of pairs of functions of degrees $\leqslant \phi$ and $\leqslant \theta - \phi$, respectively.

Consider in $Z_{\kappa,\theta}$ the term

$$(\theta^{\sigma_0}, \theta - 1^{\sigma_1}, \theta - 2^{\sigma_2}, \ldots)\ B_\theta^{\sigma_0} B_{\theta-1}^{\sigma_1} B_{\theta-2}^{\sigma_2} \ldots,$$

where the literal part is equal to

$$(B'_\phi B''_{\theta-\phi})^{\sigma_0} (B'_{\phi-1} B''_{\theta-\phi} + B'_\phi B''_{\theta-\phi-1})^{\sigma_1} (B'_{\phi-2} B''_{\theta-\phi} + B'_{\phi-1} B''_{\theta-\phi-1} + B'_\phi B''_{\theta-\phi-2})^{\sigma_2} \ldots$$

Of this consider the portion

$$B'^{\sigma_0}_{\phi}\, B'^{\sigma_1}_{\phi-1}\, B'^{\sigma_2}_{\phi-2} \ldots B''^{\sigma_0+\sigma_1+\sigma_2+\ldots}_{\theta-\phi}.$$

The weight of $B'^{\sigma_0}_{\phi}\, B'^{\sigma_1}_{\phi-1}\, B'^{\sigma_2}_{\phi-2} \ldots$

is $\phi\,(\sigma_0+\sigma_1+\sigma_2+\ldots)-\sigma_1-2\sigma_2-\ldots = \kappa',$

and that of $B''^{\sigma_0+\sigma_1+\sigma_2+\ldots}_{\theta-\phi}$

is $(\theta-\phi)(\sigma_0+\sigma_1+\sigma_2+\ldots) = \kappa'',$

where $\kappa'+\kappa'' = \kappa.$

Hence the literal portion considered must arise in the dexter of the identity in the product

$$Z_{\kappa',\phi}\, Z_{\kappa'',\theta-\phi},$$

as

$$(\phi^{\sigma_0}\phi-1^{\sigma_1}\phi-2^{\sigma_2}\ldots)\, B'^{\sigma_0}_{\phi}\, B'^{\sigma_1}_{\phi-1}\, B'^{\sigma_2}_{\phi-2} \ldots (\theta-\phi^{\sigma_0+\sigma_1+\sigma_2+\ldots})\, B''^{\sigma_0+\sigma_1+\sigma_2+\ldots}_{\theta-\phi}.$$

On the sinister side the *whole* coefficient of

$$B'^{\sigma_0}_{\phi}\, B'^{\sigma_1}_{\phi-1}\, B'^{\sigma_2}_{\phi-2} \ldots B''^{\sigma_0+\sigma_1+\sigma_2+\ldots}_{\theta-\phi}$$

must be the sum of the monomial functions obtained by the multiplication of the two functions

$$(\phi^{\sigma_0}\phi-1^{\sigma_1}\phi-2^{\sigma_2}\ldots), \quad (\theta-\phi^{\sigma_0+\sigma_1+\sigma_2+\ldots}),$$

and the first of these in dictionary order is

$$(\theta^{\sigma_0}\theta-1^{\sigma_1}\theta-2^{\sigma_2}\ldots).$$

Hence this function together with other functions subsequent to it in dictionary order must be equal to the product

$$(\phi^{\sigma_0}\phi-1^{\sigma_1}\phi-2^{\sigma_2}\ldots)(\theta-\phi^{\sigma_0+\sigma_1+\sigma_2+\ldots}).$$

In other words, the function is reducible, and the actual reduction is given by the identity. For a given value of ϕ symmetric functions are, in general, reducible in more ways than one.

Ex. gr.—Take $\kappa = 6,\ \theta = 4,\ \phi = 2.$

$$Z_{6,4} = Z'_{6,2}+Z'_{5,2}Z''_{1,2}+Z'_{4,2}Z''_{2,2}+Z'_{3,2}Z''_{3,2}+Z'_{2,2}Z''_{1,2}+Z'_{1,2}Z''_{5,2}+Z''_{6,2},$$

single and double accents being introduced to distinguish forms which, from the circumstance that ϕ and $\theta-\phi$ are equal, would be otherwise indistinguishable.

$$Z_{6,4} = (42)\,B_4B_2 + (41^2)\,B_4B_1^2 + (3^2)\,B_3^2 + (321)\,B_3B_2B_1 + (31^3)\,B_3B_1^3$$
$$+ (2^3)\,B_2^3 + (2^21^2)\,B_2^2B_1^2 + (21^4)\,B_2B_1^4 + (1^6)\,B_1^6,$$

$$Z'_{6,2} = (2^3)\,B_2'^3 + (2^21^2)\,B_2'^2B_1'^2 + (21^4)\,B_2'B_1'^4 + (1^6)\,B_1'^6,$$

$$Z'_{5,2} = (2^21)\,B_2'^2B_1' + (21^3)\,B_2'B_1'^3 + (1^5)\,B_1'^5, \qquad Z''_{1,2} = (1)\,B_1'';$$

$$Z'_{4,2} = (2^2)\,B_2'^2 + (21^2)\,B_2'B_1'^2 + (1^4)\,B_1'^4, \qquad Z''_{2,2} = (2)\,B_2'' + (1^2)\,B_1''^2;$$

$$Z'_{3,2} = (21)\,B_2'B_1' + (1^3)\,B_1'^3, \qquad Z''_{3,2} = (21)\,B_2''B_1'' + (1^3)\,B_1''^3;$$

$$Z'_{2,2} = (2)\,B_2' + (1^2)\,B_1'^2, \qquad Z''_{4,2} = (2^2)\,B_2''^2 + (21^2)\,B_2''B_1''^2 + (1^4)\,B_1''^4;$$

$$Z_{1,2} = (1)\,B_1', \qquad Z''_{5,2} = (2^21)\,B_2''^2B_1'' + (21^3)\,B_2''B_1''^3 + (1^5)\,B_1''^5;$$

$$Z''_{6,2} = (2^3)\,B_2''^3 + (2^21^2)\,B_2''^2B_1''^2 + (21^4)\,B_2''B_1''^4 + (1^6)\,B_1''^6,$$

and the relations

$$B_4 = B_2'B_2'',$$
$$B_3 = B_2'B_1'' + B_1'B_2'',$$
$$B_2 = B_2' + B_1'B_1'' + B_2'',$$
$$B_1 = B_1' + B_1''.$$

Comparison of the coefficients (1) of $B_2'^2B_2''$, (2) of $B_2'B_1'B_2''B_1''$ on the sides of the resulting identity yields the reductions

$$(42) + 3\,(2^3) = (2^2)(2),$$

$$(42) + 2\,(41^2) + 2\,(3^2) + 2\,(321) + 6\,(2^3) + 4\,(2^21^2) = (21)^2.$$

A similar process with regard to the term $B_2'B_1'^2B_2''$ yields

$$(41^2) + (321) + 2\,(2^21^2) = (21^2)(2).$$

Reductions of forms of lower degrees are also given by the same identity. It is not necessary to give them, because they can be obtained more simply by formation of the identities for the data

$$(\kappa, \theta, \phi) = (6, 3, 2),$$

$$(\kappa, \theta, \phi) = (6, 2, 1).$$

We thus obtain the reductions

$$(3^2) + (321) + (2^21^2) = (2^2)(1^2),$$

$$(321) + 3\,(2^3) + 2\,(2^21^2) = (2^21)(1),$$

$$(321) + 3\,(31^3) + 3\,(2^3) + 4\,(2^21^2) + 6\,(21^4) = (21^2)(1^2),$$

$$(321) + 3\,(31^3) + 2\,(2^21^2) + 4\,(21^4) = (21)(1^3),$$

$$(31^3)+2\,(2^21^2)+4\,(21^4) = (21^3)(1),$$

$$(31^3)+(21^4) = (2)(1^4),$$

$$(2^3)+2\,(2^21^2)+6\,(21^4)+20\,(1^6) = (1^3)^2,$$

$$(2^21^2)+4\,(21^4)+15\,(1^6) = (1^4)(1^2),$$

$$(21^4)+6\,(1^6) = (1^5)(1).$$

The identity manifestly also involves a theorem for the multiplication of any two symmetric functions whatever.

I pass on to the discussion of the reduction of non-unitariants, viz., those symmetric functions the parts of whose partitions are all greater than unity.

If a non-unitariant be reducible *quâ* non-unitariants, it must obviously be reducible by means of products of pairs of non-unitariants; this fact follows from the circumstance that the product of two non-unitariants is itself a non-unitariant; the forms, in fact, constitute a closed system. It should be observed that this would not be the case with some other systems that might present themselves for consideration.

If we had to discuss the functions which contain no part 2 in their partitions, we have no closed system; for two forms, such as (31) and (41), which are included in the system, give rise to forms, containing a part 2, which are exterior to the system. Suppose that the quantities β above considered are not all independent, but are connected by a relation

$$f(B_1, B_2, B_3, \ldots B_\theta) = 0;$$

then the expression $Z_{\kappa,\theta}$

will not involve the complete system of symmetric functions of the quantities

$$\alpha_1, \alpha_2, \alpha_3, \ldots,$$

for certain of the products $B_{p_1}^{\pi_1} B_{p_2}^{\pi_2} \ldots$

can be eliminated between the relations

$$f(B_1, B_2, B_3, \ldots B_\theta) = 0,$$

$$Z_{\kappa,\theta} = \Sigma\,(p_1^{\pi_1} p_2^{\pi_2} \ldots)\, B_{p_1}^{\pi_1} B_{p_2}^{\pi_2} \ldots,$$

and, as a consequence, the number of symmetric functions

$$(p_1^{\pi_1} p_2^{\pi_2} \ldots)$$

in the expression of $Z_{\kappa,\theta}$ will suffer a reduction. We would then have a *particular* system of symmetric functions under consideration, whose nature we may take to be exactly defined by the conditional relation

$$f(B_1, B_2, B_3, \ldots B_\theta) = 0.$$

For the theory of the reducibility of this system we are led to the identity

$$\begin{aligned} & 1+\mu B_1+\mu^2 B_2+\mu^3 B_3+\ldots+\mu^\theta B_\theta \\ = \; & (1+\mu B_1'+\mu^2 B_2'+\mu^3 B_3'+\ldots+\mu^\phi B_\phi') \\ & \times (1+\mu B_1''+\mu^2 B_2''+\mu^3 B_3''+\ldots+\mu^{\theta-\phi} B_{\theta-\phi}''), \end{aligned}$$

ϕ being any integer equal or less than $\frac{1}{2}\theta$, with the three conditions

$$f(B_1, B_2, B_3, \ldots B_\theta) = 0,$$
$$f(B_1', B_2', B_3', \ldots B_\phi', 0, 0, \ldots) = 0,$$
$$f(B_1'', B_2'', B_3'', \ldots B_{\theta-\phi}'', 0, 0, 0, \ldots) = 0.$$

Hence there are $\theta-1$ independent quantities B,

$$\phi-1 \quad ,, \quad ,, \quad B',$$
$$\theta-\phi-1 \quad ,, \quad ,, \quad B'',$$

and, since $(\theta-1)-(\phi-1)-(\theta-\phi-1) = 1,$

the satisfaction of the identity necessitates another relation between

$$B_1, B_2, B_3, \ldots B_\theta,$$

say $$\psi_\phi(B_1, B_2, B_3, \ldots B_\theta) = 0.$$

Write this for brevity $$\psi_\phi = 0.$$

This is the condition of reduction for a given value of the integer ϕ. Considering merely this particular mode of reduction, we find that the condition

$$\psi_\phi = 0$$

causes a further diminution in the number of symmetric functions appearing in the expression of

$$Z_{\kappa,\theta}.$$

These disappearing functions are those which cannot be reduced in the particular manner we are considering.

If $\phi_1, \phi_2, \ldots \phi_s$ be s particular values of ϕ, the condition

$$\psi_{\phi_1}\psi_{\phi_2}\ldots\psi_{\phi_s} = 0$$

leads to the functions that cannot be reduced in any of the modes defined by the integers

$$\phi_1, \phi_2, \ldots \phi_s.$$

For complete irreducibility we have the condition

$$\psi_1 \psi_2 \ldots \psi_{\leqslant \frac{1}{2}\theta} = 0.$$

Non-unitariants constitute the simplest restricted system that it is possible to devise. They are the solutions of the partial differential equation

$$d_1 u = \left(\frac{d}{da_1} + a_1 \frac{d}{da_2} + a_2 \frac{d}{da_3} + \ldots \right) u = 0.$$

It will subsequently appear that other restricted systems corresponding to other differential equations may be usefully considered, but, for the present, non-unitariants are alone under view.

Hence $$f(B_1, B_2, \ldots B_v) = B_1,$$

and therefore $$B_1 = B_1' = B_1'' = 0,$$

$Z_{\theta,\theta}$ now only involves non-unitariants

$$Z_{2,\theta} = (2) B_2,$$
$$Z_{3,\theta} = (3) B_3,$$
$$Z_{4,\theta} = (4) B_4 + (2^2) B_2^2,$$
$$Z_{5,\theta} = (5) B_5 + (32) B_3 B_2,$$
$$\&c.$$

In order that $$1 + \mu^2 Z_{2,\theta} + \mu^3 Z_{3,\theta} + \ldots$$

may be broken up into factors, involving non-unitariants only, we must have

$$1 + \mu^2 B_2 + \mu^3 B_3 + \ldots + \mu^\theta B_\theta$$
$$= (1 + \mu^2 B_2' + \mu^3 B_3' + \ldots + \mu^\phi B_\phi')$$
$$\times (1 + \mu^2 B_2'' + \mu^3 B_3'' + \ldots + \mu^{\theta-\phi} B_{\theta-\phi}''),$$

for some value of $\phi \leqslant \frac{1}{2}\theta$.

It is easy to see that

$$\psi_\phi = \Pi (\beta_1 + \beta_2 + \ldots + \beta_\phi),$$

when on the right we have a symmetrical function.

Hence the complete condition of reduction is

$$\prod_{\phi=1}^{\phi=\frac{1}{2}\theta} \{\Pi(\beta_1+\beta_2+\ldots+\beta_\phi)\} = 0.$$

The weight of this condition in the quantities B_2, B_3, ... is as shown by Stroh and Cayley, and, as it is easy to verify,

$$2^{\theta-1}-1 = w_\theta.$$

This condition causes one B product, containing a factor B_θ, of weight w_θ, to disappear from

$$Z_{w_\theta, \theta},$$

and indicates the irreducibility of the corresponding non-unitariant.

The same procedure as was adopted for the unrestricted system shows that every other form not thus shown to be irreducible is in fact reducible.

It is now easy to show that the number of perpetuants of degree θ and weight w is given by the coefficient of x^w in

$$\frac{x^{2^{\theta-1}-1}}{(1-x^2)(1-x^3)\ldots(1-x^\theta)}.$$

Passing to the simplest particular cases, I put $\mu = \frac{1}{x}$, and consider the factorizations of the polynomial

$$x^\theta + B_2 x^{\theta-2} + B_3 x^{\theta-3} + \ldots + B_\theta,$$

which exhibit it as the product of two polynomials each wanting the second term.

Degree 2, $\theta = 2$.

We have $$x^2 + B_2,$$

and a factor, if it exist, must be simply x, which necessitates

$$\beta_1\beta_2 = B_2 = 0.$$

Hence, in the reducing identity above considered, the terms in B_2 vanish, and no symmetric function which appears as a coefficient of any power of B_2 can be exhibited in a reduced form; such functions are comprised in the series

$$(2),\ (2^2),\ (2^3),\ \ldots,$$

which therefore are all irreducible.

Hence (2^λ)

expresses all perpetuants of degree 2, and the generating function is

$$\frac{x^2}{1-x^2}.$$

Degree 3, $\theta = 3$.

We have $x^3 + B_2 x + B_3$,

and one factor must be x.

Thence $\beta_1\beta_2\beta_3 = -B_3 = 0.$

All terms of the form $(3^{\kappa+1}2^\lambda)\, B_3^{\kappa+1} B_2^\lambda$

disappear from the reducing identity.

The series of perpetuants of degree 3 are included in

$$(3^{\kappa+1}2^\lambda),$$

and the generating function is

$$\frac{x^3}{(1-x^2)(1-x^3)}.$$

Degree 4, $\theta = 4$.

We have $x^4 + B_2 x^2 + B_3 x + B_4$.

The factors may have the forms

$$x,\ x^2 + P.$$

For the factor x, we have

$$\beta_1\beta_2\beta_3\beta_4 = B_4 = 0,$$

while, for the factor $x^2 + P$,

$$\Pi(\beta_1 + \beta_2) \equiv B_3 = 0,$$

or the whole condition is

$$\beta_1\beta_2\beta_3\beta_4\, \Pi(\beta_1 + \beta_2) \equiv B_4 B_3 = 0.$$

Hence all the terms of the form

$$(4^{\kappa+1}3^{\lambda+1}2^\mu)\, B_4^{\kappa+1} B_3^{\lambda+1} B_2^\mu$$

disappear from the reducing identity, and the whole series of perpetuants of degree 4 are comprised in the expression

$$(4^{\kappa+1}3^{\lambda+1}2^\mu),$$

and the generating function is

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)}.$$

Degree 5, $\theta = 5$.

We have $x^5+B_2x^3+B_3x^2+B_4x+B_5$,

and the required factors can assume the forms

$$x,\ x^2+P.$$

For the factor x, $\Pi\beta_1 = -B_5 = 0$,

the condition for the factor x^2+P is clearly the eliminant of

$$x^5+B_2x^3+B_3x^2+B_4x+B_5$$

and $x^5+B_2x^3-B_3x^2+B_4x-B_5$,

or of $x^4+B_2x^2+B_4$

and $B_3x^2+B_5$,

which is $\Pi(\beta_1+\beta_2) \equiv B_5^2-B_5B_3B_2+B_4B_3^2$;

hence the complete condition is

$$\Pi\beta_1\Pi(\beta_1+\beta_2) \equiv B_5^3-B_5^2B_3B_2+B_5B_4B_3^2 = 0.$$

On the left-hand side of the reducing identity, we have, with others, the three terms

$$(5^3)\,B_5^3+(5^232)\,B_5^2B_3B_2+(543^2)\,B_5B_4B_3^2,$$

and each term separately would be reducible but for the condition

$$B_5^3-B_5^2B_3B_2+B_5B_4B_3^2 = 0,$$

which indicates that we can eliminate from the reducing identity *either* of the products

$$B_5^3,\quad B_5^2B_3B_2,\quad B_5B_4B_3^2,$$

and thus obtain two instead of three reducible non-unitariants from the three

$$(5^3),\quad (5^232),\quad (543^2).$$

Eliminating B_5^3, we have, in the reducing identity

$$\{(5^3)+(5^232)\}\,B_5^2B_3B_2+\{-(5^3)+(543^2)\}\,B_5B_4B_3^2,$$

indicating the reducibility of the non-unitariants

$$(5^3)+(5^232),$$

$$-(5^3)+(543^2).$$

If, instead, we eliminate $B_5^2B_3B_2$ and $B_5B_4B_3^2$, we obtain respectively

$$\{(5^3)+(5^232)\}B_5^3+\{(5^232)+(543^2)\}B_5B_4B_3^2,$$

and

$$\{(5^3)-(543^2)\}B_5^3+\{(5^232)+(543^2)\}B_5^2B_3B_2.$$

If we agree to consider a non-unitariant reducible, if it can be expressed in terms of non-unitariants subsequent to it in dictionary order and of compound forms of the same degree, we may regard

$$(5^3) \quad \text{and} \quad (5^232)$$

as reducible, as being capable of reduction by the aid of the form (543^2), which is subsequent to them in dictionary order. Hence we regard

$$(543^2)$$

as the exemplar perpetuant of degree 5 and weight 15.

The forms (5^3), (5^232) may be said to be non-exemplar.

All exemplar perpetuants of degree 5 are comprised in the expression

$$(5^{\kappa+1}4^{\lambda+1}3^{\mu+2}2^{\nu}).$$

For a given weight w we have a number of equations of condition between the products of the quantities B_2, B_3, B_4, B_5 equal to the number of ways of composing the number $w-15$ with the parts 2, 3, 4, 5; these are formed by multiplying the left-hand side of the equation

$$B_5^3-B_5^2B_3B_2+B_5B_4B_3^2=0$$

by each product of the quantities B_2, B_3, B_4, B_5 of weight $w-15$. There are also precisely the same number of exemplar perpetuant forms of degree 5 and weight w. From these equations of condition and the equation of reducibility of weight w, we can eliminate all the products which contain the factor

$$B_5B_4B_3^2,$$

and thus exhibit the reduction of all non-exemplar non-unitariants by the aid of the exemplar forms.

Thus the generating function for degree 5 is

$$\frac{x^{15}}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}.$$

The reducibility of forms of degree 5 by means of products of quadric and cubic forms is, as we saw, governed by the relation

$$B_5^2 - B_5 B_3 B_2 + B_4 B_3^2 = 0,$$

which proves that of a weight lower than 10 all forms are so expressible; *ex. gr.*,

$$(53) = (3^2)(2) - (3^2 2)(a^2),$$

where a^2 has been introduced and in a covariant identity would represent the square of the quantic itself.

For the weight 10, however, the relation shows that only the combinations

$$(5^2) - (43^2)\, a,$$

$$(532) + (43^2)\, a$$

are so expressible.

Moreover, the form (43^2) is not expressible by means of products of quadric and cubic forms (say by products 2 . 3), and thus is not a quintic syzygant. It immediately follows that the form (543^2) is a perpetuant, for, had this form been reducible, the operation of decapitation (see *A. M. J.*, Vol. VII.), or in other words, the performance of

$$D_5 = \frac{1}{5!}(\partial_{a_1} + a_1 \partial_{a_2} + a_2 \partial_{a_3} + \ldots)^5$$

on the two sides of the equation exhibiting the reduction, would have shown (43^2) as a quintic syzygant.

Algebraical results of this nature are not yielded by Stroh's untransformed theory.

Degree 6, $\theta = 6$.

This case has been worked out in detail by Professor Cayley (*loc. cit.*), but I do not hesitate to give it here, as I wish to introduce some new methods of arriving at the equations of condition.

The quantic is

$$x^6 + B_2 x^4 + B_3 x^3 + B_4 x^2 + B_5 x + B_6,$$

the required factor assuming either of the forms

$$x, \quad x^2 + P, \quad x^3 + Px + Q.$$

For the factor x, $\qquad \Pi\beta_1 = B_6 = 0,$

and, for the factor $x^2 + P$, the condition is the vanishing of the eliminant of

$$x^6 + B_2 x^4 + B_4 x^2 + B_6$$

and $$B_3x^2+B_5.$$

This is $$\Pi(\beta_1+\beta_2) \equiv \begin{pmatrix} 1, & B_2, & B_4, & B_6 \\ & B_3, & B_5 & \end{pmatrix}$$
$$\equiv B_6B_3^3-B_5^3+B_5^2B_3B_2-B_5B_4B_3^2$$
$$= 0.$$

The notation $$\begin{pmatrix} 1, & B_2, & B_4, & B_6 \\ & & B_3, & B_5 \end{pmatrix}$$

for the eliminant in question will be found convenient in what follows.

The condition introduced by the third type of factor is obtainable in a variety of simple ways, of which a few of the most interesting will be given.

The condition is, of course, equivalent to

$$\Pi(\beta_1+\beta_2+\beta_3) = 0.$$

First Method.

We have the identity

$$x^6+B_2x^4+B_3x^3+B_4x^2+B_5x+B_6 = (x^3+Px+Q)(x^3+Rx+S),$$

leading to the relations

$$B_2-P-R = B_3-Q-S = B_4-PR = B_5-PS-QR = B_6-QS = 0.$$

Multiplying the two zero determinants

$$\begin{vmatrix} P & R & 0 \\ Q & S & 0 \\ 1 & 1 & 0 \end{vmatrix}, \qquad \begin{vmatrix} R & P & 0 \\ S & Q & 0 \\ 1 & 1 & 0 \end{vmatrix},$$

we obtain $$\begin{vmatrix} 2PR & PS+QR & P+R \\ PS+QR & 2QS & Q+S \\ P+Q & Q+S & 2 \end{vmatrix} = 0,$$

or $$\begin{vmatrix} 2B_4 & B_5 & B_2 \\ B_5 & 2B_6 & B_3 \\ B_2 & B_3 & 2 \end{vmatrix} = 0,$$

or $$(B_2^2-4B_4)(B_3^2-4B_6)-(B_2B_3-2B_5)^2 = 0,$$

the condition required (*cf.* Cayley, *l.c.*).

Second Method.

The quantic $x^6+B_2x^4+B_3x^3+B_4x^2+B_5x+B_6$,

equated to zero, determines the abscissæ of the six points of intersection of the conic

$$y^2+(B_2x+B_3)\,y+B_4x^2+B_5x+B_6=0$$

with the cubic curve $y-x^3=0$.

If the conic break up into two right lines

$$(y+Px+Q)(y+Rx+S)=0,$$

the quantic considered is replaceable by

$$(x^3+Px+Q)(x^3+Rx+S).$$

Hence the discriminant of the conic with regard to y must be a perfect square.

This discriminant is

$$(B_2^2-4B_4)\,x^2+2\,(B_2B_3-2B_5)\,x+B_3^2-4B_6.$$

Hence $$(B_2^2-4B_4)(B_3^2-4B_6)-(B_2B_3-2B_5)^2=0,$$

the same result as before.

This may be written

$$4B_4B_6-B_2^2B_6-B_5^2+B_5B_3B_2-B_4B_3^2=0.$$

The complete condition is thus

$$\begin{aligned}&\Pi\beta_1\,\Pi\,(\beta_1+\beta_2)\,\Pi\,(\beta_1+\beta_2+\beta_3)\\ &\equiv B_6\,(B_6B_3^3-B_5^3+B_5^2B_3B_2-B_5B_4B_3^2)\\ &\quad\times(4B_6B_4-B_6B_2^2-B_5^2+B_5B_3B_2-B_4B_3^2)\\ &=0,\end{aligned}$$

or, as calculated by Professor Cayley,

$$\begin{aligned}&4B_6^3B_4B_3^3-B_6^3B_3^3B_2^2-4B_6^2B_5^3B_4+B_6^2B_5^3B_2^2+4B_6^2B_5^2B_4B_3B_2\\ &-B_6^2B_5^2B_3^3-B_6^2B_5^2B_3B_2^3-4B_6^2B_5B_4^2B_3^2+B_6^2B_5B_4B_3^2B_2^2\\ &+B_6^2B_5B_3^4B_2-B_6^2B_4B_3^5+B_6B_5^5-2B_6B_5^4B_3B_2+2B_6B_5^3B_4B_3^2\\ &+B_6B_5^3B_3^2B_2^2-2B_6B_5^2B_4B_3^3B_2+B_6B_5B_4^2B_3^4=0.\end{aligned}$$

The last term, in dictionary order, being $B_6 B_5 B_4^2 B_3^4$, we see that the non-unitariant

$$(654^2 3^4)$$

is the exemplar perpetuant of degree 6 and weight 31. Eliminating the remaining terms in succession between the equation of condition and the equation of reduction, we obtain the reduction of

$$6^3 43^3 - 4\,(654^2 3^4),$$

$$(6^3 3^3 2^2) + (654^2 3^4),$$

$$(6^2 5^3 4) + 4\,(654^2 3^4),$$

&c.

Altogether of weight 31 there are 16 non-exemplar forms reducible by the aid of $(654^2 3^4)$.

All perpetuants of degree 6 are included in the expression

$$(6^{\kappa+1} 5^{\lambda+1} 4^{\mu+2} 3^{\nu+4} 2^{\tau}),$$

and the enumeration is given by the generating function

$$\frac{x^{31}}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)}.$$

Degree θ.

At this point it will be convenient to determine the general expression for all exemplar perpetuants of degree θ.

Expressed in terms of $\beta_1, \beta_2, \ldots \beta_\theta$ the equation of condition is

$$J_\theta = \overset{\theta}{\Pi}\beta_1 \overset{\theta}{\Pi}(\beta_1+\beta_2)\overset{\theta}{\Pi}(\beta_1+\beta_2+\beta_3)\ldots\overset{\theta}{\Pi}(\beta_1+\beta_2+\ldots+\beta_m),$$

where $$m \leqslant \tfrac{1}{2}\theta.$$

When $\beta_m = 0$,

$$\frac{J_\theta}{B_\theta} = B_{\theta-1}\left\{\overset{\theta-1}{\Pi}(\beta_1+\beta_2)\overset{\theta-1}{\Pi}(\beta_1+\beta_2+\beta_3)\ldots\overset{\theta-1}{\Pi}(\beta_1+\beta_2+\ldots+\beta_n)\right\}^2,$$

where $$n \leqslant \tfrac{1}{2}(\theta-1);$$

therefore $J_\theta = \dfrac{B_\theta}{B_{\theta-1}} J_{\theta-1}^2 +$ terms involving higher powers of B_θ.

Let P_θ denote the B product which corresponds to the simplest exemplar perpetuant of degree θ. Then

$$P_\theta = \frac{B_\theta}{B_{\theta-1}} P_{\theta-1}^2,$$

and, assuming $P_{\theta-1} = B_{\theta-1}B_{\theta-2}B_{\theta-3}^2B_{\theta-4}^4B_{\theta-5}^8 \dots B_3^{2^{\theta-5}}$,

we find $P_\theta = B_\theta B_{\theta-1}B_{\theta-2}^2B_{\theta-3}^4B_{\theta-4}^8 \dots B_3^{2^{\theta-4}}$,

justifying the assumption and establishing that the exemplar perpetuant of degree θ, and of weight $2^{\theta-1}-1$ is, when $\theta > 2$,

$$(\theta, \theta-1, \theta-2^2, \theta-3^4 \dots 3^{2^{\theta-4}}),$$

where, commencing from the left, $\theta-2$ *different* symbols are written down to make up the partition.

Also the general form of exemplar perpetuants is

$$(\theta^{\kappa_1+1}, \theta-1^{\kappa_2+1}, \theta-2^{\kappa_3+2}, \theta-3^{\kappa_4+4}, \dots 3^{\kappa_{\theta-2}+2^{\theta-4}}, 2^{\kappa_{\theta-1}}).$$

If we know the whole of the non-exemplar perpetuants of degree $\theta-1$, we can derive the whole of those non-exemplar perpetuants of degree θ which involve in their partitions the number θ unrepeated.

For $J_\theta = \dfrac{B_\theta}{B_{\theta-1}} J_{\theta-1}^2 + \text{terms involving higher powers of } B_\theta$, there-

fore,
$$\begin{aligned} J_6 &= \frac{B_6}{B_5}\,(B_5^3 - B_5^2B_3B_2 + B_5B_4B_3^2)^2 + \dots \\ &= B_6B_5^5 - 2B_6B_5^4B_3B_2 + 2B_6B_5^3B_4B_3^2 \\ &\quad + B_6B_5^3B_3^2B_2^2 - 2B_6B_5^2B_4B_3^3B_2 + B_6B_5B_4^2B_3^4 \\ &\quad + \dots, \end{aligned}$$

which a reference to the value of J_6, already calculated, shows to be correct.

A considerable portion of J_θ may be written down from the results already obtained; for

$$J_\theta = \frac{B_\theta}{B_{\theta-1}} J_{\theta-1}^2 + \dots$$

leads to
$$J_\theta = \frac{B_\theta B_{\theta-1}B_{\theta-2}^2B_{\theta-3}^4 \dots B_{\mu+1}^{2^{\theta-\mu-2}}}{B_\mu^{2^{\theta-\mu-1}}} J_\mu^{2^{\theta-\mu}} + \dots,$$

whence
$$J_\theta = \frac{B_\theta B_{\theta-1}B_{\theta-2}^2B_{\theta-3}^4 \dots B_7^{2^{\theta-8}}}{B_6^{2^{\theta-7}}} J_6^{2^{\theta-6}} + \dots,$$

when we know the complete value of J_6.

The simplest *non-exemplar* perpetuant of degree θ is easily found, for

$$J_\theta = \ldots - 2B_6B_5^2B_4B_3^3B_2 + B_6B_5B_4^2B_3^4,$$

and the term in J_θ which precedes the B product corresponding to the exemplar form, in dictionary order, is

$$\frac{B_\theta B_{\theta-1}B_{\theta-2}^2B_{\theta-3}^4 \ldots B_7^{2^\theta-8}}{B_6^{2^\theta-7}} B_6B_5^2B_4B_3^3B_2 (B_6B_5B_4^2B_3^4)^{2^\theta-6-1},$$

or $$B_\theta B_{\theta-1}B_{\theta-2}^2B_{\theta-3}^4 \ldots B_7^{2^\theta-8}B_6^{2^\theta-7}B_5^{2^\theta-6+1}B_4^{2^\theta-5-1}B_3^{2^\theta-4-1}B_2,$$

for $$\theta > 6.$$

The simplest non-exemplar perpetuant thus has the partition

$$(\theta\,\theta-1\,\theta-2^2\,\theta-3^4 \ldots 7^{2^\theta-8}\,6^{2^\theta-7}\,5^{2^\theta-6+1}\,4^{2^\theta-5-1}\,3^{2^\theta-4-1}\,2),$$

for $$\theta > 6,$$

and this gives for

$\theta = 7$ $\quad (765^3 4^3 3^7 2),$

$\theta = 8$ $\quad (876^2 5^5 4^7 3^{15} 2),$

$\theta = 9$ $\quad (987^2 6^4 5^9 4^{15} 3^{31} 2),$

and so on.

The calculation of the complete value of J_7 is a very laborious matter, as it contains several hundreds of terms. Moreover, special methods of elimination lead to extraneous factors which are very troublesome.

In a similar manner it is possible to find the perpetuant solutions of the partial differential equation

$$\frac{d}{da_\lambda} + a_1 \frac{d}{da_{\lambda+1}} + a_2 \frac{d}{da_{\lambda+2}} + \ldots = 0.$$

An Extension of Vandermonde's Theorem. *By* F. H. Jackson. Read March 14th, 1895. Received, in revised form, May 14th, 1895.

1. The function

$$\underset{\kappa=\infty}{L} \frac{(a-n+1)(a-n+2)\dots(a-n+\kappa)}{(a+1)(a+2)\dots(a+\kappa)}\kappa^n \equiv \frac{\Gamma(a+1)}{\Gamma(a-n+1)} \quad \dots\dots(1).$$

If n be a positive integer,

$$\Gamma(a+1) = a(a-1)(a-2)\dots(a-n+1)\,\Gamma(a-n+1),$$

and

$$\frac{\Gamma(a+1)}{\Gamma(a-n+1)} = a(a-1)(a-2)\dots(a-n+1).$$

Similarly, if n be a negative integer ($=-m$), function (1) reduces to

$$\frac{1}{(a+m)(a+m-1)\dots(a+1)}.$$

2. Let a_n denote the product of n related quantities,

$$a,\ a-1,\ a-2,\ \dots\ a-n+1,$$

then such expressions as $a_{\frac{1}{2}}$, a_{-n}, $a_{p/q}$ seem to be without meaning. Exactly the same might have been written concerning $a^{\frac{1}{2}}$, a^{-n}, $a^{p/q}$, so long as a^n was regarded as the product of n factors each equal to a. As soon as the general law

$$a^m \times a^n = a^n \times a^m = a^{m+n}$$

was assumed in the Theory of Indices, fractional and negative powers were interpreted, and the Binomial Theorem was shown to be true (with certain restrictions) for negative and fractional values of the index. Vandermonde's Theorem is a finite algebraical identity analogous to the Binomial Theorem for positive integral indices. We shall show that

$$(a+b)_n = a_n + n a_{n-1} b_1 + \frac{n.n-1}{2!} a_{n-2} b_2 + \dots + \frac{n.n-1\dots n-r+1}{r!} a_{n-r} b_r + \dots \quad \dots\dots(2),$$

where n is not restricted to being a positive integer, and a_n denotes the function (1).

3. Firstly, writing

$$a_n = a\,(a-1)(a-2)\ldots(a-n+1),$$

$$a_m = a\,(a-1)(a-2)\ldots(a-m+1),$$

(m and n being positive integers), we have

$$a_m \times (a-m)_n = a_n \times (a-n)_m = a_{m+n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

Let us assume these to be general laws in a manner analogous to the assumption

$$a^m \times a^n = a^n \times a^m = a^{m+n}$$

in the Theory of Indices, then we must find in general functions a_m and a_n which satisfy the relation (A), m and n being unrestricted.

Function (1), namely $\dfrac{\Gamma(a+1)}{\Gamma(a-n+1)}$, is such a function, for, on writing

$$a_n = \frac{\Gamma(a+1)}{\Gamma(a-n+1)},$$

we get $$a_n \times (a-n)_m = \frac{\Gamma(a+1)}{\Gamma(a-n+1)} \frac{\Gamma(a-n+1)}{\Gamma(a-n-m+1)}$$

$$= \frac{\Gamma(a+1)}{\Gamma(a-n-m+1)} = a_{m+n}.$$

In the same way $$a_m \times (a-m)_n = a_{m+n}.$$

Of course the relations (A) would be satisfied if we assumed

$$a_n = \frac{f(a)}{f(a-n)},$$

$f(a)$ denoting any function of a whatever, but the function a_n must be such as will reduce to

$$a\,(a-1)(a-2)\ldots(a-n+1)$$

if n be a positive integer. We therefore take

$$a_n = \frac{\Gamma(a+1)}{\Gamma(a-n+1)},$$

which function, we know, reduces to

$$a\,(a-1)(a-2)\ldots(a-n+1),$$

if n be a positive integer.

An extended form of the relation (A) is

$$a_p \times (a-p)_q \times (a-p-q)_r \times (a-p-q-r)_s \times \ldots = a_{p+q+r+s+\ldots}.$$

Let each of the m quantities p, q, r, s, ... be equal to $\frac{n}{m}$, where n and m are both integers; then $(a)_{n/m}$ will be a function such that

$$(a)_{n/m} \times \left(a-\frac{n}{m}\right)_{n/m} \times \left(a-\frac{2n}{m}\right)_{n/m} \times \ldots \times \left(a-\frac{m-1\,.\,n}{m}\right)_{n/m}$$

$$= a_{n/m+n/m+\ldots \text{ to } m \text{ terms}}$$

$$= a_n.$$

The function (1) satisfies this relation.

Putting $n = 0$ in $\qquad a_n = \dfrac{\Gamma(a+1)}{\Gamma(a-n+1)}$,

we get $\qquad a_0 = 1.$

4. Let $F_1(\alpha, \beta, \gamma)$ denote the hypergeometric series in which the element x is equal to unity; then

$$\frac{\Pi(\gamma-1)\,\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)} = F_1(\alpha, \beta, \gamma) \quad \ldots\ldots\ldots\ldots (B),$$

where Π denotes Gauss's Π function. In Gamma Functions this may be written

$$\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} = F_1(\alpha, \beta, \gamma) \quad \ldots\ldots\ldots\ldots\ldots\ldots (C),$$

For α substitute $-n$,

,, β ,, $-b$,

,, γ ,, $a-n+1$.

Then the equation (C) becomes

$$\frac{\Gamma(a-n+1)\,\Gamma(a+b+1)}{\Gamma(a+1)\,\Gamma(a+b-n+1)} = F_1(-n, -b, a-n+1)$$

$$= 1 + \frac{(-n)(-b)}{1!\,(a-n+1)} + \frac{(-n)(-n+1)(-b)(-b+1)}{2!\,(a-n+1)(a-n+2)} + \ldots$$

$$+ \frac{(-n)(-n+1)\ldots(-n+r-1)(-b)\ldots(-b+r-1)}{r!\,(a-n+1)(a-n+2)\ldots(a-n+r)} + \ldots$$

$$= 1 + \frac{n_1 b_1}{1!\,(a-n+1)_1} + \frac{n_2 b_2}{2!\,(a-n+2)_2} + \ldots + \frac{n_r b_r}{r!\,(a-n+r)_r} + \ldots$$

$$\ldots\ldots\ldots\ldots (D).$$

Now $\frac{a_{n-1}}{a_n} = \frac{\Gamma(a+1)}{\Gamma(a-n+2)} \frac{\Gamma(a-n+1)}{\Gamma(a+1)} = \frac{\Gamma(a-n+1)}{\Gamma(a-n+2)} = \frac{1}{(a-n+1)}$,

$$\frac{a_{n-2}}{a_n} = \frac{1}{(a-n+2)_2},$$

$$\frac{a_{n-r}}{a_n} = \frac{1}{(a-n+r)_r},$$

and $$\frac{\Gamma(a-n+1)}{\Gamma(a+1)} \frac{\Gamma(a+b+1)}{\Gamma(a+b-n+1)} = \frac{(a+b)_n}{a_n};$$

therefore we have

$$\frac{(a+b)_n}{a_n} = \frac{a_n}{a_n} + \frac{n_1}{1!} \frac{a_{n-1} b_1}{a_n} + \frac{n_2}{2!} \frac{a_{n-2} b_2}{a_n} + \ldots + \frac{n_r}{r!} \frac{a_{n-r} b_r}{a_n} + \ldots .$$

Multiplying both sides by a_n, we have

$$(a+b)_n = a_n + n_1 a_{n-1} b_1 + \frac{n.n-1}{2!} a_{n-2} b_2 + \ldots + \frac{n.n-1 \ldots n-r+1}{r!} a_{n-r} b_r + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (E),$$

subject to the convergence of the infinite series on the right side of the above equation.

5. Denoting the general term of the series (E) by a_r, the ratio

$$\frac{u_{r+1}}{u_r} \equiv \frac{n-r+1}{r} \frac{b-r+1}{a-n+r},$$

which approaches unity when r increases without limit.

Using the general test

$$\operatorname{Lim}_{r=\infty} \left[\left\{ r \left(\frac{u_r}{u_{r+1}} - 1 \right) - 1 \right\} \log r \right] > 1 \text{ (convergent series)},$$

$$< 1 \text{ (divergent series)},$$

we find the condition of convergence is

$$a+b+1 > 0.$$

Thursday, April 4th, 1895.

Major P. A. MACMAHON, R.A., F.R.S., President, in the Chair.

The Rev. T. C. Simmons read a paper on "A New Theorem in Probability." Messrs. Bryan, Cunningham, the President, and Dr. C. V. Burton (a visitor) joined in a discussion on the paper.

The President (Mr. Kempe, Vice-President, in the Chair) communicated a Note on "The Linear Equations that present themselves in the Method of Least Squares."

The President then read the title of a paper by the Rev. W. R. W. Roberts, viz., "On the Abelian System of Differential Equations, and their Rational and Integral Algebraic Integrals, with a discussion of the Periodicity of Abelian Functions."

The following presents were received:—

Miller, W. J. C.—"Mathematical Questions and Solutions," Vol. LXII., 8vo; London, 1895.

"Smithsonian Report, 1893," 8vo; Washington, 1894.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XIX., St. 3; Leipzig, 1895.

"Mittheilungen der Mathematischen Gesellschaft im Hamburg," Bd. III., Heft 5, 1895.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXIV., Heft 1; Jahrgang 1892; Berlin, 1895.

"Archives Néerlandaises," Tome XXVIII., Livr. 5; Harlem, 1895.

"The Silver Question: Injury to British Trade and Manufactures," papers by G. Jamieson, T. H. Box, and D. O. Croal, 8vo; London, 1895.

"Bulletin de la Société Mathématique de France," Tome XXIII., No. 1; Paris, 1895.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Serie III., Vol. I., Fasc. 1, 2; 1895.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Heft 1; 1895.

Braune, W., and O. Fischer.—"Der Gang des Menschen," Th. 1, royal 8vo; Leipzig, 1895.

Bruns, H.—"Das Eikonal," R. 8vo; Leipzig, 1895.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1., Vol. IV., Fasc. 5; Roma, 1895.

"Educational Times," April, 1895.

"Acta Mathematica," XIX., 1; Stockholm, 1895.

"Journal für die reine und angewandte Mathematik," Bd. CXIV., Heft 4; Berlin, 1895.

"Annals of Mathematics," Vol. IX., No. 2, January, 1895; University of Virginia.

"Indian Engineering," Vol. XVII., Nos. 8, 9, 10.

A New Theorem in Probability. *By* Rev. T. C. Simmons, M.A.

Read April 4th, 1895. Received, in revised form, June 6th, 1895.

1. "*If an event happen on the average once in* m *times,* m *being greater than unity, then it is more likely to happen less than once in* m *times than it is to happen more than once in* m *times.*" In the present paper I undertake to prove this novel proposition, which may be enunciated more explicitly thus:—"If an event may happen in b ways and fail in a ways, a being greater than b, and all these ways are equally likely to occur, then, μ trials being made, where μ is any multiple of $a+b$, large or small, or any *random* number, the event is more likely to happen less than $\frac{\mu b}{a+b}$ times than it is to happen more than $\frac{\mu b}{a+b}$ times." Moreover, if the ratio of a to b be greater than 4, I shall venture to assert and prove a wider proposition, viz., that the event is *more likely than not* to happen less than $\frac{\mu b}{a+b}$ times. This amounts to saying that if a die, for instance, be thrown any number of times, large or small, chosen at random, the number of appearances of the ace is more likely than not to be less than $\frac{1}{6}$ of the number of throws. For reasons which will be stated in Art. 32, I am compelled *at present* to qualify the foregoing statements by the limitation that $b = 1$.

2. The first suggestion of such a proposition arose in this way. At the beginning of the present year I was engaged, for a purpose to be elsewhere recorded, in the collection and examination of upwards of 40,000 random digits; and was considerably surprised to find that, aggregating the results, each digit presented itself, with unexpected

frequency, *less* than $\frac{1}{10}$ of the number of times. For instance, in 100 sets of 150 digits each, I found that a digit presented itself in a set more frequently under 15 times than over 15 times; similarly in the case of 80 sets each of 250 digits, and also in other aggregations. Attempts to get rid of the discrepancy proved futile, it reappearing with such persistency as to demand an explanation. Doubts arose at first as to the randomness of the digits; but subsequent laborious calculations of each separate term of $(\frac{9}{10}+\frac{1}{10})^{250}$ and $(\frac{9}{10}+\frac{1}{10})^{150}$, to six places of decimals, elicited the fact that the discrepancy was to be expected; and, a like result persistently appearing in other numerical expansion tests, the idea was suggested of examining closely the relation between the sum of the first n terms and the sum of the last na terms in the general expansion of $\left(\frac{a}{a+1}+\frac{1}{a+1}\right)^{n(a+1)}$.

Strange to say, although mathematicians of the highest eminence have, ever since the discovery of the binomial theorem, devoted themselves to the scientific treatment of probability, and even in some cases to its unscientific treatment (in such instances as the credibility of witnesses, or the fallibility of juries, or discussions as to whether the acquisition of £100 is of greater or less "importance" or "value" to a man possessing £10,000 than is the acquisition of £10 to a man possessing £1000!), there seems no trace of previous investigations of this particular matter. It is true that, when n is very large, it has been constantly assumed that the two sums of terms are almost exactly equal; but even this, I believe, has never been strictly demonstrated,* nor have attempts been made to show to what closeness of approximation the equality may be relied on. Moreover, as we shall see in Art. 22, it is necessary that not n only, but $\frac{n}{a+1}$ also, should be a large number, if the two sums of terms are to be approximately equal at all.

3. In what follows, we shall presume throughout that n, a, b are positive integers, and a always greater than b; *i.e.*, in no case less

* The assumption has generally been based (so far as I have observed) on the fact that, when n is infinite and r finite, the r^{th} term before the $(nb+1)^{\text{th}}$ in $\left(\frac{a}{a+b}+\frac{b}{a+b}\right)^{na+nb}$ differs only infinitesimally from the r^{th} term after it; no account being taken, (i.) of what happens when r becomes equal, for instance, to $\frac{1}{2}nb$ or thereabouts, nor of the fact (ii.) that the sum of an infinite number of infinitesimal differences may itself represent a *finite* difference, nor (iii.) that the terms after the $(nb+1)^{\text{th}}$ infinitely exceed, in number, the terms before it.

than 2. In the expansion of $\left(\frac{a}{a+b}+\frac{b}{a+b}\right)^{na+nb}$ in descending powers of a, the greatest term is the $(nb+1)^{\text{th}}$, which may also be called the neutral term. The first nb terms may conveniently be denominated the *short-side*, and the last na terms the *long-side*, of the expansion.

When the odds against an event are a to b, and the number of trials is a multiple of $a+b$, we shall call it a *complete set* of trials; when not a multiple of $a+b$, it may be called a *broken set*.

We will investigate the most important case first—it is

I. *Complete Sets.*

4. Theorem.—*In the expansion of* $\left(\frac{a}{a+b}+\frac{b}{a+b}\right)^{na+nb}$ *in descending powers of* $\frac{a}{a+b}$, *the sum of the first* nb *terms always exceeds the sum of the last* na *terms; the excess is a maximum when* $n=1$, *and constantly diminishes as* n *increases, lying always between* $\frac{1}{3}\frac{a-b}{a+b}$ *times the greatest term in* $\left(\frac{a}{a+b}+\frac{b}{a+b}\right)^{na+nb}$ *and* $\frac{1}{3}\frac{a-b}{a+b}$ *times the greatest term in* $\left(\frac{a}{a+b}+\frac{b}{a+b}\right)^{(n+1)(a+b)}$; *its ultimate value, when* n *is very large, being equal to* $\frac{1}{3}\frac{a-b}{\sqrt{2\pi nab\,(a+b)}}$.

Formulæ.—(1) If the excess be developed in powers of $\frac{1}{n}$, its first two terms are

$$\frac{1}{3}\frac{a-b}{a+b}\left\{1-\frac{4}{45n}\frac{(a+2b)(2a+b)}{ab\,(a+b)}\right\}G_n,$$

where G_n is the greatest term for the index $na+nb$.

(2) If the sum of the first nb terms of the expansion be similarly developed, its first three terms are

$$\frac{1}{2}-\frac{a+2b}{3\,(a+b)}\left\{1+\frac{2}{45n}\frac{(a-b)(2a+b)}{ab\,(a+b)}\right\}G_n,$$

with a corresponding formula for the sum of the last na terms.

5. For the sake of brevity, we will employ the symbol $(a, b)^r$ to denote the r^{th} power of $\frac{a}{a+b} + \frac{b}{a+b}$, and the symbol $(a, 1)^r$ to denote the r^{th} power of $\frac{a}{a+1} + \frac{1}{a+1}$.

Now it will easily be seen, if we multiply the separate terms of $(a, b)^r$ by the separate terms of $(a, b)^1$, that the sum of the first p terms of $(a, b)^{r+1}$ is less than the sum of the first p terms of $(a, b)^r$, by $\frac{b}{a+b}$ times the p^{th} term of $(a, b)^r$; and the sum of the first $p+1$ terms of $(a, b)^{r+1}$ is greater than the sum of the first p terms of $(a, b)^r$, by $\frac{a}{a+b}$ times the $(p+1)^{\text{th}}$ term of $(a, b)^r$.

This being premised, and employing S_{na+n+t} to denote the short-side (*i.e.*, the sum of the first $n+1$ terms) of $(a, 1)^{na+n+t}$, where t has any value from 1 to $a+1$, both inclusive; and S_{na+n} to denote the short-side (*i.e.*, the sum of the first n terms) of $(a, 1)^{na+n}$, we have

$$(n+1)^{\text{th}} \text{ term of } (a, 1)^{na+n} = G_n,$$

therefore $$S_{na+n} - S_{na+n+1} = -\frac{a}{a+1} G_n;$$

$$(n+1)^{\text{th}} \text{ term of } (a, 1)^{na+n+1} = \frac{a}{a+1} \cdot \frac{na+n+1}{na+1} G_n,$$

therefore $$S_{na+n+1} - S_{na+n+2} = \frac{1}{a+1} \cdot \frac{n+\frac{1}{a+1}}{n+\frac{1}{a}} G_n;$$

$$(n+1)^{\text{th}} \text{ term of } (a, 1)^{na+n+2} = \frac{a^2}{(a+1)^2} \cdot \frac{(na+n+1)(na+n+2)}{(na+1)(na+2)} G_n,$$

therefore $$S_{na+n+2} - S_{na+n+3} = \frac{1}{a+1} \cdot \frac{\left(n+\frac{1}{a+1}\right)\left(n+\frac{2}{a+1}\right)}{\left(n+\frac{1}{a}\right)\left(n+\frac{2}{a}\right)} G_n;$$

and so on, till we arrive at

$$S_{na+n+a} - S_{(n+1)(a+1)}.$$

Hence, by addition, $S_{n(a+1)}-S_{(n+1)(a+1)}$

$$=-\frac{a}{a+1}G_n+\frac{1}{a+1}G_n\left\{\frac{n+\frac{1}{a+1}}{n+\frac{1}{a}}+\frac{\left(n+\frac{1}{a+1}\right)\left(n+\frac{2}{a+1}\right)}{\left(n+\frac{1}{a}\right)\left(n+\frac{2}{a}\right)}+\ldots \text{ to } a \text{ terms}\right\},$$

the r^{th} term inside the bracket being

$$T_r\equiv\frac{\left(n+\frac{1}{a+1}\right)\left(n+\frac{2}{a+1}\right)\ldots\left(n+\frac{r}{a+1}\right)}{\left(n+\frac{1}{a}\right)\left(n+\frac{2}{a}\right)\ldots\left(n+\frac{r}{a}\right)}.$$

We shall consequently have to find the value of

$$T_1+T_2+\ldots+T_a.$$

6. Employing a well-known formula (see, for instance, Todhunter's *Algebra*, p. 557, Ex. 155), we have

$$T_r=\frac{1+\frac{r(r+1)}{2n(a+1)}+\frac{(r-1)r(r+1)(3r+2)}{24n^2(a+1)^2}+\ldots}{1+\frac{r(r+1)}{2na}+\frac{(r-1)r(r+1)(3r+2)}{24n^2a^2}+\ldots}.$$

Denote this by $1-\frac{x}{n}+\frac{y}{n^2}$;

multiplying, and equating coefficients, we easily obtain x and y, giving

$$T_r=1-\frac{1}{n}\frac{r(r+1)}{2a(a+1)}+\frac{1}{n^2}\frac{r(r+1)}{24a^2(a+1)^2}\{3r^2+r(8a+7)+4a+2\}$$

$$+\text{terms in } \frac{1}{n^3},\ \frac{1}{n^4},\ \&c.$$

Summing the values of this expression from $r=1$ to $r=a$, a process which, though somewhat tedious, presents no difficulty, we obtain

$$T_1+T_2+\ldots+T_a=a-\frac{1}{n}\frac{a+2}{6}+\frac{1}{n^2}\frac{(a+2)(13a^2+21a+6)}{120a(a+1)}+\ldots,$$

so that $S_{(n+1)(a+1)}-S_{n(a+1)}$

$$=\frac{a+2}{6(a+1)}\left\{\frac{1}{n}-\frac{1}{n^2}\frac{13a^2+21a+6}{20a(a+1)}+\text{terms in } \frac{1}{n^3},\ \&c.\right\}G_n.$$

7. If G_{n+1} denote the greatest term in $(a, 1)^{(n+1)(a+1)}$, we have

$$\frac{G_{n+1}}{G_n} = \frac{\{n(a+1)+1\}\{n(a+1)+2\} \ldots \{n(a+1)+a+1\}}{\{na+1\}\{na+2\} \ldots \{na+a\}(n+1)} \frac{a^a}{(a+1)^{a+1}}$$

$$= \frac{\left(n+\frac{1}{a+1}\right)\left(n+\frac{2}{a+1}\right) \ldots \left(n+\frac{a}{a+1}\right)}{\left(n+\frac{1}{a}\right)\left(n+\frac{2}{a}\right) \ldots \left(n+\frac{a}{a}\right)}.$$

Substituting a for r in the value of T_r given in Art. 6, we have consequently

$$G_{n+1} = \left\{1 - \frac{1}{2n} + \frac{1}{n^2} \frac{11a^2+11a+2}{24a(a+1)} + \text{terms in } \frac{1}{n^3}, \text{ \&c.}\right\} G_n.$$

8. Let now

$$S_{n(a+1)} = \phi(a) + \left(x + \frac{y}{n} + \frac{z}{n^2} + \ldots\right) G_n,$$

where x, y, z are functions of a, not containing n. Then

$$S_{(n+1)(a+1)} - S_{n(a+1)}$$

$$= \left\{x + \frac{y}{n+1} + \frac{z}{(n+1)^2} + \ldots\right\} \left\{1 - \frac{1}{2n} + \frac{1}{n^2} \frac{11a^2+11a+2}{24a(a+1)} + \ldots\right\} G_n$$

$$- \left\{x + \frac{y}{n} + \frac{z}{n^2} + \ldots\right\} G_n,$$

by the preceding article. Expanding $\frac{1}{n+1}$, and equating coefficients of $\frac{1}{n}$ and $\frac{1}{n^2}$ in the resultant expression with the corresponding coefficients in the expression at the end of Art. 6, we easily obtain the values of x and y, giving

$$S_{n(a+1)} = \phi(a) - \frac{a+2}{3(a+1)} \left\{1 + \frac{1}{n} \frac{2(a-1)(2a+1)}{45a(a+1)} + \ldots\right\} G_n.$$

9. Let Δ_n denote the excess of the short-side over the long-side of $(a, 1)^{na+n}$. Then, the sum of all the terms in the expansion being

equal to unity, we have

$$\Delta_n = 2S_{n(a+1)} - 1 + G_n$$

$$= \psi(a) + \frac{a-1}{3(a+1)} \left\{ 1 - \frac{1}{n} \frac{4(a+2)(2a+1)}{45a(a+1)} + \dots \right\} G_n,$$

where $$\psi(a) = 2\phi(a) - 1.$$

It seems fairly evident *a priori* that $\psi(a)$ must $= 0$, but attempts at a rigid demonstration have given more trouble than anything else in the present paper. The unknown terms in $\frac{1}{n^2}$, $\frac{1}{n^3}$, &c., preclude any tests by numerical examples when n is finite, and when n is infinite numerical tests are not possible. How do we know but that $\psi(a)$ is some quantity which for any given value of a is infinitesimal, and not absolute zero? The difficulty, by abstract methods, lies in not "begging the question" in the very act of proving it. And, if we left it uncertain, many of the conclusions which are about to follow would be inadmissible, since our present subject consists largely in the comparison, by addition and subtraction, of other infinitesimal quantities, among which the existence of an unknown infinitesimal $\psi(a)$ would work sad havoc; and we must not make the unauthorized assumption which we adversely criticised in the footnote to page 291.

It may be sufficient here, however, to assure the reader that we shall prove, by a very indirect but perfectly conclusive method in Art. 38, that $\psi(a) = 0$ absolutely. Assuming it in the meanwhile, we obtain

$$S_{n(a+1)} = \frac{1}{2} - \frac{a+2}{3(a+1)} \left\{ 1 + \frac{1}{n} \frac{2(a-1)(2a+1)}{45a(a+1)} + \dots \right\} G_n.$$

10. The method of Arts. 5–8 may be utilized to find the sum of the short-side terms of $(a, b)^{na+nb}$. As in Art. 5, we obtain, for the excess of the first nb terms of $(a, b)^{na+nb}$ over the first $nb+1$ terms of $(a, b)^{(n+1)(a+b)}$, the expression

$$-\frac{a}{a+b} G_n + \frac{b}{a+b} \left\{ \frac{n + \frac{1}{a+b}}{n + \frac{1}{a}} + \frac{\left(n + \frac{1}{a+b}\right)\left(n + \frac{2}{a+b}\right)}{\left(n + \frac{1}{a}\right)\left(n + \frac{2}{a}\right)} + \dots \right\} G_n.$$

The series inside the bracket will have to be summed to $a+b-1$ instead of a terms, leading to an expression much more complicated

than the one for $T_1 + \ldots + T_a$ in Art. 6. It will also be noted that the short-side of $(a, b)^{(n+1)(a+b)}$ consists of $nb+b$ terms in all: we must therefore express the sum of the remaining $b-1$ terms in the form of the product of G_n by some function of a, b, and n. Combining this with the former expression, the result will be obtained. There is no difficulty about the method, but the expressions become extremely cumbrous and unwieldy. My own attempt resulted finally in a coefficient of $\frac{1}{n^2}$ containing 19 terms involving powers of b from b^5 downwards combined with powers of a from a^3 downwards. This kind of work seems needless, however, as a little consideration will, I think, in the next article suffice to show that the formula for $(a, b)^{na+nb}$ can be immediately deduced from that for $(a, 1)^{na+n}$.

11. The formula of Art. 9 is, of course, applicable when h is substituted for a, and nb for n, giving

$$S_{nb\,(h+1)} = \frac{1}{2} - \frac{h+2}{3\,(h+1)} \left\{ 1 + \frac{1}{n} \frac{2\,(h-1)(2h+1)}{45h\,(h+1)} + \ldots \right\} G_{nb}.$$

This expresses the sum of the first nb terms of the expansion of $(h, 1)^{nbh+nb}$ in the form of a constant, together with a function of h and n, a factor of the latter being the $(nb+1)^{\text{th}}$ term of the expansion. Now, if h, instead of being an integer, were a fraction, each of the first $nb+1$ terms of the expansion would be of the same algebraical form, and therefore we may infer that any relation connecting them, by way of addition or multiplication, would remain of the same algebraical form also. The argument is the same as in what is called Euler's proof of the binomial theorem. Putting therefore

$$h = \frac{a}{b},$$

we have for the sum of the first nb terms of

$$\left(\frac{a}{b},\ 1\right)^{nb.(a/b)+nb},$$

that is, for the short-side of the expansion of

$$\left(\frac{a}{a+b} + \frac{b}{a+b}\right)^{na+nb},$$

the expression

$$\frac{1}{2}-\frac{a+2b}{3(a+b)}\left\{1+\frac{2}{45n}\,\frac{(a-b)(2a+b)}{ab(a+b)}+\ldots\right\}G,$$

where G denotes the greatest term of the expansion.

12. We can, in fact, *demonstrate* the validity of the above inference in the case of one particular fractional value of h. For the sum of the first n terms of $(b,\,1)^{nb+n}$ is, by Art. 9,

$$\frac{1}{2}-\frac{b+2}{3(b+1)}\left\{1+\frac{1}{n}\,\frac{2(b-1)(2b+1)}{45b(b+1)}+\ldots\right\}G_n.$$

Adding G_n to this, and subtracting from 1, we have, for the sum of the remaining nb terms, that is, for the sum of the first nb terms of

$$\left(\frac{1}{1+b}+\frac{b}{1+b}\right)^{nb+n},$$

or of $\cdots$ $$\left\{\frac{\frac{1}{b}}{\frac{1}{b}+1}+\frac{1}{\frac{1}{b}+1}\right\}^{nb\,(1+1/b)},$$

in descending powers of $\dfrac{\frac{1}{b}}{\frac{1}{b}+1}$, the expression

$$\frac{1}{2}-\frac{1+2b}{3(b+1)}\left\{1-\frac{1}{n}\,\frac{2(b-1)(2+b)}{45b(b+1)}+\ldots\right\}G_n,$$

or $$\frac{1}{2}-\frac{\frac{1}{b}+2}{3\left(\frac{1}{b}+1\right)}\left\{1+\frac{1}{n}\,\frac{2\left(\frac{1}{b}-1\right)\left(\frac{2}{b}+1\right)}{45\,\frac{1}{b}\left(\frac{1}{b}+1\right)}+\ldots\right\}G_n,$$

which proves the validity of substituting $\dfrac{1}{b}$ for h in the formula of Art. 11.

13. Hence, if Δ_n denote the excess of the short-side over the long-side of $(a,\,b)^{na+nb}$, we have, as in Art. 9,

$$\Delta_n=\frac{a-b}{3(a+b)}\left\{1-\frac{1}{n}\,\frac{4(a+2b)(2a+b)}{45ab(a+b)}+\ldots\right\}G_n,$$

G_n being the greatest term of the expansion. Moreover, if G_{n+1} be the greatest term in $(a, b)^{(n+1)(a+b)}$, we have, as in Art. 7,

$$G_{n+1} = \left\{ 1 - \frac{1}{2nb} + \ldots \right\} G_n.$$

Now $\frac{1}{2b}$ is, for all integral values of a and b, where $a > b$, greater than $\frac{4(a+2b)(2a+b)}{45ab(a+b)}$. Therefore Δ_n lies between $\frac{1}{3}\frac{a-b}{a+b} G_n$ and $\frac{1}{3}\frac{a-b}{a+b} G_{n-1}$; and, subject to a reservation, the theorem of Art. 4 is proved.

14. The reservation is this: that the terms involving $\frac{1}{n^3}$, $\frac{1}{n^4}$, &c., which are infinite in number, although of no importance when n is large, may, for anything we know at present, form such seriously disturbing elements when n is small as to render the above formulæ for such a case practically useless. And each coefficient being some unknown function of a and b, one would imagine that such disturbance would increase when the ratio of a to b became larger. No one would anticipate *a priori* that, when $n = 1$, and $a = \infty$, for instance, the formula of Art. 9 would be of any value whatever.

But it will be found on trial that, even for the severe test-case of $n = 1$, the formula does, in a surprising manner, give a very fairly close approximation to the actual value of Δ_n, obtained by independent calculation. With a view to demonstrate this, I have prepared a table, whose second column gives, for assigned values of a, the value of

$$\frac{1}{3}\frac{a-1}{a+1} G_n,$$

the third column the value of

$$\frac{1}{3}\frac{a-1}{a+1}\left\{ 1 - \frac{1}{n}\frac{4(2a+1)(a+2)}{45\,a\,(a+1)} \right\} G_n,$$

the fourth column the independently obtained value of Δ_n, and the last column the ratio of the "error" to the exact value of Δ_n. Great care has been taken in the calculations, but, as I have been unable to get them tested by an independent mind, inaccuracies may here and there have crept in. These would not, however, if existent, affect the general conclusions deducible from the table as a whole.

15. From $a=4$ to $a=8$, the calculation is for $(a+1)^{na+n}$; the "proportion of error" is, however, the same as if the denominators were included.

$n=1, \quad b=1.$

	First Approximation.	Second Approximation.	Actual Value.	Proportion of Error.
$a=2$	·049383	·034751	·037037	$\frac{1}{16}$
$a=3$	·070313	·052084	·054687	$\frac{1}{21}$
$a=4$	256	$194\frac{14}{25}$	203	$\frac{1}{24}$
$a=5$	$4166\frac{2}{3}$	$3216\frac{4}{81}$	3344	$\frac{1}{26}$
$a=6$	77760	60645	62921	$\frac{1}{27}$
$a=7$	1647086	1294139	1340730	$\frac{1}{28\cdot8}$
$a=8$	$39146837\frac{1}{3}$	30940834	32009911	$\frac{1}{29\cdot9}$
$a=9$	·103312	·081986	·084777	$\frac{1}{30\cdot3}$
$a=24$	·1151261	·0933978	·0962064	$\frac{1}{34\cdot2}$
$a=100$	·120797	·099002	·101813	$\frac{1}{36\cdot2}$
$a=1000$	·122443	·100642	·103454	$\frac{1}{36\cdot79}$
$a=\infty$	·12262648	·10082622	·10363832	$\frac{1}{36\cdot856}$

The above results show, I venture to think, beyond all question, that the formula-value never differs from the actual value by more than $\frac{1}{16}$, and, when a is greater than 8, never by more than $\frac{1}{30}$; and that, whenever a is not less than 100, it gives almost exactly $\frac{35}{36}$ or $\frac{36}{37}$ of the actual value. For the extremely severe test-case of $n=1$, this approximation is closer than the most sanguine investigator would have dared to hope beforehand.

16. The following are other miscellaneous test-cases, calculated with all possible care; in some instances it has been necessary to use ten-figure logarithms; and many of the calculations in the fourth column are extremely laborious.

$b = 1$.

	First Approximation.	Second Approximation.	Actual Value.	Proportion of Error.
$a=2$, $n=2$	·036580	·031164	·031550	$\frac{1}{82}$
„ $n=3$	·030348	·027351	·027485	$\frac{1}{205}$
„ $n=4$	·0264940	·02453148	·02459540	$\frac{1}{384}$
„ $n=5$	·02381189	·02240082	·02243662	$\frac{1}{627}$
„ $n=6$	·02180755	·0207306	·0207529	$\frac{1}{930}$
„ $n=7$	·0202365	·0193799	·0193949	$\frac{1}{1290}$
„ $n=8$	·0189624	·0182601	·0182706	$\frac{1}{1730}$
$a=3$, $n=2$	·0519100	·0451800	·0456240	$\frac{1}{103}$
„ $n=3$	·0430100	·0392997	·0394536	$\frac{1}{256}$
„ $n=4$	·0375332	·0351012	·03517326	$\frac{1}{488}$
„ $n=5$	·03372185	·03197331	·03201413	$\frac{1}{784}$
$a=9$, $n=2$	·076047942	·068199043	·068673713	$\frac{1}{145}$
$a=99$, $n=2$	·0888641163	·080845800	·081324387	$\frac{1}{169}$
$a=\infty$, $n=2$	·090223523	·082203654	·082682266	$\frac{1}{173}$

The foregoing table exhibits a regularity which cannot be the result of accident, and which leads to the sure conviction that the Δ_n formula, which we have strictly demonstrated merely for large values of n, holds not only for the foregoing cases, but for other hitherto untested cases where n is of moderate magnitude. We may, I think, confidently assert that, when $n = 2$ and $a > 2$, the "error" in the formula is never greater than $\frac{1}{100}$; when $n = 3$, it is never greater than $\frac{1}{200}$; when $n = 4$ and $a > 3$, it is never greater than $\frac{1}{500}$; when $n = 5$ and $a > 3$, it is never greater than $\frac{1}{800}$; when $n = 6$ and $a > 2$, it is never greater than $\frac{1}{1000}$; and so on, the approximation of course always becoming closer as n increases. It will be observed how much closer the second approximation is than the first.

17. In order that there may be as little doubt as possible of the applicability of the formula to small values of n, I add yet three more tables, calculated (i.) for $a=10$, (ii.) for $a=\infty$, and (iii.) for miscellaneous values, chosen at haphazard, of a and b.

(i.) $b=1, \quad a=10.$

	First Approximation.	Second Approximation.	Actual Value.	Proportion of Error.
$n=1$	300000...	238909...	24688...	$\frac{1}{31}$
$n=2$	630000...	565855...	56973...	$\frac{1}{147}$
$n=3$	14880...	13870...	13909...	$\frac{1}{357}$
$n=4$	370230...	351382...	351902...	$\frac{1}{676}$
$n=5$	9487530...	9101129...	9109467...	$\frac{1}{1092}$
$n=6$	24779664...	23938657...	2395353...	$\frac{1}{1610}$

The above calculation is for $(a+1)^{n\,(a+1)}$, without the denominators, and only the first few figures of each number are given. Some of the latter calculations necessitate the finding of $\log 11$ to at least ten places of decimals.

(ii.) $b=1, \quad a=\infty.$

	First Approximation.	Second Approximation.	Actual Value.	Proportion of Error.
$n=1$	·12262648	·10082622	·10363832	$\frac{1}{36{\cdot}856}$
$n=2$	·090223523	·082203654	·082682266	$\frac{1}{173}$
$n=3$	·074680603	·070255085	·070421970	$\frac{1}{422}$
$n=4$	·065122268	·062227945	·062307055	$\frac{1}{787}$
$n=5$	·058489123	·056409510	·056453940	$\frac{1}{1270}$
$n=6$	·053541047	·051954646	·051982423	$\frac{1}{1871{\cdot}3}$

In calculating the above table, I have utilized the values of e^{-p} given in Part II. of *Mathematical Tracts*, by Professor F. W. Newman.

(iii.)

	First Approximation.	Second Approximation.	Actual Value.	Proportion of Error.
$(3+2)^5$	72	$60\frac{4}{75}$	61	$\frac{1}{64}$
$(3+2)^{10}$	163296	149749	150223	$\frac{1}{300}$
$(5+2)^7$	37500	$32357\frac{1}{7}$	32707	$\frac{1}{93}$
$(9+2)^{11}$	180796...	159694...	160820...	$\frac{1}{143}$
$(9+2)^{22}$	372635...	350888...	351508...	$\frac{1}{567}$
$(10+3)^{26}$	301247...	288614...	288844...	$\frac{1}{1250}$

In the above, the denominators of a and b have, for the sake of simplicity, been omitted; and in the last three cases, the numbers being very large, only their first few digits are given.

18. It is now, I think, abundantly manifest that the Δ_n formula may be confidently applied, within limits such as those indicated in Arts. 15 and 16, to small as well as large values of n. It is curious to note that the G_{n+1} formula in terms of G_n (see Art. 7), from which the Δ_n formula is partly derived, does not, for small values of n, give approximations nearly so close to the actual values as does the Δ_n formula itself. For instance, when $a = \infty$, $b = 1$, $n = 2$, we obtain

First Approximation.	Second Approximation.	Actual Value of G_{n+1}.
·203003	·234017	·224042

so that the second approximation (to three terms) differs by more than $\frac{1}{28}$ from the actual value, a much wider deviation than for the corresponding case of Δ_n (see Art. 16), where the "proportion of error" is only $\frac{1}{173}$.

19. The closeness of the approximation of the Δ_n formula, and the fact of its becoming closer as a increases, may enable us to surmise

the nature of the terms that follow. Since Δ_n clearly vanishes altogether when $a = b$, it is fairly evident that $a-b$ must be a factor of each separate term. Again, the interchange of a and b will give the excess of the long-side over the short-side, which is $-\Delta_n$; therefore the remaining factors of each term must form a symmetric function of a and b. A little consideration will, moreover, show that the substitution of $\frac{1}{a}$ for a, and $\frac{1}{b}$ for b, and nab for n, will also give the excess of long-side over short-side, or $-\Delta_n$ (compare Art. 12). Therefore the above substitution in any term must lead to the same result as the substitution of b for a, and a for b. This would seem to imply that the coefficient of $\frac{1}{n^p}$ is of the form

$$\frac{(a-b)\,\phi\,(a,\,b)}{a^p b^p\,\psi\,(a,\,b)},$$

where $\phi\,(a,\,b)$ is a symmetric homogeneous function of a and b, of $p-1$ dimensions higher than the other symmetric homogeneous function $\psi\,(a,\,b)$. Again, the substitution of $\frac{a}{a+b}$ for a, $\frac{b}{a+b}$ for b, and $n\,(a+b)$ for n, would make no alteration in any term of the original expansion, and therefore could make no alteration in the formula for Δ_n; whence it would follow that $(a+b)^p$ is a factor of $\psi\,(a,\,b)$. Once more, the denominator of the coefficient of $\frac{1}{n^p}$ in the formula for T_r in Art. 6 will be found, on consideration, to contain $a^p\,(a+1)^p$ as a factor, and no other factor involving a; nor can such a factor be introduced by the summation, from $r = 1$ to $r = a$, of any rational integral function of r. Observing the formula of Art. 7, we see therefore that the denominator of the coefficient of $\frac{1}{n^p}$ in the Δ_n formula of Art. 9 cannot contain a function of a of higher dimensions than $a^p\,(a+1)^{p+1}$. Combining this conclusion with those obtained above, it would seem clear that the coefficient of $\frac{1}{n^p}$ in the Δ_n formula of Art. 13 must be of the form

$$\frac{(a-b)\,\phi\,(a,\,b)}{k a^p b^p\,(a+b)^{p+1}},$$

where k is some integer, and $\phi\,(a,\,b)$ is a symmetric homogeneous

function of a and b of $2p$ dimensions. We may therefore suppose that

$$\Delta_n = \frac{a-b}{3(a+b)}\left\{1-\frac{1}{n}\,\frac{8a^2+\dots}{45ab(a+b)}+\frac{1}{n^2}\,\frac{l_2a^4+\dots}{k_2a^2b^2(a+b)^2}\right.$$
$$\left.\pm\frac{1}{n^3}\,\frac{l_3a^6+\dots}{k_3a^3b^3(a+b)^3}+\dots\right\}G_n,$$

where the omitted terms contain the residue of symmetric homogeneous functions of a and b.

Now, if the quantities $1, -\frac{8}{45}, \frac{l_2}{k_2}, \frac{l_3}{k_3}, \frac{l_4}{k_4}$ *... form a converging series, we shall at once account for the fact that the first two terms give a very close approximation to the value of the whole series, even when* $n=1$. It would also explain why the approximation is closer for large than for small values of a (in comparison with b), if we were to suppose that the coefficients of omitted terms in the residue of the numerators would lead to series converging less rapidly.

I have no doubt that the above hypothesis (which we shall have an opportunity of testing in Art. 40) is correct; and no doubt also that the coefficients of the G_{n+1} formula are subject to the same law, *but that they give rise to a series which converges less rapidly.*

20. By observation of the table of Art. 16, and the second table of Art. 17 (the first table of Art. 17 being for this purpose clearly inapplicable), it will be seen how the "actual value" of Δ_r in those cases always lies about midway between the "first approximations" for $n=r$ and $n=r+1$. In several other instances where $b=1$, I have also tested that, for small values of n, Δ_n always lies about midway between $\frac{1}{3}\,\frac{a-b}{a+b}\,G_n$ and $\frac{1}{3}\,\frac{a-b}{a+b}\,G_{n+1}$; the matter seems sufficiently clear without being pursued any further.

The reservation made at the end of Art. 13 may now be removed, and I trust that I may be allowed to assert confidently that the theorem of Art. 4 is completely proved; and that the formulæ there given may be unhesitatingly applied, within close approximations, for small as well as for large values of n.

When n is very large, the term involving $\frac{1}{n}$ may be omitted; and we then obtain, by a well-known formula for G_n,

$$\Delta_n = \frac{1}{3}\,\frac{a-b}{\sqrt{2\pi nab(a+b)}}.$$

21. The applicability of the theorem to the scientific theory of gambling is manifest. It proves that, at the end of any complete set of n $(a+1)$ trials, a man who gives odds of a to 1 in a fair wager is more likely to have made a net gain than a net loss. When n is very large, the excess of his whole chance of gaining over his whole chance of losing is, of course, extremely minute; but that it should be always measurable and always *positive* is fairly subversive of hitherto accepted notions.

But is the largeness of n the *only* condition necessary for the minuteness of the excess? Let us see. Suppose $a = 99$, $n = 10$. The formula (corrected in accordance with Art. 16) gives $\Delta_n = \cdot 04068$, whence short-side $= \cdot 4570$, long-side $= \cdot 4162$, the neutral term G_n being $\cdot 1268$. Here the difference between $\cdot 45$ and $\cdot 41$ is very considerable; showing, as it does, that a man who bets 99 to 1 in a fair wager is, after 1000 trials, more likely, *by so much as* 10 *per cent.*, to have made a net gain than a net loss. Odds of 99 to 1 may be considered rather large; but insurance companies, in the course of daily business, are constantly having to deal with odds far greater.

Let us take another instance. My house is insured for a premium at the rate of 1*s.* 6*d.* per £100, which, translated into probability language, means that the insurance company bets me about 1333 to 1 that my house will not be burnt down within the next 12 months. But the company have to allow for office expenses and profits, so that the real odds, in their opinion, are greater than this; let us suppose 1499 to 1, implying an event the probability of whose occurrence is $\frac{1}{1500}$. Now 12,000 may, in probability language, be fairly considered a "very large number"; and if, six months ago, any one had asked me what was likely to happen to a millionaire who kept on making bets of 1499 shillings to 1 shilling against such an event, I should have confidently replied that after 12,000 trials he was just as likely to have made a net loss as a net gain. Well, let us put $a = 1499$, $b = 1$, $n = 6$ in the formula. We obtain

$$\Delta_n = \cdot 06136, \quad G_n = \cdot 18816;$$

whence probability of net gain $= \cdot 43660$, probability of net loss $= \cdot 37524$, or probability of net gain exceeds, *by so much as* 17 *per cent.*, the probability of net loss.

22. The preceding considerations lead to an important conclusion. It is this: that, in order to secure the approximate balancing of gains and losses, *it is not only necessary that the number of trials should be a*

large number, but that the product of the number of trials by the probability of the event should also be a large number. This, once pointed out, may be seen to be true on other grounds; but, strange to say, I am unable to find it mentioned by any previous writer.

On the contrary, a writer of deservedly high repute tells us: "If the probability of an event be p, then, out of N cases in which it is in question, it will happen pN times, N being any very large number."* The necessary condition, of course, ought to be, not that N is a very large number, but that pN is a very large number. A similar objection applies to the statement of another writer, to whom all students of probability are greatly indebted, viz.: "The value of a given chance of obtaining a given sum of money is the chance multiplied by that sum; for *in a great number of trials* this would give the sum actually realized."†

23. We must now pass on to another portion of our subject. The examination of broken-period sets of trials will be found to lead to conclusions quite as interesting as, and still more curious than, those we have already discussed.

For want of a better term, let us use the word "advantage" to denote, after a certain number of events, in the case of a gambler who *gives odds*, the excess of the probability that he has made a net gain over the probability that he has made a net loss. We will make a further remark on the word in Art. 41. The symbol A_r may conveniently be employed to denote the "advantage" after r events.

II. *Broken-period Sets.*

24. Consider first the expansion of $(a, b)^{na+nb+1}$. From Art. 5, we have

$$S_{na+n+1} - S_{na+n} = \frac{a}{a+b} G_n,$$

and, in like manner, $L_{na+n+1} - L_{na+n} = \frac{b}{a+b} G_n$,

Therefore, by subtraction,

$$A_{na+n+1} = A_{na+n} + \frac{a-b}{a+b} G_n,$$

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* Professor Chrystal, *Algebra*, Vol. II., Chap. XXXVI., § 1.

† Professor Crofton, Article "Probability" in the *Encyclopædia Britannica*. The italics are my own. I have been most unwilling to cite particular authors, for almost every writer on probability has made statements more or less equivalent; a few months ago I should certainly have done so myself.

showing that the "advantage" after $na+n+1$ trials *exceeds* that after $na+n$ trials by $\frac{a-b}{a+b}G_n$.

25. Take next the expansion of $(a, b)^{na+nb-1}$. Here, observing that G_n the $(nb+1)^{\text{th}}$ term of $(a, b)^{na+nb}$ is also the nb^{th}, and likewise the $(nb+1)^{\text{th}}$ term of $(a, b)^{na+nb-1}$, we have

$$S_{na+nb}-S_{na+nb-1}=-\frac{b}{a+b}G_n,$$

and, similarly, $$L_{na+nb}-L_{na+nb-1}=-\frac{a}{a+b}G_n.$$

Whence, by subtraction,

$$A_{na+nb}=A_{na+nb-1}+\frac{a-b}{a+b}G_n,$$

that is to say, the "advantage" after $na+nb-1$ trials *falls short* of the "advantage" after $na+nb$ trials by $\frac{a-b}{a+b}G_n$; therefore, according to the formula of Art. 13, it is a *negative quantity*. Hence, if a broken-period set consist of one trial less than a complete set, the man who *takes* odds, and not the man who *gives* odds, has the so-called "advantage." This, however, in no way weakens the general conclusions already deduced from Art. 4, as we proceed to show.

26. The algebraical sum of the two "advantages" after $na+nb+1$ and $na+nb-1$ trials is $2\Delta_n$, which is positive; thus the "advantage" in the former case exceeds the "disadvantage" in the latter. And if a gambler makes $na+nb-1$, or $na+nb$, or $na+nb+1$ trials, all equally likely, his probable "advantage," *i.e.*, the *excess* of his gain-chance over his loss-chance (but not the *ratio* of the chances), will be the same as if he were to make $na+nb$ trials for certain. The matter is sufficiently interesting to allow of illustration by numerical examples, calculated independently of our formulæ.

(i.) $$\left(\frac{2}{3}+\frac{1}{3}\right)^5 \text{ "Advantage" } =-\frac{19}{243},$$

$$\left(\frac{2}{3}+\frac{1}{3}\right)^6 \quad \text{"} \quad =+\frac{23}{729},$$

$$\left(\frac{2}{3}+\frac{1}{3}\right)^7 \quad \text{"} \quad =+\frac{103}{729},$$

(ii.) $$\left(\frac{5}{6}+\frac{1}{6}\right)^5 \text{ "Advantage" } = -\frac{763}{3888},$$

$$\left(\frac{5}{6}+\frac{1}{6}\right)^6 \quad \text{"} \quad = +\frac{1672}{23328},$$

$$\left(\frac{5}{6}+\frac{1}{6}\right)^7 \quad \text{"} \quad = +\frac{7922}{23328}.$$

Note that, in each group, the middle "advantage" is the algebraical mean of the first and last.

From (ii.) we may deduce that if 139968 persons throw dice, each betting always (with some person or persons outside) 5 to 1 against the ace, and that if 46656 stop after 5 throws, 46656 after 6 throws, and 46656 after 7 throws, then we may "expect," in the aggregate, 65625 to make a net gain, 55593 a net loss, and 18750 to end as they began. If, on the contrary, the whole 139968 had thrown 6 times and no more, we must have "expected" 46875 to make a net gain, 36843 a net loss, and 56250 to end as they began. The difference, but not the ratio, between the gainers and the losers is the same in both cases.

27. Putting $b = 1$, we will now consider broken-period sets represented by $n(a+1)\pm p$, where p lies between 1 and $a+1$. It is very easy, but superfluous, to show that the algebraical sum of the "advantages" after $n(a+1)+2$ and after $n(a+1)-2$ trials is greater than twice Δ_n. The case is included in the proposition we are about to prove, viz., that the sum of the "advantages" after $n(a+1)+p$ and after $n(a+1)-p$ trials is greater than the like sum after $n(a+1)+(p-1)$ and after $n(a+1)-(p-1)$ trials. It may be remarked that the "advantage," for any given value of n, is always greatest when $p = 1$, and goes on diminishing as p increases, becoming negative somewhere in the latter half of the period, as p approaches the value a.

28. Denote by X the n^{th} term of

$$(a, 1)^{na+n-p},$$

which is $$\frac{(na+n-p)(na+n-p-1)\ldots(na-p+2)}{(n-1)!}\ \frac{a^{na-p-1}}{(a+1)^{na+n-p}}.$$

Then we have $$S_{na+n-p+1} - S_{na+n-p} = -\frac{X}{a+1},$$

and, similarly, $$L_{na+n-p+1} - L_{na+n-p} = \frac{X}{a+1}.$$

Therefore, by subtraction,

$$A_{na+n-p+1}-A_{na+n-p}=-\frac{2X}{a+1}.$$

Again, denote by Y the $(n+1)^{\text{th}}$ term of

$$(a,\,1)^{na+n+p-1},$$

which is $\dfrac{(na+n+p-1)(na+n+p-2)\ldots(na+p)}{n!}\,\dfrac{a^{na+p-1}}{(a+1)^{na+n+p-1}}$.

Then, as before, $A_{na+n+p}-A_{na+n+p-1}=-\dfrac{2Y}{a+1}$.

We require therefore to prove that X is greater than Y, which is the same thing as proving the coefficient of z^{n-1} in $\left(\dfrac{a}{a+1}+\dfrac{z}{a+1}\right)^{na+n-p}$ greater than the coefficient of z^n in $\left(\dfrac{a}{a+1}+\dfrac{z}{a+1}\right)^{na+n+p-1}$.

Let C_n, C_{n-1}, C_{n-2} denote the respective coefficients of z^n, z^{n-1}, z^{n-2} in $(a+z)^{na+n-p}$. Then we have to show that C_{n-1} is greater than the coefficient of z^n in

$$\frac{(a+z)^{2p-1}}{(a+1)^{2p-1}}\left\{\ldots+C_{n-2}z^{n-2}+C_{n-1}z^{n-1}+C_nz^n+\ldots\right\},$$

which, since

$$C_n=\frac{na-p+1}{na}C_{n-1},\quad C_{n-2}=\frac{na-a}{na-p+2}C_{n-1},\quad \&c.,$$

is the same as showing that $(a+1)^{2p-1}$ is greater than the coefficient of z^n in

$$(a+z)^{2p-1}\left\{\ldots+\frac{(na-a)(na-2a)}{(na-p+2)(na-p+3)}z^{n-3}+\frac{na-a}{na-p+2}z^{n-2}+z^{n-1}\right.$$
$$\left.+\frac{na-p+1}{na}z^n+\ldots\right\},$$

that is to say, $(a+1)^{2p-1}>a^{2p-1}\dfrac{na-p+1}{na}$

$$+(2p-1)\,a^{2p-2}+\binom{2p-1}{2}a^{2p-3}\frac{na-a}{na-p+2}$$
$$+\binom{2p-1}{3}a^{2p-4}\frac{(na-a)(na-2a)}{(na-p+2)(na-p+3)}+\ldots,$$

where the symbol $\binom{l}{r}$ denotes $\frac{l!}{r!\,(l-r)!}$. Expanding $(a+1)^{2p-1}$, and comparing the two sides term by term, it is obvious that the required inequality obtains, so long as $na-p+2$ is not less than $na-a$; that is, so long as p is not greater than $a+2$.

Thus we have proved $A_{na+n+2}+A_{na+n-2} > A_{na+n+1}+A_{na+n-1}$, *i.e.*, $> 2\Delta_n$. Similarly, $A_{na+n+3}+A_{na+n-3}$ is still further $> 2\Delta_n$, and so on, till we arrive at $A_{na+n+(a+2)}+A_{na+n-(a+2)}$. Our present object is, however, attained when we arrive at $A_{na+n+p}+A_{na+n-p}$, where p is, as nearly as possible, equal to $\frac{1}{2}(a+1)$.

29. Let us again take a particular example. Suppose a number of persons each to bet 5 to 1 against the ace, and two of them to stop when the die has been thrown once, two when it has been thrown twice, two at three times, two at four times, and so on, until the number of persons is infinite. Divide them into equal groups: let group I. consist of one person who stops at 3 throws, one who stops at 9, and all who stop at intermediate numbers; group II. of the other who stops at 9, one who stops at 15, and all who stop at intermediate numbers, and so on. The n^{th} group will contain 12 persons, ranged equally on both sides of the two persons who stop at $6n$ throws.

By the preceding article, the sum of the "advantages" of these 12 persons (*i.e.*, the excess of the sum of their 12 chances of net gain over the sum of their 12 chances of net loss) is greater than $12\Delta_n$, or than $\frac{8}{3}\left(1-\frac{154}{675n}\right)G_n$. Two persons only have a chance of ending as they began, and the sum of the chances that this will happen is $2G_n$. Subtracting this, we see that the excess of the sum of the 12 chances of net gain over the 12 chances of not making a net gain is $\left(\frac{2}{3}-\frac{1232}{2025n}\right)G_n$. Therefore, if a person in the group be chosen at random, the excess of his chance of gaining over his chance of not gaining is $\frac{1}{12}\left(\frac{2}{3}-\frac{1232}{2025n}\right)G_n$, which for all values of n is positive; therefore he is more likely than not to make a net gain. The same result will ensue, whichever group we choose, also for the 5 persons we omitted before the groups began. Therefore, *any person chosen at random out of the whole infinite multitude is more likely than not to have made a net gain.* This, in other words, is the proposition we undertook in Art. 1 to prove, *i.e.*, that if a die be thrown any number of

times chosen at random, the number of appearances of the ace is more likely than not to be less than $\frac{1}{6}$ of the number of throws. The conclusion will perhaps be as surprising to the reader as it was, in the first instance, to the present writer.

30. Let us investigate the general case which includes the preceding article. The sum of the "advantages" in any complete group of $a+1$ sets of trials, whose centre is a set of $n(a+1)$ trials, is, by Art. 28, greater than $(a+1)\Delta_n$,

i.e., greater than $\frac{1}{3}(a-1)\left\{1-\frac{1}{n}\frac{4(2a+1)(a+2)}{45a(a+1)}\right\}G_n$,

i.e., ,, $G_n+\frac{1}{3}(a-4)G_n-\frac{1}{n}\frac{4(a-1)(2a+1)(a+2)}{135a(a+1)}G_n$,

which, whenever $a>4$, is easily seen to be $>G_n$.

Thus, by the reasoning of the previous article, if a person bets more than 4 to 1 in a fair wager, and is undecided as to how many trials he has made, or will make, he is more likely than not finally to make a net gain. This surely is the case of the ordinary persistent gambler who gives odds, provided that his stakes are always the same, and small compared with his means: for it may be assumed that no gambler ever kept a systematic record, perfectly accurate from boyhood, of the exact *number* of transactions he had entered upon.

31. It will be observed that in Arts. 27–30 we have assumed that $b=1$. I made many persistent attempts, by the method of Art. 11, and otherwise, occupying more time than I should like to confess, to extend the proposition of Art. 28 to the general case of $(a, b)^{na+nb+p}$, with the result of discovering at the end that it is not true except when $b=1$. The method of Art. 28 for forming the successive "advantage"-differences is not valid when $b>1$, since the number of terms in the short-side increases gradually in a most irregular and confusing way from nb to $nb+b$, as we proceed from expansion to expansion between $(a, b)^{na+nb}$ and $(a, b)^{na+nb+a+b}$. The demonstration of Art. 28, in fact, though apparently valid until $p=a+2$, will be found, on careful examination, to be valid, even for the case of $(a, 1)^{na+n+p+1}$, only until $p=a-1$. When $p=a$, we can see on other grounds that

$$\Delta_{n+1}+\Delta_{n-1}>2\Delta_n,$$

and therefore $$A_{(n+1)(a+1)}+A_{(n-1)(a+1)}>2A_{na+a};$$

but I doubt very much if it is true always that

$$A_{(n+1)(a+1)+1}+A_{(n-1)(a+1)-1}>2A_{na+a}.$$

As to b being greater than unity, put $a=3$, $b=2$, $n=1$, $p=3$.

We can prove at once, by numerical calculation, that the sum of the "advantages" for $(\frac{3}{5}+\frac{2}{5})^2$ and $(\frac{3}{5}+\frac{2}{5})^8$, so far from being greater than twice the "advantage" for $(\frac{3}{5}+\frac{2}{5})^5$, is a *negative quantity*. Therefore, for this and certain other cases at any rate, $A_{na+nb-p}+A_{na+nb-p}$ is not greater than $2A_{na+nb}$. I am, in fact, inclined to think that the sum is nearly always negative when pb exceeds by unity a multiple of $a+b$.

32. The above conclusion is very disappointing, as I have been thereby prevented from proving our results for every case; and, although Art. 4 is demonstrated for all integral values of b, it has been necessary to qualify the first paragraph of Art. 1 by the somewhat clumsy limitation that $b=1$. There can be little doubt, I think, that the paragraph in question, as well as Art. 30, though necessarily subject to this limitation so far as Art. 28 is concerned, is susceptible of proof by some other method when b is any integer less than a. "If an event happen on the average once in m times, m being greater than unity, then it is more likely to happen less than once in m times than it is to happen more than once in m times." To my mind, at least, it seems incredible that such a proposition should be true for all integral values of m, and not true for all improper-fractional values of m. There the matter must be left at present.

33. It may be interesting to summarize roughly, by means of a figure, the foregoing conclusions. We will confine ourselves to the case when $b=1$. Neglecting higher powers of $\frac{1}{n}$, we may put

$$\Delta_n = hG_n - \frac{k}{n}G_n,$$

where h is less than $\frac{1}{3}$, and k less than $\frac{1}{15}$. The variations of G_n, as n changes, are difficult to represent; but we know at least that G_n is always less than unity, and continually decreases, G_{n+1} being, roughly speaking, equal to $G_n\left(1-\frac{1}{2n}\right)$; moreover, when n is large, G_n varies as $n^{-\frac{1}{2}}$. Therefore, except when n is very small, we may put

$$\Delta_n = \frac{C}{n^{\frac{1}{2}}} - \frac{D}{n},$$

where both C and D are considerably less than unity, and D much smaller than C.

Take a horizontal line as axis of x, and a vertical line as axis of y. When x has any value r (always integral) let the corresponding value of y (positive or negative) denote the "advantage," at the end of r trials, of a person who gives odds of a to 1. For values of x

which are multiples of $a+1$, we thus obtain points on a curve $M_1M_2M_3\ldots$ *above* the axis of x, always approaching the axis, and convex on the lower side.

When x has any value $n(a+1)+1$, the corresponding value of y will lie between four times and five times its value when

$$x = n(a+1);$$

ultimately, when n is very large, approaching the former limit. The sum of the ordinates for

$$x = n(a+1)+1 \quad \text{and} \quad x = n(a+1)-1$$

will always be exactly double of the ordinate when

$$x = n(a+1).$$

As x ranges from $n(a+1)+1$ to $n(a+1)+a$, y will continually diminish; lying, I have reason to think, on a curve of very slight convexity turned downwards, the convexity being, however, so slight that, unless the figure is drawn on a very large scale, it is hardly distinguishable from a straight line. From

$$x = n(a+1)+a \quad \text{to} \quad x = n(a+1)+a+2,$$

the curve will proceed upwards in an absolutely straight line, and so on, repeating itself in similar fashion for the next period.

For large values of x, the highest points of these zigzags will lie very nearly on the curve

$$y = 4ax^{-\frac{1}{2}},$$

and the lowest points very nearly on the curve

$$y = -2ax^{-\frac{1}{2}},$$

the complete-period curve $M_1M_2M_3\ldots$ being

$$y = ax^{-\frac{1}{2}},$$

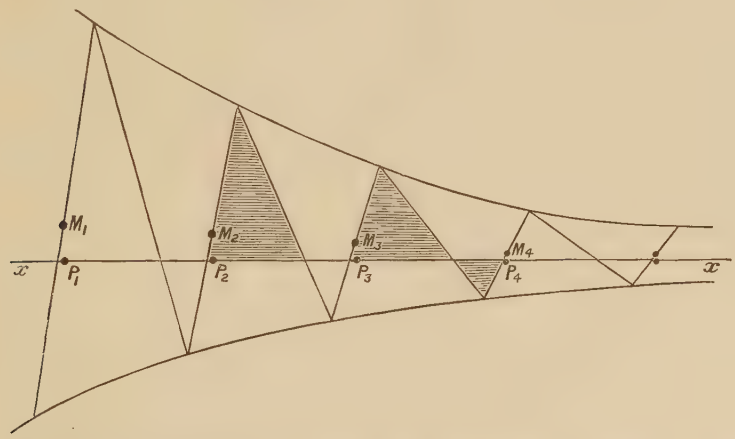
where a is some positive quantity, a fixed function of a.

If it be borne in mind that the vertical proportions are, of necessity, enormously magnified, we may obtain from the subjoined figure a very clear idea of the average "advantage" for any given range of trials. If x, the number of trials, is known to lie between two values represented on the x-line by the points H and K, all we have to do is to draw vertical lines HH' and KK' through H and K. The average "advantage" for that range of trials is very nearly obtained by subtracting all the lower shaded areas between HH' and KK' from all the upper shaded areas between HH' and KK',* and dividing by the length of HK. We perceive at once how, when HK is greater than any complete period-length P_1P_2, the average "advantage" is always

* By an oversight, some of the shading in the figure has been omitted. *All* the upper triangles ought to be shaded, as well as all the lower ones.

positive; but that it may be negative when HK is very small and situated close to the left of any of the points P_1, P_2, P_3, &c. We perceive also at once how the average "advantage" diminishes numerically, (i.) if HK is moved bodily to the right, over any distance which is a multiple of P_1P_2; or (ii.) if, H remaining fixed, K is moved to the right over any distance which is a multiple of P_1P_2. These are the main facts which it has been the object of the present paper to prove.

When x is very large, it is interesting to note that any shaded triangle above the axis has clearly four times the area of either adjacent shaded triangle below the axis, which would seem to show, especially when a as well as x is large, that the sum of the positive "advantages" for any complete period, from $x = n(a+1)$ to $x = (n+1)(a+1)$, is four times the sum of the negative "advantages" for the same period.



The reader can easily construct for himself the beginning of the "advantage"-curve for any given value of a. For instance, when $a = 4$, the first thirteen values of y, from $x = 1$ to $x = 13$, are ·6, ·28, ·024, −·1808, ·06496, ·31072, ·1534, ·00662, −·12758, ·05361, ·23480, ·116690, ·003294. Note that here y is positive in every case except when $x = 4$ or 9; and I have no doubt that in fact y is always positive except when x is of the form $5m-1$.

Again, when $a = 5$ (the case of a die), the first nine values of y are ·6, ·38, ·148, −·0355, −·1962, ·0717, ·3396, ·2093, ·085; and I have no doubt that for *every* complete period there are four positive values of y and two negative ones. The proportions of the curve for these two cases are not quite the same as those in the figure, the latter being illustrative of *large* values of x.

34. In this and the next three articles, without assuming any of the results of pp. 294–315, we will consider the whole matter from another point of view.

If every term of $(a, b)^{na+nb}$ be divided by the $(nb+1)^{\text{th}}$ term G_n, we may, commencing from the middle of the expansion, write, for the short-side and long-side,

$$[S]_{na+nb} = \frac{n}{n+\frac{1}{a}} + \frac{n\left(n-\frac{1}{b}\right)}{\left(n+\frac{1}{a}\right)\left(n+\frac{2}{a}\right)} + \frac{n\left(n-\frac{1}{b}\right)\left(n-\frac{2}{b}\right)}{\left(n+\frac{1}{a}\right)\left(n+\frac{2}{a}\right)\left(n+\frac{3}{a}\right)} + \ldots$$

to nb terms,

$$[L]_{na+nb} = \frac{n}{n+\frac{1}{b}} + \frac{n\left(n-\frac{1}{a}\right)}{\left(n+\frac{1}{b}\right)\left(n+\frac{2}{b}\right)} + \ldots \quad \text{to } na \text{ terms.}$$

If the excess of the former expression over the latter could be expanded in a series of the form $B+Cn^{-1}+Dn^{-2}+En^{-3}+\&c.$, all the results of the present paper would be obtained. The fact that

$$B = \frac{1}{3}\,\frac{a-b}{a+b}$$

ought, one would think, by putting $n=\infty$, to admit of a fairly simple straightforward proof; I have made repeated attempts, in vain, to obtain one myself.

35. If s_r denote the r^{th} term of the former series, and l_r the r^{th} term of the latter, and if K_r denote the ratio of s_r to l_r, we easily obtain

$$K_{r+1} = \frac{a^2}{b^2}\,\frac{(2nb+1)^2-(2r+1)^2}{(2na+1)^2-(2r+1)^2}K_r,$$

showing that K_r, which at first is greater than unity, diminishes with increasing rapidity at every subsequent term, till it becomes zero. As soon, then, as any term s_ρ is less than l_ρ, every succeeding short-side term is less than the corresponding long-side term.

Again, if $T_1, T_2, \ldots T_{na+nb+1}$ be the successive terms of $(a, b)^{na+nb}$, and a person bet odds of a shillings to b shillings in a fair wager, after $na+nb$ trials his expectation will be

$$nb\,(a+b)\,T_1+(nb-1)(a+b)\,T_2+(nb-2)(a+b)\,T_3+\ldots$$
$$\ldots+(a+b)\,T_{nb}+0\,.\,T_{nb+1}$$
$$-(a+b)\,T_{nb+2}-2\,(a+b)\,T_{nb+3}-\ldots-na\,(a+b)\,T_{na+nb+1}.$$

But this expectation is zero. Hence

$$T_{nb}+2T_{nb-1}+\ldots+(nb-1)\,T_2+nbT_1=T_{nb+2}+2T_{nb+3}+\ldots+naT_{na+nb+1},$$

i.e., $$s_1+2s_2+3s_3+\ldots+nb\,.\,s_{nb}=l_1+2l_2+3l_3+\ldots+na\,.\,l_{na}.$$

36. From the preceding article we can deduce at once that Δ_n is always positive.*

Let s_p be the first short-side term which is less than l_p, the corresponding long-side term. Then, since

$$s_1+2s_2+3s_3+\ldots+nb\,.\,s_{nb}=l_1+2l_2+3l_3+\ldots+na\,.\,l_{na},$$

and

$$s_p+2s_{p+1}+3s_{p+2}+\ldots+(nb-p+1)\,s_{nb}<l_p+2l_{p+1}+\ldots+(na-p+1)\,l_{na},$$

we have, by subtraction,

$$s_1+2s_2+\ldots+(p-2)\,s_{p-2}+(p-1)(s_{p-1}+\ldots+s_{nb})$$
$$>l_1+2l_2+\ldots+(p-1)(l_{p-1}+\ldots+l_{na}).$$

But, $s_1, s_2, \ldots s_{p-1}$ being severally greater than $l_1, l_2, \ldots l_{p-1}$,

$$(p-2)\,s_1+(p-3)\,s_2+\ldots+2s_{p-3}+s_{p-2}>(p-2)\,l_1+(p-3)\,l_2+\ldots+l_{p-2}.$$

Therefore, by addition,

$$(p-1)(s_1+s_2+\ldots+s_{p-2}+\ldots+s_{nb})>(p-1)(l_1+l_2+\ldots+l_{p-2}+\ldots+l_{na}),$$

or $$s_1+s_2+s_3+\ldots+s_{nb}>l_1+l_2+l_3+\ldots+l_{na}.$$

37. The desirability of an algebraical proof of the last equation of Art. 35 has been suggested; and I venture to offer the following.

The $(nb)^{\text{th}}$ and $(nb+1)^{\text{th}}$ terms of $(a, b)^{na+nb-1}$ are both equal to G_n. Dividing the whole expansion by G_n, the short-side, in reversed order, may be written

$$[S]_{na+nb-1}=1+\frac{n-\frac{1}{b}}{n+\frac{1}{a}}+\frac{\left(n-\frac{1}{b}\right)\left(n-\frac{2}{b}\right)}{\left(n+\frac{1}{a}\right)\left(n+\frac{2}{a}\right)}$$
$$+\frac{\left(n-\frac{1}{b}\right)\left(n-\frac{2}{b}\right)\left(n-\frac{3}{b}\right)}{\left(n+\frac{1}{a}\right)\left(n+\frac{2}{a}\right)\left(n+\frac{3}{a}\right)}+\ldots \text{ to } nb \text{ terms.}$$

* This proof has gradually evolved itself out of some remarks in a letter received, subsequently to April 4th, from Dr. Biddle, of Kingston-on-Thames, to whom I am in consequence greatly indebted. Dr. Biddle's own proof, which is somewhat different, will appear in *Educational Times Reprint*, Vol. LXIII., Quest. 12686.

Subtract from each term the corresponding term in $[S]_{na+nb}$, and we have

$$[S]_{na+nb-1}-[S]_{na+nb}=\frac{1}{na}\frac{n}{n+\frac{1}{a}}+\frac{2}{na}\frac{n\left(n-\frac{1}{b}\right)}{\left(n+\frac{1}{a}\right)\left(n+\frac{2}{a}\right)}$$

$$+\frac{3}{na}\frac{n\left(n-\frac{1}{b}\right)\left(n-\frac{2}{b}\right)}{\left(n+\frac{1}{a}\right)\left(n+\frac{2}{a}\right)\left(n+\frac{3}{a}\right)}+\ldots$$

$$=\frac{1}{na}\left(s_1+2s_2+3s_3+\ldots+nb\,.\,s_{nb}\right).$$

But, by Art. 5, $[S]_{na+nb-1}-[S]_{na+nb}=\frac{b}{a+b}$;

therefore* $s_1+2s_2+3s_3+\ldots+nb\,.\,s_{nb}=\frac{nab}{a+b}$.

Similarly,* $l_1+2l_2+3l_3+\ldots+na\,.\,l_{na}=\frac{nab}{a+b}$;

therefore $s_1+2s_2+3s_3+\ldots+nb\,.\,s_{nb}=l_1+2l_2+3l_3+\ldots+na\,.\,l_{na}$.

38. The long-side of $(a,\,b)^{na+nb-1}$, after division by G_n, may be written

$$[L]_{na+nb-1}=1+\frac{n-\frac{1}{a}}{n+\frac{1}{b}}+\frac{\left(n-\frac{1}{a}\right)\left(n-\frac{2}{a}\right)}{\left(n+\frac{1}{b}\right)\left(n+\frac{2}{b}\right)}+\ldots \text{ to } na \text{ terms.}$$

Each of the first nb terms of this is greater than the corresponding term in $[S]_{na+nb-1}$ above. Hence $S_{na+nb-1}-L_{na+nb-1}$ is negative, *i.e.*, $A_{na+nb-1}$ is negative, whatever be the values of a, b, n. This proof is of more value than the one given in Art. 25, for it is obtained independently of the Δ_n formula, and leads to the only rigid demonstration I have been able to contrive to show that the function $\psi(a)$ in Art. 9 is absolute zero.

Let us try and put the matter clearly. By Art. 36, A_{na+nb} or Δ_n is always positive; by the present article, $A_{na+nb-1}$ is always negative; and, by Art. 5 (see also Art. 25), $A_{na+nb-1}$ always $=A_{na+nb}-\frac{a-b}{a+b}G_n$. Now, if $\psi(a)$ be not absolute zero, it must, for any given value of a,

* These two interesting relations have perhaps been proved before; but I am unable to find them anywhere.

be either positive or negative, and cannot be both. Again, observing the formula of Art. 9, it is evident that the terms following $\psi(a)$, both alone and when diminished by $\frac{a-b}{a+b} G_n$, can be made as small as we please, and therefore smaller than $\psi(a)$, by taking n large enough. Hence, ultimately, A_{na+nb} and $A_{na-nb-1}$ are both of the same sign as $\psi(a)$. Therefore, when n is infinite, $\psi(a)$ is both positive and negative; but $\psi(a)$ is independent of n, and therefore would be for all values of n both positive and negative, which is impossible. Therefore $\psi(a)$ must be absolute zero, and Δ_n can contain no term independent of n.

39. My task is now completed; but it may be permissible to add three more articles by way of supplement.

The first explorer of a hitherto untrodden region has not always the good fortune to discover the best road at starting; certainly I had not; and an account of the various paths pursued to open out the way in our present subject may be of interest, as saving trouble to future investigators. My only ambition at first was to prove that Δ_n is always positive, the original attempt being by what still seems to me the most natural and straightforward method, *i.e.*, to show that, when $b = 1$, the first series in Art. 34 always exceeds the second. Repeated failures in this direction were probably due to lack of skill, for the two series certainly look as if they ought to be manageable, and they may be commended to the reader's attention.

Next, putting $a = 2$, the expansion of $(2+1)^{3n} \times (2+1)^3$ was compared with that of $(2+1)^{3n+3}$, leading to the result

$$\Delta_n - \Delta_{n+1} = \frac{2^{2n+1}}{3^{3n+3}} \frac{(3n)!}{n!\,(2n+2)!} (3n-1);$$

which, finally, after a good deal of trouble, gave $\Delta_n =$ the constant term in the expansion of

$$\frac{y}{y+4} \left\{ \frac{4^n}{3^{3n}} (1+y)^{3n} - 1 \right\},$$

which, on making n infinite, is ultimately equal to $\frac{1}{9} \frac{4^n}{3^{3n}}$ times the coefficient of y^n in $(1+y)^{3n}$, or $\frac{1}{9}$ of the greatest term in $(\frac{1}{3}+\frac{2}{3})^{3n}$.

This not only proved Δ_n to be positive when n is infinite, and to increase as n diminishes, which had been looked for; but it gave the ultimate value of Δ_n, which had not been looked for. The same

method was much more difficult, but still manageable, when applied to $(3+1)^{4n}$; and gave

$$\Delta_n - \Delta_{n+1} = \frac{3^{3n+1}}{4^{4n+3}} \frac{(4n)!}{n!\,(3n+3)!}(48n^2+14n-7),$$

positive again; also Δ_n ultimately equal to $\frac{1}{6}$ of the greatest term in $(\frac{1}{4}+\frac{3}{4})^{4n}$. If now $3n-1$ and $48n^2+14n-7$ could have been expressed in the form $\psi(2, n)$, and $\psi(3, n)$, and ψ determined, it would have been allowable to put

$$\Delta_n - \Delta_{n+1} = \frac{4^{4n+1}}{5^{5n+4}} \frac{(5n)!}{n!\,(4n+4)!}\psi(4, n)$$

for the expansion of $(4+1)^{5n}$. Efforts to this end proved fruitless; moreover, the direct method, in its application to $a = 4$ and greater values, was utterly unmanageable, and for a long time the work was at a standstill. It seemed clear, however, that, for infinite values of n, Δ_n must be always some simple multiple of G_n.

At length another method was found, which, proving Δ_n ultimately $= \frac{1}{5}G_n$, when $a = 4$, and $= \frac{2}{9}G_n$ when $a = 5$, suggested the result

$$\Delta_n = \frac{1}{3}\,\frac{a-1}{a+1}\,G_n$$

for the general case. The method proved, in fact, applicable to the general case when $n = \infty$, but resolutely refused to prove $\Delta_n - \Delta_{n+1}$ always positive when n is finite.

Again work was at a standstill, and recourse was had to a large number of numerical examples, in doubt whether the theorem, after all, was universally true. These examples showing, quite by accident, that Δ_n always lies about half-way between

$$\frac{1}{3}\,\frac{a-1}{a+1}\,G_n \quad \text{and} \quad \frac{1}{3}\,\frac{a-1}{a+1}\,G_{n+1},$$

exploration was suggested on a new track. This was to expand Δ_n in a series of ascending powers of $\frac{1}{n}$; which, by a most cumbrous and intricate method, gave at length the second approximation formula for Δ_n, with its surprising applicability to small values of n. This, together with the investigation of broken-period sets, formed the basis of the communication on April 4th. The method was still based on the comparison of the two expansions of $(a+1)^{(n+1)(a+1)}$ and $(a+1)^{n(a+1)} \times (a+1)^{a+1}$, and was about five times the length of the one here given. Moreover, it was utterly inapplicable to the expansion

of $(a+b)^{na+nb}$, whose Δ_n formula was given on that occasion without demonstration.

Not until May, *i.e.*, after three months' hard work on the theorem, was it perceived that a fundamental and yet quite natural mistake had prevailed throughout; which was imagining that the expansion of $(a+1)^{n(a+1)}$ was more easily manageable than the expansion of

$$\left(\frac{a}{a+1}+\frac{1}{a+1}\right)^{n(a+1)}.$$

A new and far shorter method, inapplicable to the former case, was found applicable to the latter, and is the one here given. By the indulgence of the Society and the referees, I have been permitted to entirely re-write the paper, finally adding Arts. 36 and 38. It is curious that the apparently simple matter of proving $\psi(a)=0$ should have proved the most baffling task of all; possibly the reader may be able to devise a simpler and more direct method.

Moreover, we are now able to add the third approximation formula for Δ_n, impossible by the former method. Arts. 14–19 still seem necessary, even after this addition, so they have been left nearly in their original form.

40. It will be found that the coefficient of $\frac{1}{n^3}$ in the expression for T_r in Art. 6 is

$$-\frac{r^2(r+1)^2}{48a^3(a+1)^3}\{r^2+r(8a+5)+12a^2+16a+6\}.$$

Summing this from $r=1$ to $r=a$, a very tiresome process, we obtain

$$S_{(n+1)(a+1)}-S_{n(a+1)}=\frac{a+2}{6(a+1)}\left[\frac{1}{n}-\frac{1}{n^2}\,\frac{13a^2+21a+6}{20a(a+1)}\right.$$
$$\left.+\frac{1}{n^3}\,\frac{407a^4+1306a^3+1225a^2+398a+24}{840a^2(a+1)^2}+\ldots\right]G_n.$$

Proceeding in Art. 7 to another term, we shall now find that

$$G_{n+1}=\left\{1-\frac{1}{2n}+\frac{1}{n^2}\,\frac{11a^2+11a+2}{24a(a+1)}-\frac{1}{n^3}\,\frac{7a^2+7a+2}{16a(a+1)}+\ldots\right\}G_n;$$

whence, by an extension of the method of Art. 8, we obtain

$$S_{n(a+1)}=\frac{1}{2}-\frac{a+2}{3(a+1)}\left\{1+\frac{2}{45n}\,\frac{(a-1)(2a+1)}{a(a+1)}\right.$$
$$\left.-\frac{4}{945n^2}\,\frac{(a-1)(2a+1)(a^2+a+1)}{a^2(a+1)^2}+\ldots\right\}G_n.$$

Observing that $945 = 3^3 . 5 . 7$, we thus have

$$\Delta_n = \frac{1}{3}\,\frac{a-1}{a+1}\left\{1 - \frac{1}{n}\,\frac{2^2(a+2)(2a+1)}{3^2 . 5a\,(a+1)} + \frac{1}{n^2}\,\frac{2^3(a+2)\,(2a+1)\,(a^2+a+1)}{3^3 . 5 . 7a^2\,(a+1)^2} + \ldots\right\} G_n.$$

This may be compared with the two expressions for $\Delta_n - \Delta_{n+1}$ given in Art. 39. The reader, by substituting numerically, will moreover find the "proportion of error" now, generally speaking, about $\frac{1}{4}$ of what it was in the second approximation, and again less for large than for small values of a. It is disappointing to find, however, that this third approximation is, in all the cases I have tried, *less* than the actual value, suggesting the presumption that the coefficient of $\frac{1}{n^3}$ will be again positive, and destroying, in one detail, the symmetry of the series.

I have since proved, by an independent method, that, when $a = 2$, the next term inside the bracket is $+\frac{56}{3^7 . n^3}$, and, when $a = 3$, it is $+\frac{169}{4 . 3^7 . n^3}$. Both of these results conform with the formula

$$+\frac{1}{n^3}\,\frac{2^4(a+2)(2a+1)(a^2+a+1)^2}{3^4 . 5 . 7 . a^3\,(a+1)^3},$$

suggesting the possibility of this being, for all values of a, the next term inside the bracket. On testing numerically, I find this fourth approximation, both for $a = 2$ and $a = 3$, greater than the actual value of Δ_n, implying apparently that the coefficient of $\frac{1}{n^4}$ will be for those cases, and therefore also for the general case, negative.

It is interesting to note that the quantities $1, -\frac{2^3}{3^2 . 5}, \frac{2^4}{3^3 . 5 . 7}, \frac{2^5}{3^4 . 5 . 7}$ form a converging series; so that, if the above formula gives the actual coefficient of $\frac{1}{n^3}$, a still stronger presumption is afforded of the correctness of the hypothesis of Art. 19.

The regularity of form of the successive coefficients further suggests the possibility of expanding either of the series of Art. 34, and their difference, in such manner as to obtain an expression for the *general* term. Whether, in fact, our theorem and formulæ admit of a simple, straightforward proof, or remain for generations

apparently incapable of it, like Bernoulli's Theorem, will be left for the future to decide.

41. In conclusion, it is hardly necessary to assure the reader that my object has been in no sense to prove that the gambler who gives odds is in a *more advantageous position* than the one who takes odds. In fact, I have proved afresh, in Art. 37, that in a fair wager the gambler's "expectation" is always zero, whether after 10 trials or after 10,000,000. The vocabulary of the English language, and probably of all languages, is strikingly deficient in words exactly suitable for the scientific treatment of probability; and the word "advantage," to denote excess of probability of net gain over probability of net loss, is the best I could devise. But the fact that it has always been carefully guarded within inverted commas may suffice to show that it is throughout employed in a purely technical sense, and not in the ordinary one.

Thursday, May 9th, 1895.

Major P. A. MACMAHON, R.A., F.R.S., President, in the Chair.

Mr. William Henry Metzler, A.B. (Toronto), Associate-Professor, Syracuse University, Syracuse, New York, and Mr. Frederick William Russell, B.A., formerly Scholar of Trinity College, Cambridge, assistant-master in University College School, London, were elected members. Mr. P. H. Cowell was admitted into the Society.

The following communications were made:—

On the most General Solution of Given Degree of Laplace's Equation: Dr. Hobson.

A Property of a Skew-Determinant, and on the Geometrical Meaning of a Form of the Orthogonal Substitution: Prof. M. J. M. Hill.

The Spherical Catenary: Prof. Greenhill and Mr. T. I. Dewar (a model was exhibited of this catenary, formed by a chain wrapped on a terrestrial globe).

Mr. Heppel exhibited a set of Napier's Bones, of date 1746, and explained how they were used in calculation.

The following papers were taken as read:

On those Orthogonal Substitutions that can be generated by the Repetition of an Infinitesimal Orthogonal Substitution: Dr. H. Taber.

Notes on the Theory of Groups of Finite Order (*continuation*): Prof. W. Burnside.

Applications of Trigraphy: Mr. J. W. Russell.

The Reciprocators of Two Conics: Messrs. J. W. Russell and A. E. Jolliffe.

The following presents were received:—

Queen's College, Galway, "Calendar for 1894–95," 8vo; Dublin, 1895.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XIX., St. 4; Leipzig, 1895.

"Cambridge Philosophical Society, Proceedings," Vol. VIII., Pt. 4; October, 1894.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang 40, Heft 1; 1895.

Kluyver, J. C.—"Invarianten-Theorie," pamph., 8vo (offprint).

"Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," III., 1894.

"Archives Néerlandaises," Tome XXIX., Liv. 1; Harlem, 1895.

"Proceedings of the Royal Society," Vol. LVII., Nos. 343–344.

"Proceedings of the Physical Society of London," Nos. 55, 56; April–May, 1895.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. I., Nos. 6–7. New York.

"Société des Antiquaires de l'Ouest—Bulletin," Trim. IV.; 1894.

"Bulletin des Sciences Mathématiques," Série 2, Tome XIX., Avril, 1895; Paris.

"Rendiconto dell' Accademia della Scienze Fisiche e Matematiche," Serie 3ª, Vol. I., Fasc. 3; Napoli, 1895.

"Mémoires de la Société des Sciences de Bordeaux," Tome III., Cahier 2; Tome IV., Cahier 1, 2; Paris, 1893.

Perot, A.—"Sur l'Existence et la Propagation des Oscillations Electro-magnétiques dans l'Air," pamph.

Rayet, G.—Observations Pluviométriques et Thermométriques, 8vo; Bordeaux, 1893.

Thomae, J.—"Untersuchungen über zweizweideutige Verwandschaften und einige Erzeugnisse derselben," roy. 8vo; Leipzig, 1895.

"Educational Times," May, 1895.

"Atti della Reale Accademia dei Lincei—Rendiconti," 1 Sem., Vol. IV., Fasc. 6–7; Roma, 1895.

"Annales de la Faculté des Sciences de Marseille," Tome IV., Fasc. 1, 2, 3.

"Journal de l'Ecole Polytechnique," Cahier 64; Paris, 1894.

"Indian Engineering," Vol. XVII., Nos. 11–15.

"Acta Mathematica," XIX., No. 2.

"The Mathematical Magazine," edited by Dr. A. Martin, Washington, D.C., 1895; Vol. II., No. 9, January, 1895.

"List of Members of the London Mathematical Society," from January 15th, 1866, to November 9th, 1893; 30 sessions (14th November, 1867, called 4th session, 2 previous lists).

Notes on the Theory of Groups of Finite Order (continued).

By W. BURNSIDE.

Received May 7th, 1895. Read May 9th, 1895.

The first of the two notes in the present communication deals with certain properties of groups whose order is even. It is shown that if 2^m is the highest power of 2 contained in the order of a group, and if the sub-groups of order 2^m are cyclical, the group cannot be simple; so that, in particular, no group whose order is divisible by 2, but not by 4, can be simple. When the highest power of 2 which divides the order of a group is either 2^2 or 2^3 it is shown that, unless the group contains a smaller number of distinct conjugate sets of operations of orders 2 or 4 than the sub-groups of orders 2^2 and 2^3 respectively contain, the group cannot be simple. In the first case, this condition cannot be satisfied unless 3 is a factor of the order; nor can it be satisfied in the second case unless either 3 or 7 is a factor of the order, and, therefore, no group of even order can be simple unless its order is divisible by 12, 16, or 56. It seems extremely probable that this property may be extended to the more general form that, if the order of a group be

$$N = 2^m n$$

where n is odd, and if N is relatively prime to $2^m-1, 2^{m-1}-1, \ldots 2^2-1$, the group cannot be simple; but I have not hitherto succeeded in proving this more general result.

In the second note, Dr. Cole's and Herr Hölder's determination of all simple groups whose orders do not exceed 660 is carried on from 660 to 1092, the order of the next known simple group, with the result of showing that no simple groups exist in the interval.

VIII. *On Groups of Even Order ; and, in particular, those whose Orders are divisible by no higher power of* 2 *than* 2^3.

Let $$N = 2^m n,$$

where n is odd, be the order of a group ; and let the sub-groups of order 2^m be cyclical. Then, if the group contains an operation S of odd order which is not permutable with any operation of order 2, S must be one of a set of $2^m\mu$ conjugate operations, where μ is odd. If the group is simple, it can be represented as a transitive permutation-group arising from the permutations of the $2^m\mu$ conjugate operations among themselves, when they are transformed by the N operations of the group. If the set of conjugate operations be transformed by an operation of order 2^m, the resulting substitution of the permutation-group must consist of μ cycles of 2^m symbols each ; for, if any cycle consisted of 2^r ($r<m$) symbols only, the corresponding 2^r operations conjugate to S would be permutable with a group of order 2^{m-r}, which is supposed not to be the case. Now a substitution consisting of μ (odd) cycles of 2^m symbols each is equivalent to an odd number of transpositions, and a group containing such a substitution cannot be simple.

If, on the other hand, the group contains no operation of odd order which is not permutable with an operation of order 2, an operation of order 2 must itself be contained self-conjugately in the group, which again cannot be simple. Hence a group whose order is $2^m n$ (n odd), in which the sub-groups of order 2^m are cyclical, cannot be simple. In particular, a group whose order is even, but not divisible by 4, cannot be simple.

The number of possible different types of sub-group of order 2^m increases very rapidly with m, but, when m is either 2 or 3, it is not difficult to determine under what limitations it is possible for a group to be simple.

If the order is $$N = 2^2 m,$$

where m is odd, and the sub-groups of order 2^2 are not cyclical, each such sub-group contains 3 operations of order 2. Suppose that a sub-group of order 2^2 is contained self-conjugately in a sub-group of order $2^2 m_1$, where $m = m_1 m_2$. If 3 is a factor of m_1, the 3 operations of order 2 in this sub-group may form a single conjugate set, and then all the operations of order 2 in the main group form a single conjugate set. Suppose now that this is not the case, so that the

group contains 3 different conjugate sets of operations of order 2. Every operation of order 2 is certainly self-conjugate in a sub-group of order $2^2 m_1$, and may be self-conjugate in a more extensive sub-group. Let, then, S, an operation of order 2 be self-conjugate within a group of order $2^2 m_1 \mu$. An operation of this sub-group whose order is odd, and which is not contained in the sub-group of order $2^2 m_1$, is permutable with S, and with no operation of order 2 which is not conjugate to S. Hence it must form one of a set of $2r$ conjugate operations, where r is odd. If now the group be represented as a permutation-group, consisting of the permutations of these $2r$ operations among themselves which arise by transforming them by all the operations of the group, any operation of order 2 which is not conjugate to S will give a substitution in the permutation-group, consisting of r transpositions, *i.e.*, an odd substitution, and, therefore, the group cannot be simple. Hence, the group is certainly not simple unless the maximum sub-group, which contains an operation of order 2 self-conjugately, is of order $2^2 m_1$; and when this condition is satisfied every operation of order 2 is permutable with just 2 other operations of order 2, and with no more.

But, now, if A, B are two operations of order 2 belonging to different conjugate sets, and if

$$(AB)^n = 1,$$

A, B generate a dihedral group of order $2n$. If n were odd, A and B would be conjugate, which is not the case. Hence, n must be even, and then $(AB)^{\frac{1}{2}n}$ is an operation of order 2 which is permutable with n distinct pairs of operations of order 2. But this is in direct contradiction to what has just been proved, so that this case cannot occur. It follows that, if a group whose order is $2^2 m$ (m odd) contains 3 different conjugate sets of operations of order 2, it cannot be simple.

If, next, the order is $N = 2^3 m$,

where m is odd, and if, as in the case just dealt with, the main group contains the same number of conjugate sets of operations of orders 2 and 4 as are contained in a sub-group of order 2^3, it may again be shown that the group cannot be simple. In this case, however, putting aside the cyclical groups of order 2^3 which have already been dealt with, there are 4 other possible types of sub-groups

of order 2^3. These are the groups which may be generated as follows:—

(i) $A^2 = B^2 = C^2 = 1$, $AB = BA$, $AC = CA$, $BC = CB$;

(ii) $A^2 = B^4 = 1$, $AB = BA$;

(iii) $A^2 = B^4 = 1$, $AB = B^3A$;

(iv) $A^4 = B^4 = 1$, $AB = B^3A$, $A^2 = B^2$,

Of these (i) is an Abelian group containing 7 operations of order 2, each of which is self-conjugate. Group (ii) is again Abelian, and contains 4 operations of order 4, and 3 operations of order 2, each one of the 7 being self-conjugate. In the case of group (iii) there are 2 conjugate operations of order 4, and 5 operations of order 2, one of which is self-conjugate, while the remainder form 2 conjugate sets of 2 each. Group (iv) contains a single self-conjugate operation of order 2, and 6 operations of order 4 forming 3 conjugate sets of 2 each.

If, now, in the group of order

$$N = 2^3m \quad (m \text{ odd})$$

the sub-groups of order 2^3 are of type (i), and if the main group contains 7 conjugate sets of operations of order 2, let A, B be two such operations chosen from different sets. The group generated by A and B must be a dihedral group of order $4n$, where n is odd. If this sub-group is not self-conjugate within a sub-group of order 2^3n, it must form one of a set of $2r$ conjugate sub-groups, where r is odd, and, when these are transformed among themselves by operations of order 2 of conjugate sets other than those contained in the dihedral group, the corresponding substitution of the permutation-group will consist of r transpositions, which involves that the group is composite.

If the dihedral group is contained self-conjugately in a group of order 2^3n, the cyclical sub-group of order n which it contains must be transformed into itself by the operations of a group of order 2^3. Let S be the operation of order n generating the cyclical sub-group, and let A, B, C be the generating operations of the group of order 2^3, A and B belonging to the dihedral group. Then

$$ASA = S^{-1} \quad \text{and} \quad BSB = S^{-1}.$$

If, now, (α) $CSC = S^{-1}$,

S is permutable with the sub-group formed by 1, BC, CA, AB and if

(β) $CSC = S$,

S is permutable with the sub-group formed by 1, C, AB, ABC. Hence, in either case, S is one of a set of $2s$ conjugate operations, where s is odd; and it follows as before that the group is composite.

If next, the sub-groups of order 2^3 are of type (ii), and if the main group contains 3 distinct sets of conjugate operations of order 2, one of these sets contains exclusively operations which are the squares of operations of order 4, and the other two sets those that are not.

Let, now, A be an operation of order 2 which is the square of an operation of order 4, and let B be an operation of order 2 belonging to a different conjugate set from A. Then A and B must generate a dihedral group of order $4n$, where n is odd. Suppose that AB is an operation of this group of order $2n$, and write

$$(AB)^n = C, \quad (AB)^2 = S_n,$$

so that C is an operation of order 2, and S_n an operation of order n. The operation C must clearly belong to a different conjugate set from both A and B. Now

$$AS_nA = S_n^{-1}, \quad BS_nB = S_n^{-1}, \quad CS_nC = S_n.$$

If A^1 is any operation contained in the sub-group within which the cyclical sub-group generated by S_n is self-conjugate, and belonging to the same conjugate set as A, then

$$A^1S_nA^1 = S_n^{-1},$$

and, therefore, S_n cannot certainly be permutable with any operation of order 4, since it is not permutable with the square of any such operation. The operation S_n therefore forms one of a set of $4r$ conjugate operations, where r is odd; and, when these are transformed by any operation of order 4, the resulting substitution of the permutation-group consists of r cycles of 4 symbols each. This is an odd substitution, and therefore, again, in this case, the group cannot be simple.

A group of order 8 of type (iii), generated by A and B, where

$$A^2 = 1, \quad B^4 = 1, \quad AB = B^3A,$$

contains 5 operations of order 2, viz.,

$$A, \quad B^2, \quad AB, \quad AB^2, \quad AB^3,$$

of which B^2 is self-conjugate, while A, AB^2 and AB, AB^3 form conjugate sets. From these 5 operations and identity 2 groups of order 4 may be formed, viz.,

$$1, \quad B^2, \quad A, \quad AB^2$$

and

$$1, \quad B^2, \quad AB, \quad AB^3.$$

If, now, the sub-groups of order 2^3 contained in a group of order 2^3m (m odd) are of this type, and if the main group contains 3 distinct sets of conjugate sub-groups of order 2, one of these sets consists of the squares of operations of order 4, and the other two sets of operations of order 2, which are not such squares. Moreover, the above analysis of the operations of such a group of order 2^3 shows that no operation of one of the two latter sets can be permutable with any operation of the other. Now each set contains $2r$ conjugate operations, where r is odd, and, if one set is transformed by an operation of the other set, the resulting substitution consists of r transpositions, and is therefore an odd substitution. Once, again, in this case, the group, then, cannot be simple.

Finally, when the group of order 2^3 is of type (iv), and the main group contains 3 distinct sets of conjugate operations of order 4, the number of operations contained in each set must be of the form $4\mu+2$.

If such a set is transformed by one of its own operations, the resulting substitution will keep 2 symbols unchanged, and interchange the remainder in $2r$ cycles of 2 and $(\mu-r)$ cycles of 4 each, where r is some number less than μ. If the set is transformed by an operation of order 4 belonging to another conjugate set, the resulting substitution will consist of $2r^1+1$ cycles of 2 and $(\mu-r^1)$ cycles of 4 each. Now, since there is only a simple conjugate set of operations of order 2 in this case, the squares of these two substitutions must be of the same type, and therefore $r = r^1$. Hence, one of the two substitutions is necessarily odd, and it follows again in this last case that the group must be composite.

The conditions under which it has been shown that groups of order 2^2m and 2^3m, m being odd, cannot be simple may now be shown to hold necessarily if in the one case 3, and in the other 3 and 7, are not factors of m. For this purpose I prove the following theorem.

If, p^m being the highest power of a prime p which divides the order of a group G, a sub-group h of order p^m is Abelian, and if H be the greatest sub-group that contains h self-conjugately, the number of distinct sets of conjugate operations whose orders are powers of p in G is the same as the number in H.

Let P be any operation of h, and let it be one of x conjugate operations of H. Then P is permutable in a sub-group of H of order $\frac{n_H}{x}$, n_H being the order of H. Hence, if the order of the greatest sub-group within which P is permutable is $\frac{n_H x'}{x}$, P must belong to x'

different groups of order p^m. Hence, summing for the distinct sets of conjugate operations whose orders are powers of p contained in G,

$$\Sigma \frac{n_G x}{n_H x'} x' = (p^m - 1) \frac{n_G}{n_H},$$

n_G being the order of G, or

$$\Sigma x = p^m - 1,$$

which proves the theorem.

Now, if the order of H is relatively prime to $p^m - 1$, $p^{m-1} - 1$, ... $p - 1$, every operation of h is self-conjugate in H, and G contains $p^m - 1$ distinct sets of conjugate operations whose orders are powers of p.

When $p^m = 2^2$, this condition will be satisfied if the order of H does not contain 3; and, when $p^m = 2^3$, it will be satisfied if the order of H contains neither 3 nor 7 as a factor.

A group of order $2^2 m$ is therefore certainly composite if the odd number m is not divisible by 3; and a group of order $2^3 m$, in which the sub-groups of order 2^3 are Abelian, is certainly composite if m is divisible by neither 3 nor 7.

Suppose, next, that in a group of order $2^3 m$ the sub-groups of order 2^3 are of type (iii), given by

$$A^2 = 1, \quad B^4 = 1, \quad AB = B^3 A;$$

and suppose that A and B^2 are conjugate operations in the group. Then A must be the square of some operation B' of order 4, and the sub-group formed by

$$1, \quad B^2, \quad B'^2, \quad B^2 B'^2$$

occurs in the two sub-groups of order 2^3 which contain B and B'. In the first B'^2 and $B^2 B'^2$ are conjugate operations, and in the second B^2 and $B^2 B'^2$ are conjugate. Hence B^2, B'^2, $B^2 B'^2$ form a single conjugate set in the sub-group that contains the group

$$1, \quad B^2, \quad B'^2, \quad B^2 B'^2$$

self-conjugately. The order of this sub-group is therefore divisible by 3; and hence, unless m is divisible by 3, A and B^2 cannot be conjugate operations.

The operation A must enter into an odd number n' of sub-groups of order 2^3. If, then, A and AB belong to the same conjugate set, each operation of the set enters into n' sub-groups of order 2^3; while the number of operations in the set is $2n''$, n'' being odd.

Hence $2n''n'$ is the total number of these operations, distinct or not, which enter in the conjugate set of sub-groups of order 2^3. But, since 4 enter into each sub-group of order 2^3, this is impossible; and therefore A and AB cannot be conjugate.

Suppose, now, lastly, that in a group of order 2^3m the sub-groups of order 2^3 are of type (iv), given by

$$A^4 = 1, \quad B^4 = 1, \quad A^2 = B^2, \quad AB = B^3A;$$

and suppose that A and B are conjugate, so that

$$S^{-1}AS = B,$$

$$S^{-1}A^2S = B^2 = A^2,$$

and S occurs in the group g within which A^2 is permutable. Let the order of this sub-group be $2^3n_1n_2$, and let it contain n_2 sub-groups of order 2^3. Since, within g, A and B are conjugate, it cannot contain 3 distinct conjugate sets of cyclical sub-groups of order 4. Suppose, now, that the sub-group of g of order 2^3n_1 which contains a sub-group of order 2^3 self-conjugately also contains each of its 3 sub-groups of order 4 self-conjugately. Then any sub-group of order 4 will be self-conjugate within a sub-group of g of order $2^3n_1n_2'$, and will form one of n_2'' conjugate sub-groups within g, and each of these will enter in n_2' of the $n_2(=n_2'n_2'')$ sub-groups of g of order 2^3.

Hence $$3n_2 = \Sigma n_2'n_2'',$$

where the summation is extended to the different distinct sets of conjugate sub-groups of order 4 contained in g. This is impossible, since the number of these sets does not exceed 2; and therefore the 3 sub-groups of order 4 contained in the sub-group of g of order 2^3n_1 are, in this sub-group, conjugate to each other. Hence n_1 must be divisible by 3; and, unless this condition obtains, A and B cannot be conjugate.

Hence a group whose order is 2^3m (m odd) in which the sub-groups of order 2^3 are not Abelian cannot be simple unless m is divisible by 3.

Combining now all the results, they give the theorem that a group whose order is even cannot be simple unless the order contains either 12, 16, or 56 as a factor.

IX. *On the non-Existence of Simple Groups whose Orders lie between 660 and 1092.*

In Vol. xv of the *American Journal of Mathematics*, Dr. Cole has carried on from 201 to 660 a discussion of the possibility of a simple group corresponding to a given order, which was begun and taken as far as 200 by Herr Hölder (*Math. Ann.*, Vol. xlii). The simple group of next smallest order to 660 that is known to exist is a group of order 1092; and it appears a not uninteresting application of the tests for the simplicity of a group, which depend on its order, that have been given in these notes and elsewhere, to determine how many of the 432 numbers from 661 to 1092 inclusive are at once shown to have no simple group corresponding to them. These tests may now be stated as follows. There are no simple groups whose orders are

(i) the power of a prime,

(ii) the product of two or three prime factors,

(iii) the product of four prime factors (with the exception of the order $2^2.3.5$),

(iv) the product of five prime factors (with the exceptions of the orders $2^3.3.7$, $2^2.3.5.11$, $2^2.3.7.13$),

(v) of the forms $p_1^m p_2$ (p_1, p_2 primes in ascending order),

(vi) even, but not divisible by 12, 16, or 56.

These tests imply that, if there are simple groups whose orders are odd, none can be of smaller order than $3^4.5^2$ or 2025, so that in the interval in question there can be no simple groups of odd order. One further test that may be given here for the sake of completeness is that there are no simple groups whose orders are $p_1^2 p_2^m$ or $p_1^3 p_2^m$.†

These tests applied to the 432 orders from 661 to 1092 dispose of all cases except the following sixteen, viz. :—

$*672 = 2^5.3.7$, $800 = 2^5.5^2$, $*880 = 2^4.5.11$, $960 = 2^6.3.5$,

$720 = 2^4.3^2.5$, $*816 = 2^4.3.17$, $900 = 2^2.3^2.5^2$, $1040 = 2^4.5.13$,

$756 = 2^2.3^3.7$, $840 = 2^3.3.5.7$, $*912 = 2^4.3.19$, $1056 = 2^5.3.11$,

$*784 = 2^4.7^2$, $864 = 2^5.3^3$, $*936 = 2^3.3^2.13$, $1080 = 2^3.3^3.5$.

Of these the six that are marked with a star are immediately shown,

† Unless $p_2 = 3$, a group of order $p_1^3 p_2^m$, if simple, would necessarily contain p_1^3 conjugate sub-groups of order p_2^m. If the operations of these were all distinct, the sub-group of order p_1^3 would be self-conjugate. If, on the other hand, two sub-

each by a simple application of Sylow's theorem, not to correspond to a simple group. That none of the remaining ten correspond to a simple group may be shown by considering them individually.

$$N = 720 = 2^4 . 3^2 . 5.$$

A simple group of this order would contain either 16 or 36 conjugate sub-groups of order 5. If there are 16, each is self-conjugate in a group of order $3^2 . 5$. Such a group is necessarily Abelian, and cannot be expressed in 15 symbols. There must therefore be 36 conjugate sub-groups of order 5, and each is then self-conjugate in a sub-group of order $2^2 . 5$. If this sub-group contains an operation of order 4, it must, when expressed in 36 symbols, consist of either 5, 7 or 8 cycles. If it has 5 or 7, it is an odd substitution, and the group cannot be simple. If it has 8, it is one of 45 conjugate cyclical sub-groups of order 4 whose squares are all distinct. No one of these can transform another, or the square of another, into itself, and therefore, when expressed in 45 symbols, these operations of order 4 consist of 11 cycles, and are odd operations, making the group composite. If the sub-group of order $2^2 . 5$ contains no operation of order 4, it must contain an operation of order 10. The corresponding operation of order 2, which is permutable with an operation of order 5, must, if it is an even substitution of 36 symbols, consist of 10 transpositions. Such an operation is permutable with 16 distinct sub-groups of order 5, and is therefore permutable in a sub-group of order $2^4 . 5$ at least, which makes the group composite.

$$N = 756 = 2^2 . 3^3 . 7.$$

If simple, the group must contain $3^3 . 7$ operations of even order and $6 . 36$ operations of order 7, leaving 350 operations whose orders are powers of 3. There are 28 sub-groups of order 3^3, and, if any two of these have a common sub-group of order 3^2, the group is certainly composite. The sub-groups of order 3 which are common to two sub-groups of order 3^3 form a single conjugate set; and when the group is expressed in 28 symbols a simple calculation will show that each sub-group of order 3^3 must contain 8 operations keeping 1 symbol unchanged and 18 keeping 4 symbols unchanged. There are therefore 63 sub-groups of order 3 which occur in more than one sub-group of order 3^3. On the other hand, such a sub-group of

groups of order p_2^m had a maximum common sub-group of order p_2^r, this would (see Note VI) be self-conjugate in a group of order $p_1^3 p_2^r$, and therefore in the main group. See also the recent investigations of Herr Frobenius in the *Berliner Sitzungsberichte*.

order 3 must (Note VI) be permutable in a sub-group of order $2^2 . 3^2$ at least; and must therefore be one of a set of 21 conjugate sub-groups at most. The supposition that the group is simple thus leads to a contradiction.

$$N = 800 = 2^5 . 5^2.$$

A simple group of this order must contain 16 conjugate sub-groups of order 5^2, each self-conjugate in a group of order $2 . 5^2$. Expressed in 16 symbols, the sub-group of order 5^2 must contain 3 sub-groups of order 5, each of which consists of 3 cycles, and 3 each of which consist of 2 cycles. Hence, in the sub-group of order $2 . 5^2$ an operation of order 2 must be permutable with an operation of order 5, which consists of 3 cycles. The operation of order 2 therefore must consist of 5 transpositions, and, this being an odd substitution, the group cannot be simple.

$$N = 840 = 2^3 . 3 . 5 . 7.$$

There must be 8, 15, or 120 conjugate sub-groups of order 7. That there should be 8 is clearly impossible if the group is simple; while, if there are 120, there can only be $2^3 . 3 . 5$ operations whose orders are not 7. Now (method of Note V), the group contains at least $3 . 5 . 7$ operations of even order, so that in this case there would only remain 15 operations of orders 1, 3, and 5. This is clearly impossible if the group be simple. Hence, there must be 15 conjugate sub-groups of order 7, each contained self-conjugately in a sub-group of order $2^3 . 7$. Such a sub-group necessarily contains an Abelian sub-group of order $2^2 . 7$, and this cannot be represented in 14 symbols. The group is therefore composite.

Before dealing with the next case, it will be convenient to prove the following lemma:—

If p^m is the highest power of a prime p, which divides the order of a group, and if h is a sub-group of order p^m, the number of sub-groups conjugate to h that have a sub-group of order p^r, but no sub-group of order p^{r+1}, in common with h is zero or a multiple of p^{m-r}.

If there are any such sub-groups, let h' be one, and let

$$P_1 (= 1),\ P_2,\ \ldots\ P_{p^m},$$

be the operations of h. Then, of the sub-groups

$$P_1^{-1} h' P_1,\quad P_2^{-1} h' P_2,\quad \ldots\quad P_{p^m}^{-1} h' P_{p^m},$$

just p^{m-r} are distinct, and each has in common with h a sub-group of

order p^r, and none of higher order. If these do not exhaust the sub-groups conjugate to h which have in common with it a sub-group of order p^r, and none of higher order, let h'' be another such sub-group. Then, of the sub-groups

$$P_1^{-2}h''P_1, \quad P_2^{-2}h''P_2, \quad \ldots \quad P_{p^m}^{-1}h''P_{p^m},$$

no one can be the identical with any one of the previous set, and just p^{m-r} are distinct. This process can be continued till the set is exhausted, and the lemma is thus proved.* A theorem which is equivalent to the above is given without proof in a note by M. E. Maillet (*Comptes Rendus*, CXVIII, pp, 1187, 1188).

$$N = 864 = 2^5 . 3^3.$$

There is no transitive group of 9 symbols of this order. (*Cf.* Dr. Cole, *Bull. New York Math. Soc.*, Vol. II, No. 10.) Hence, if the group is simple, there must be 27 conjugate sub-groups of order 2^5. Let h be one of them; then there must be $2x_1$, $4x_2$, $8x_3$, and $16x_4$ groups conjugate to h, and having in common with it sub-groups of order 2^4, 2^3, 2^2, 2 respectively, and no sub-groups of respectively higher orders. Hence,

$$1 + 2x_1 + 4x_2 + 8x_3 + 16x_4 = 27,$$

and therefore x_1 must be different from zero.

But a sub-group of order 2^4 which is common to the sub-groups of order 2^5 must (Note VI) be self-conjugate in a sub-group of order $2^5 . 3$ at least. Hence the group must be isomorphous to a transitive group of 9 symbols; and, therefore, since the isomorphism must be merihedric, the group cannot be simple.

$$N = 900 = 2^2 . 3^2 . 5^2.$$

There must be 36 sub-groups of order 5^2. If a sub-group of order 5 were contained self-conjugately in a more extensive sub-group than one of order 5^2, it must necessarily be in one of order $6 . 5^2$, and the sub-group would then be one of 6 conjugate sub-groups, which would make the group composite. If this is not the case, all the operations of the 36 groups of order 5^2, except identity, are distinct; so that there are only $2^2 . 3^2$ operations whose orders are not powers of 5. But the group must contain at least $3^2 . 5^2$ operations of even order; so that this latter supposition is impossible.

* From this lemma it follows at once that a group of order $p_1^m p_2^2$ cannot in any case be simple unless $p_2 \equiv 1 \pmod{p_1^2}$. *Cf.* Note VI.

$$!N = 960 = 2^6 . 3 . 5.$$

The group must contain 15 conjugate sub-groups of order 2^6. Let T be an operation of order 2 which is contained self-conjugately in a sub-group of order 2^6. If the group is simple, T, when expressed in 15 symbols, must consist of 6 or 4 transpositions. Represent the symbols by 1, 2, ... 14, 15, and consider that sub-group of order 2^6 which keeps 15 unchanged. Let T keep 13, 14, and 15 unchanged. Then, if T is self-conjugate in the sub-group, every one of its operations must either keep 13 and 14 unchanged, or must interchange them. Now T belongs to 3 different sub-groups of order 2^6, and therefore the sub-group that keeps 15 unchanged must contain 3 operations of the conjugate set to which T belongs. These 3 operations must all keep 13, 14, and 15 unchanged, as otherwise T would be self-conjugate in a group of greater order than 2^6. Hence the 15 conjugate operations consist of 5 sets of 3 each, each set keeping 3 of the 15 symbols unchanged. The group is therefore imprimitive in 5 sets of 3 symbols each, and, if simple, must be expressible as a transitive group of 5 symbols. This is impossible for a group whose order contains the factor 2^6. The case in which T consists of 4 transpositions may be treated in a similar manner.

$$N = 1040 = 2^4 . 5 . 13.$$

If simple, the group must have 26 sub-groups of order 5, each contained self-conjugately in a sub-group of order $2^3 . 5$. Such a sub-group necessarily contains an operation of order 10; and the corresponding operation of order 2, which is permutable with an operation of order 5, must, if expressed as an even substitution of 26 symbols, consist of 10 transpositions. It must therefore occur in 6 sub-groups of order $2^3 . 5$, and be permutable with 6 sub-groups of order 5. But, since 6 is not a factor of the order of the group, this is impossible.

$$N = 1056 = 2^5 . 3 . 11.$$

There must be 12 conjugate sub-groups of order 11, each self-conjugate in a sub-group of order $2^2 . 11$. But such a sub-group necessarily contains an operation of order 22, and this cannot be expressed in 12 symbols.

$$N = 1080 = 2^3 . 3^3 . 5.$$

There must be 6, 36, or 216 sub-groups of order 5; and the first supposition is clearly impossible for a simple group. If there were 36, each would be self-conjugate in a group of order $2 . 3 . 5$. This group would contain a sub-group of order 15, and a sub-group of order 3 self-conjugately. The latter would necessarily be self-conjugate in a sub-group of order $2 . 3^2 . 5$, and would be therefore one of at most 12 conjugate sub-groups. But in a group of degree 12 an operation of order 15 would contain a single cycle of 5 symbols, so that this case cannot occur. There must therefore be 216 sub-groups of order 5; leaving only $2^3 . 3^3$ operations whose orders are not 5. Now, since 7 is not a factor of the order of the group, there must be more than one conjugate set of operations whose orders are 2, or powers of 2, and corresponding to each there must be a distinct set of either $3^3 . 5$ or $2 . 3^2 . 5$ operations of even order in the group (method of Note V). Hence this case certainly cannot occur, and this group must be composite.

$$N = 1092 = 2^2 . 3 . 7 . 13.$$

If simple, a group of this order must contain 14 sub-groups of order 13, each being self-conjugate in a group of order $6 . 13$. Since a group of degree 14 cannot contain operations of order 26 or 39, this latter sub-group must be metacyclical in type. Again, there must be 78 sub-groups of order 7, each self-conjugate in a sub-group of order $2 . 7$; and this must be dihedral in type, as otherwise the 78 sub-groups would contain $78 . 12$ distinct operations. Hence the distribution of the operations of the group in conjugate sets is necessarily identical with that of the known simple group of this order.

On the Geometrical Meaning of a Form of the Orthogonal Transformation.

By M. J. M. Hill, M.A., D.Sc., F.R.S., Professor of Mathematics at University College, London. Received and read May 9th, 1895.

The orthogonal transformation in space of three dimensions has been put by Lipschitz in a publication entitled *Untersuchungen über die Summen von Quadraten*, published at Bonn, in 1886, into the form

$$\left.\begin{aligned} x+\nu y-\mu z &= \quad X-\nu Y+\mu Z \\ -\nu x+\ y+\lambda z &= \quad \nu X+\ Y-\lambda Z \\ \mu x-\lambda y+\ z &= -\mu X+\lambda Y+\ Z \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where the new axes X, Y, Z are derived by a right-handed rotation through an angle θ from the old axes x, y, z, the constants λ, μ, ν being defined thus:

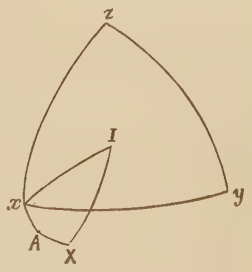
$$\lambda = \tan\tfrac{1}{2}\theta\cos\xi, \quad \mu = \tan\tfrac{1}{2}\theta\cos\eta, \quad \nu = \tan\tfrac{1}{2}\theta\cos\zeta \ \ldots\ldots (2),$$

where ξ, η, ζ are the direction angles of the axis of rotation.

The object of this note is to point out the geometrical meaning of the equations (1).

Draw a sphere whose centre is at the origin O, cutting the axes of x, y, z at x, y, z; the axes of X, Y, Z at X, Y, Z; the axis of rotation at I.

Draw great circle arcs perpendicular to Ix, IX at x and X, respectively, meeting at A.



Then the first of equations (1) is obtained by projecting the coordinates of any point in the two systems along OA.

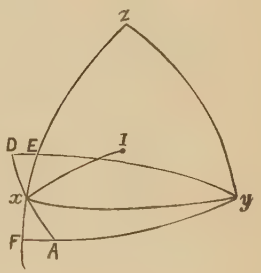
Let the direction angles of OA with regard to Ox, Oy, Oz be α, β, $\pi-\gamma$; so that in the annexed figure α, β, γ are all acute angles.

To find the direction angles of OA with regard to OX, OY, OZ imagine the figure rotated about I until IX coincides with Ix; then

A moves to a point D, so that

$$xD = xA.$$

Let yA cut zx at F, and yD cut zx at E. Then the spherical triangles xED, xFA are equal in all respects. Therefore



$$ED = AF,$$

$$yD + yA = \pi;$$

therefore $$yD = \pi - \beta.$$

In like manner

$$zD + zA = \pi;$$

therefore $$zD = \gamma.$$

Hence the direction angles of OD with regard to Ox, Oy, Oz are α, $\pi-\beta$, γ.

Hence the direction angles of OA with regard to OX, OY, OZ are α, $\pi-\beta$, γ.

The next step is to express α, β, γ in terms of θ, ξ, η, ζ.

$$\tan \tfrac{1}{2}\theta \sin \xi = \tan AIx \sin xI = \tan xA = \tan \alpha;$$

therefore $$\cos \alpha = (1 + \sin^2 \xi \tan^2 \tfrac{1}{2}\theta)^{-\frac{1}{2}},$$

$$\begin{aligned}
\cos \beta &= \cos Ay \\
&= \cos Ax \cos xy + \sin Ax \sin xy \cos Axy \\
&= \sin \alpha \sin Ixy \\
&= \frac{\sin \alpha}{\sin \xi} \sin \xi \sin Ixy \\
&= \frac{\sin \alpha}{\sin \xi} \sin \left(\frac{\pi}{2} - Iz\right) \\
&= \cos \alpha \frac{\tan \alpha}{\sin \xi} \cos \zeta \\
&= \cos \alpha \tan \tfrac{1}{2}\theta \cos \zeta,
\end{aligned}$$

$$\begin{aligned}
\cos \gamma &= -\cos (\pi - \gamma) = -\cos Az \\
&= -(\cos Ax \cos xz + \sin Ax \sin xz \cos Axz) \\
&= -\sin \alpha \cos \left(\frac{\pi}{2} + Ixz\right) \\
&= \sin \alpha \sin Ixz
\end{aligned}$$

$$= \frac{\sin \alpha}{\sin \xi} \sin \xi \sin Ixz$$

$$= \frac{\sin \alpha}{\sin \xi} \sin \left(\frac{\pi}{2} - Iy \right)$$

$$= \frac{\sin \alpha}{\sin \xi} \cos \eta$$

$$= \cos \alpha \tan \tfrac{1}{2}\theta \cos \eta ;$$

therefore $\cos \alpha : \cos \beta : \cos \gamma = 1 : \tan \tfrac{1}{2}\theta \cos \zeta : \tan \tfrac{1}{2}\theta \cos \eta$

$$= 1 : \nu : \mu.$$

Now, projecting the coordinates x, y, z, and then X, Y, Z of any point P along OA, it follows that

$$x \cos \alpha + y \cos \beta + z \cos (\pi - \gamma) = X \cos \alpha + Y \cos (\pi - \beta) + Z \cos \gamma ;$$

therefore $$x + \nu y - \mu z = X - \nu Y + \mu Z,$$

which is the first of equations (1).

The second and third equations can be obtained in like manner.

A Property of Skew Determinants. *By* M. J. M. Hill, M.A., D.Sc., F.R.S., Professor of Mathematics at University College, London. Received and read May 9th, 1895.

It has been shown by Professor Cayley that the orthogonal transformation could be expressed thus

$$\left.\begin{aligned} x_1 &= a_{1,1} y_1 + a_{1,2} y_2 + \ldots + a_{1,n} y_n \\ \ldots \quad & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_n &= a_{n,1} y_1 + a_{n,2} y_2 + \ldots + a_{n,n} y_n \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $$a_{r,r} = \frac{2\beta_{r,r} - \Delta}{\Delta} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$a_{r,s} = \frac{2\beta_{r,s}}{\Delta} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where Δ is the skew determinant

$$\begin{vmatrix} b_{1,1} & \dots & b_{1,n} \\ \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,n} \end{vmatrix}$$

in which $b_{r,s} = -b_{s,r}, \quad r \neq s$

but $b_{r,r} = 1$ } .....(4),

and where $\beta_{r,s}$ is the co-factor of $b_{r,s}$.

To see directly that this is orthogonal, it is necessary to show that

$$\beta_{1,r}^2 + \beta_{2,r}^2 + \dots + \beta_{n,r}^2 = \Delta\beta_{r,r} \quad \dots\dots\dots\dots\dots(5),$$

and $$2(\beta_{1,r}\beta_{1,s} + \beta_{2,r}\beta_{2,s} + \dots + \beta_{n,r}\beta_{n,s}) = \Delta(\beta_{r,s} + \beta_{s,r}) \quad \dots\dots(6).$$

As the latter equation includes the former, it is sufficient to prove it.

Let $$\Delta^2 = \begin{vmatrix} c_{1,1} & \dots & c_{1,n} \\ \dots & \dots & \dots \\ c_{n,1} & \dots & c_{n,n} \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots\dots(7),$$

where $$c_{r,s} = b_{r,1}b_{s,1} + b_{r,2}b_{s,2} + \dots + b_{r,n}b_{s,n} \quad \dots\dots\dots\dots(8).$$

It should be noticed that $c_{r,s}$ contains two terms from the diagonal of Δ, viz., $b_{r,r}$ and $b_{s,s}$ which occur in the terms $b_{r,r}b_{s,r}$ and $b_{r,s}b_{s,s}$, whose sum is equal to

$$b_{s,r} + b_{r,s} = 0.$$

They may therefore be omitted or replaced by

$$b_{r,r}b_{r,s} + b_{s,r}b_{s,s},$$

which also vanishes.

Hence $$c_{r,s} = b_{1,r}b_{1,s} + b_{2,r}b_{2,s} + \dots + b_{n,r}b_{n,s} \quad \dots\dots\dots\dots\dots(9).$$

If $r = s$, $\quad c_{r,r} = b_{r,1}^2 + b_{r,2}^2 + \dots + b_{r,n}^2 = b_{1,r}^2 + b_{2,r}^2 + \dots + b_{n,r}^2$.

Also $$c_{r,s} = c_{s,r}.$$

Multiply now the matrices

$$\begin{Vmatrix} b_{1,1} & \dots & b_{1,n} \\ \dots & \dots & \dots \\ b_{r-1,1} & \dots & b_{r-1,n} \\ b_{r+1,1} & \dots & b_{r+1,n} \\ \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,n} \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} b_{1,1} & \dots & b_{1,n} \\ \dots & \dots & \dots \\ b_{s-1,1} & \dots & b_{s-1,n} \\ b_{s+1,1} & \dots & b_{s+1,n} \\ \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,n} \end{Vmatrix}.$$

The product is

$$\begin{vmatrix} c_{1,1} & \dots & c_{1,s-1} & c_{1,s+1} & \dots & c_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{r-1,1} & \dots & c_{r-1,s-1} & c_{r-1,s+1} & \dots & c_{r-1,n} \\ c_{r+1,1} & \dots & c_{r+1,s-1} & c_{r+1,s+1} & \dots & c_{r+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n,1} & \dots & c_{n,s-1} & c_{n,s+1} & \dots & c_{n,n} \end{vmatrix},$$

and is also $(-1)^{r+s}(\beta_{r,1}\beta_{s,1}+\beta_{r,2}\beta_{s,2}+\dots+\beta_{r,n}\beta_{s,n})$.

Again,

$$\Delta \cdot \beta_{r,s}$$

$$= \begin{vmatrix} b_{1,1} & \dots & b_{1,n} \\ & & \\ \vdots & \vdots & \vdots \\ & & \\ b_{n,1} & \dots & b_{n,n} \end{vmatrix} \begin{vmatrix} b_{1,1} & \dots & b_{1,s-1} & b_{1,s} & b_{1,s+1} & \dots & b_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{r-1,1} & \dots & b_{r-1,s-1} & b_{r-1,s} & b_{r-1,s+1} & \dots & b_{r-1,n} \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ b_{r+1,1} & \dots & b_{r+1,s-1} & b_{r+1,s} & b_{r+1,s+1} & \dots & b_{r+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,s-1} & b_{n,s} & b_{n,s+1} & \dots & b_{n,n} \end{vmatrix}$$

$$= \begin{vmatrix} c_{1,1} & \dots & c_{1,r-1} & b_{1,s} & c_{1,r+1} & \dots & c_{1,n} \\ & & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & & & & & \\ c_{n,1} & \dots & c_{n,r-1} & b_{n,s} & c_{n,r+1} & \dots & c_{n,n} \end{vmatrix},$$

as is seen by multiplying rows by rows, and using (8).

Also $\Delta . \beta_{s,r}$

$$= \begin{vmatrix} b_{1,1} & \dots & b_{1,n} \\ \vdots & \vdots & \vdots \\ b_{n,1} & \dots & b_{n,n} \end{vmatrix} \begin{vmatrix} b_{1,1} & \dots & b_{1,r-1} & 0 & b_{1,r+1} & \dots & b_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{s-1,1} & \dots & b_{s-1,r-1} & 0 & b_{s-1,r+1} & \dots & b_{s-1,n} \\ b_{s,1} & \dots & b_{s,r-1} & 1 & b_{s,r+1} & \dots & b_{s,n} \\ b_{s+1,1} & \dots & b_{s+1,r-1} & 0 & b_{s+1,r+1} & \dots & b_{s+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,r-1} & 0 & b_{n,r+1} & \dots & b_{n,n} \end{vmatrix}$$

$$= \begin{vmatrix} c_{1,1} & \dots & c_{1,r-1} & b_{s,1} & c_{1,r+1} & \dots & c_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n,1} & \dots & c_{n,r-1} & b_{s,n} & c_{n,r+1} & \dots & c_{n,n} \end{vmatrix},$$

as is seen by multiplying columns by columns, and using (9).

Therefore $\Delta (\beta_{r,s} + \beta_{s,r})$

$$= \begin{vmatrix} c_{1,1} & \dots & c_{1,r-1} & (b_{1,s}+b_{s,1}) & c_{1,r+1} & \dots & c_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{s,1} & \dots & c_{s,r-1} & (b_{s,s}+b_{s,s}) & c_{s,r+1} & \dots & c_{s,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n,1} & \dots & c_{n,r-1} & (b_{n,s}+b_{s,n}) & c_{n,r+1} & \dots & c_{n,n} \end{vmatrix}$$

$$= (-1)^{r+s} . 2 \begin{vmatrix} c_{1,1} & \dots & c_{1,r-1} & c_{1,r+1} & \dots & c_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{s-1,1} & \dots & c_{s-1,r-1} & c_{s-1,r+1} & \dots & c_{s-1,n} \\ c_{s+1,1} & \dots & c_{s+1,r-1} & c_{s+1,r+1} & \dots & c_{s+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n,1} & \dots & c_{n,r-1} & c_{n,r+1} & \dots & c_{n,n} \end{vmatrix}$$

$$= 2 (\beta_{r,1} \beta_{s,1} + \dots + \beta_{r,n} \beta_{s,n}).$$

Putting $r = s$, $\beta_{r,1}^2 + \beta_{r,2}^2 + \dots + \beta_{r,n}^2 = \Delta \beta_{r,r}$,

and also by symmetry

$$2(\beta_{1,r}\beta_{1,s}+\beta_{2,r}\beta_{2,s}+\ldots+\beta_{n,r}\beta_{n,s}) = \Delta(\beta_{r,s}+\beta_{s,r}),$$

$$\beta_{1,r}^2+\beta_{2,r}^2+\ldots+\beta_{n,r}^2 = \Delta\beta_{r,r}.$$

Researches in the Calculus of Variations.—Part VI., The Theory of Discontinuous or Compounded Solutions.

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Communicated (with certain additional Critical Remarks) May 10th, 1894.

In the following pages I hope to place the theory of discontinuous, or as they may more appropriately be called compounded, solutions in the calculus of variations on a satisfactory basis. The theory also leads to a rule for ascertaining whether the continuous solution given by the ordinary equations of the calculus is, or is not, the only possible solution.

1. Discontinuity presents itself in two ways in the calculus of variations :—

(1) There may be *stationary* solutions, which involve discontinuity of some fluxion of y at some point or points of the integral.

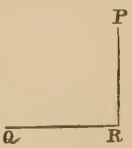
(2) There may be maximum or minimum solutions, which are not *stationary*, *i.e.*, solutions in which δU, as well as $\delta^2 U$, is capable of only one sign.

Solutions of this class appear to arise in two principal ways :—

(a) The region of integration may be restricted so that, along a certain boundary, δy is not capable of either sign. The restriction may either be *explicit*, as when we are asked to find the shortest *sea* line from one bay to another, or *implicit*, as in the case of *least* action, where the fact that the values of the variables must be *real* gives rise to a boundary. This class of problem has been sufficiently treated of by Mr. Todhunter in his Adams Prize Essay, entitled "Researches in the Calculus of Variations," and it will be unnecessary here to discuss it.

(*b*) It may be required to find the least value of an integral *when every element is taken positively* whatever its algebraic sign may be. This class of problem does not appear to have been treated of hitherto.

2. With regard to the first class of problem, considerable uncertainty may arise, for, although it is clearly the duty of the proposer of the problem to state with exactness the conditions which the solution must satisfy, yet accuracy is usually sacrificed for brevity, and not until the solution is actually presented does the question as to the true meaning of the problem arise. The difficulty is usually to know what amount of discontinuity is admissible at the point of junction of the two curves which give the compound solution. This point is fundamental in the theory. For instance, suppose that the two straight lines QR and RP are offered as a solution of the problem to make



$$\int_Q^P (\ddot{y}^2 - 2y)\, dx$$

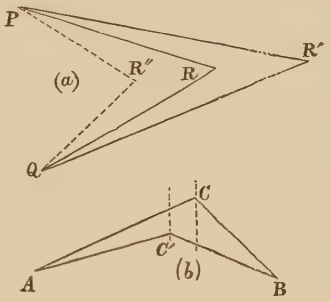
a minimum. Then the first objection would be that, since $\ddot{y}$ became infinite at R, the solution was not satisfactory without a special justification. If we endeavour to avoid this difficulty by splitting the integral into two others, thus,

$$\int_Q^P (\ddot{y}^2 - 2y)\, dx = \int_R^P (\ddot{y}^2 - 2y)\, dx + \int_Q^R (\ddot{y}^2 - 2y)\, dx,$$

so that no infinite values occur in either integral, the proposer of the problem would say that, had he contemplated such an artifice being adopted, he would have expressly excluded it by a stricter enunciation, for it is evidently a mere evasion first to lump an infinite portion of the integral at the point R, and then to omit entirely the term at R on the ground that it does not belong to either the integral *from* Q *to* R or *from* R *to* P. But, if the solution be still defended on the ground that it does conform to the problem *as actually stated*, the proposer would ask what is meant by $\ddot{y}$ in the last element of the integral from Q to R, or in the first element from R to P? If we make $\ddot{y}$ as zero throughout in each integral, this implies that *the definition of* $\ddot{y}$ *is changed at* R. For, whether we take $\ddot{y}_r dx^2$ as $y_{n+1} + y_{n-1} - 2y_n$, where n has the value $r-1$, r, or $r+1$ (and it must be one of these), $\ddot{y}$ will be infinite in *one or both integrals*, and the *raison d'être* of the division into two integrals disappears. Here,

then, we see the cardinal rule as to dividing an integral into two others, and employing different solutions in the two integrals—*the division can only be made when it is permissible to change the definition of the derived function at the point of junction*, and even then the change must be made in accordance with conditions to be laid down by the proposer of the problem.

3. There is another fundamental point which does not seem to have been sufficiently attended to hitherto. When the initial and terminal coordinates are fixed, it is usual, in dealing with continuous solutions, only to give a variation to y. But this method is defective where there is a discontinuity. For it does not enable us to compare a curve $PR'Q$ with PRQ in (*a*), because the elements to the right of R do not appear in the original integral at all, and no variation of the elements of that integral can introduce them, nor can it get rid of the elements which are omitted in passing from PRQ to $PR''Q$. Nor are we much better off where the discontinuity is like that in (*b*), for the first variation of $\dot{y}$ between the ordinates at C and C' has a *finite* value, and terms involving squares of $\delta\dot{y}$ will therefore be of the same order of magnitude as the terms of the first variation, so that the ordinary theory of the calculus cannot be applied.*



4. *Variation of a Compounded Solution.*—Suppose we are required to find the stationary value of

$$\int_{x'}^{x''} f(x, y, \dot{y}, \ddot{y})\, dx,$$

and that in calculating the values of the fluxions $\dot{y}$ and $\ddot{y}$ we are allowed to use different functions, θ and ϕ, in different parts. Then

* I have thought it necessary to dwell on the points explained in §§ 2 and 3 at considerable length, because the principles there laid down are in direct opposition to those assumed by the late Mr. Todhunter in his Adams Prize Essay on this subject. As a consequence of these fundamental differences, I am unable, speaking generally, to accept any of Mr. Todhunter's stationary solutions, except in those problems in which there is no discontinuity in the direction of the tangent, $\dot{y}$ being the highest fluxion of y involved. Fortunately this exception includes some of the most important problems in that work.

we write $$\delta U = \delta \int_{x'}^{x''} f\,dx \equiv \delta \int_{x'}^{x_1} u\,dx + \delta \int_{x'}^{x''} v\,dx,$$

u and v being the forms which f assumes when θ and φ are substituted for y. Hence, writing

$$U_0 = \frac{du}{dy}, \quad U_1 = \frac{du}{d\dot{y}}, \quad U_2 = \frac{du}{d\ddot{y}},$$

$$V_0 = \frac{dv}{dy}, \quad V_1 = \frac{dv}{d\dot{y}}, \quad V_2 = \frac{dv}{d\ddot{y}},$$

we obtain, after integrating by parts, the equation

$$\delta U = \Big|_{x'}^{x_1} \{(U_1 - \dot{U}_2)\,\delta\theta + U_2\delta\dot{\theta}\} + \Big|_{x_1}^{x''} \{(V_1 - \dot{V}_2)\,\delta\phi + V_2\delta\dot{\phi}\}$$
$$+ \Delta x_1 \Big|^{x_1} (u - v) + \int_{x'}^{x_1} (U_0 - \dot{U}_1 + \ddot{U}_2)\,\delta\theta\,dx + \int_{x'}^{x''} (V_0 - \dot{V}_1 + \ddot{V}_2)\,\delta\phi\,dx,$$

the fluxional notation being adopted throughout, and Δ being used to denote the complete variation, so that

$$\delta y_1 = \Delta y_1 - \dot{y}_1 \Delta x_1, \quad \delta\dot{y}_1 = \Delta\dot{y}_1 - \ddot{y}_1 \Delta x_1,$$

whether y has the form θ or ϕ. Hence the conditions at the terminal points are the same as in the continuous solution, and in addition, we must have

$$(U_1 - \dot{U}_1)\,\delta\theta + U_2\delta\dot{\theta} + u\Delta x - \{(V_1 - \dot{V}_1)\,\delta\phi + V_2\delta\dot{\phi} + v\Delta x\} = 0,$$

when $x = x_1$

The subsequent treatment of this equation depends on the conditions of the problem. We may, of course, assume that the solution is to consist of a continuous series of points, so that always

$$\theta - \phi = 0, \quad \Delta\theta - \Delta\phi = 0,$$

when $x = x_1$. The first of these equations gives a relation among the eight constants introduced in the solution of the differential equations for θ and ϕ, while the second reduces by one the equations which determine the constants. Similarly, if it is required that the direction of the tangent shall be continuous, we again lose one equation and one constant. Now, if there were no equation between θ and ϕ at x_1, we should in all have eleven constants to determine, viz., the eight constants of integration, and x', x_1, and x'', and we should have six equations from the x' and x'' limits, and five from the coefficients $\Delta\theta$, $\Delta\dot{\theta}$, $\Delta\phi$, $\Delta\dot{\phi}$ and Δx_1, obtained from the terms at x_1. Hence, in

every case, the problem is determinate, so far as regards the *number* of equations and disposable quantities.

5. *Number of Possible Solutions.*—It is evident that the problem is definite, no matter how many separate curves we assume the solution to be composed of, because, in every case, the number of the equations as obtained is the same as that of the disposable constants. Hence there would appear to be, in general, an indefinite number of discontinuous solutions, obtained from supposing first one point of discontinuity, then two, then three, and so on.* But it will usually be found that the curves

$$y = \theta(x), \quad y = \phi(x), \quad \&c.,$$

are really the same curve, from which we can infer that the continuous solution is the only stationary solution.

6. But, besides problems in which y_1 is explicitly restricted, there is another class in which the quantity under the integral sign may differ in sign from the physical quantity it is intended to represent, and a discontinuity is thus introduced when we seek the maximum of the physical quantity.

The example which suggested this class of problem was the following, taken from pp. 270–273 of Mr. Todhunter's *Researches*.

> "A particle [of unit mass] is to descend in a vertical plane from one fixed point to another, constrained by a smooth curve which is convex downwards; required the curve so that $\int P\,dt$ taken during the time of motion may be a minimum, where P denotes the pressure on the curve at the time t. Thus we may say that we require the whole pressure [*i.e.*, the momentum] to be a minimum."

I give the continuous solution, as obtained by the ordinary rule, in Mr. Todhunter's notation, except that I use $\dot{y}$ and $\ddot{y}$ where he uses p and q.

Taking the highest point O as origin, and the axis of x vertically downwards, and denoting by v, ρ, and s the velocity, radius of

* This is what we should expect from first principles. For, if we regard the integral as the sum of an indefinitely great number of terms, the problem becomes that of finding the stationary value of an expression involving an indefinite number of variables. That there is often but one solution is probably in part due to the fact that imaginary values are excluded by the very nature of the problem, and in part because it is an implied condition of the problem that as the successive values of x approach coincidence so also shall those of y, as well as to the special form of the equations.

curvature, and length of arc described at the time t, we get for the integral the expression

$$U \equiv \int \left(\frac{v^2}{\rho} + g \frac{dy}{ds} \right) dt$$

$$= \int \left(\frac{v\, ds}{\rho} + g \frac{dy}{v} \right) = \int \left(\frac{\dot{y} \sqrt{2gx}}{1+\dot{y}^2} + g \frac{\dot{y}}{\sqrt{2gx}} \right) dx.$$

By the usual theory we must have

$$-\frac{d}{dx}\left(\frac{1}{\sqrt{x}} - \frac{4\dot{y}\ddot{y}\sqrt{x}}{(1+\dot{y}^2)^2} \right) + \frac{d^2}{dx^2} \frac{2\sqrt{x}}{1+\dot{y}^2} = 0 \quad \ldots\ldots\ldots\ldots (1),$$

from which we get

$$\frac{\dot{y}^2}{(1+\dot{y}^2)\sqrt{x}} = C \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The term of the first order in δU which is free from the integral sign reduces to

$$\frac{2\sqrt{x}\,\delta\dot{y}}{1+\dot{y}^2},$$

since the limits of the coordinates are fixed.

Here, however, we are in a difficulty, for, though $2\sqrt{x}\,\delta\dot{y}/1+\dot{y}^2$ vanishes when $x = 0$, it does not vanish elsewhere, even though $\dot{y} = \infty$, for it is evidently $2\delta \tan^{-1} \dot{y}$, and is proportional to the change in the angle the tangent makes with the vertical, so that the solution would not be stationary, even if it were a possible one, and again, if the initial and final coordinates be given, the curve is thereby determined, and it is not possible in addition to prescribe the direction of the tangent—which might have been anticipated, because $\ddot{y}$, the highest fluxion, appears linearly in the integral.*

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* If we work the question with s as the independent variable, we get $\sqrt{\ddot{x}^2 + \ddot{y}^2}$ under the integral, and at first sight the above objection would be avoided, for the quadratic terms involving $\delta\ddot{x}$ and $\delta\ddot{y}$ in the second variation are

$$(\ddot{x}^2 + \ddot{y}^2)(\delta\ddot{x}^2 + \delta\ddot{y}^2) - (\ddot{x}^2\delta\ddot{x} + \ddot{y}\delta\ddot{y})^2$$

upon a positive factor, *i.e.*, they are

$$(\ddot{x}\delta\ddot{y} - \ddot{y}\delta\ddot{x})^2,$$

which is always positive, and therefore it would seem that the theory laid down in my paper on the discrimination of maxima and minima in this class of problem did not hold here But $\dot{x}^2 + \dot{y}^2 = 1$; therefore $\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0$, whence

$$\delta \frac{\ddot{x}}{\dot{y}} = -\delta \frac{\ddot{y}}{\dot{x}},$$

showing that $\ddot{x}\delta\ddot{y} - \ddot{y}\delta\ddot{x}$ is only of the order $\dot{x}\delta\dot{y} - \dot{y}\delta\dot{x}$.

There is, however, one case in which the stationary solution gives a true minimum, viz., when the extreme points O and P are on the same horizontal. For, writing U in the form

$$U = \sqrt{2gx}\tan^{-1}\dot{y} - \int\left(\frac{\sqrt{2g}}{2\sqrt{x}}\tan^{-1}\dot{y} - \frac{g\dot{y}}{\sqrt{2gx}}\right)dx \quad \ldots\ldots\ldots(4),$$

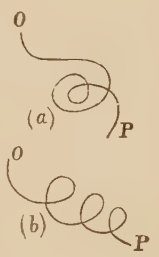
we get easily

$$\delta^2 U = -2\Big|_0^P \sqrt{2gx}\,\frac{\dot{y}\delta\dot{y}^2}{(1+\dot{y}^2)^2} + \int \frac{\sqrt{2g}}{\sqrt{x}}\,\frac{\dot{y}\delta\dot{y}^2}{(1+\dot{y}^2)}\,dx.$$

Since $x = 0$ at P and at O, the terms at the limits disappear. Again, the quantity under the integral sign is always positive, because, though $\dot{y}$ changes sign at the bottom of the curve, dx changes sign there also, so that the coefficient of $\delta\dot{y}^2$ which has the same sign as $\dot{y}\,dx$ is everywhere positive, and $\delta^2 U$ is always positive.

Observe, however, that this involves the assumption that the tangent is everywhere continuous in direction. Otherwise, instead of a single pair of limiting terms corresponding to P and O, we should have as many pairs as there were points of discontinuity, and the reasoning would fail. Hence enunciation of the problem should be modified, by the addition of the words "of continuous curvature" to "smooth curve."

The occurrence of a limiting term which can only vanish when the x coordinate of the final point vanishes seems to show that in general there can be no maximum or minimum of the integral, and evidently, by drawing the figure as in (a), we can give the integral any negative value, while, with no point of inflexion as in (b), we can give it any positive value.



The physical quantity which the proposer of the problem seems to have had in mind is the "whole pressure" as obtained from the sum of the numerical values of the elements *taken irrespective of sign*. We can represent

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Hence the most important terms in $\delta^2 U$ are not quadratic terms of the order $\delta\ddot{x}^2$, but are of the order $\delta\dot{x}\delta\ddot{x}$, and there is no maximum or minimum. Of course these correspond to the terms which in the discussion of the text are brought outside the sign of integration as $-2\sqrt{2gx}\,\dot{y}\,\delta\dot{y}^2/(1+\dot{y}^2)^2$. Again, if we integrate the equation obtained when we use s as the independent variable, we of course find that the constants introduced are not independent, and we get the same result by this method as by the other.

this analytically by saying that we have to make

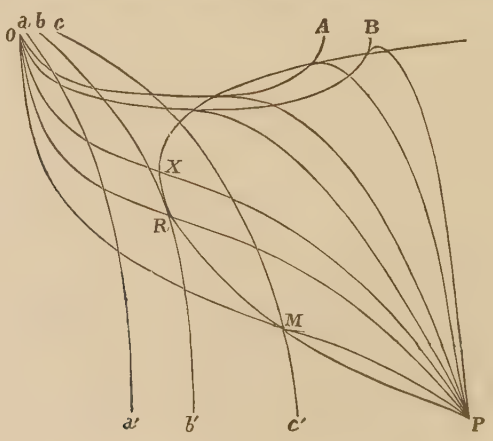
$$U = \int + \sqrt{u^2 dx^2}$$

a minimum, and thus we get a problem of a novel character.

The first condition evidently is that δU shall be zero or *positive* whatever sign δy has. The stationary solution is the only one which makes δU *zero* for all values of δy. The curve $u = 0$ is the only one which makes δU *positive* for all values of δy. Hence the curve we require must be made up of arcs of these two curves. Furthermore, as to the stationary curve, it is evident that its use is only legitimate where the elements $u\,dx$ are positive if the stationary curve gives a minimum value to the integral, or are negative if it makes the integral a maximum.

8. Applying this to Mr. Todhunter's problem, we see that, if we exclude *sudden* changes of direction, the solution must consist of what we may call the *momentum curve*, defined by (3), and arcs of parabolas touching them. We cannot say how many arcs may be required, but it is natural to begin with two, the parabola being of course the lower, for the momentum curve cannot fulfil the limiting conditions at the lower point P.

Now each parabola described by a particle projected from P with velocity due to the descent from O will touch one of the series of momentum curves OA, OB, &c., described from O. Draw the curve PMR through the points of contact. The solution, if it consists of only two arcs, must be one of those compounded curves, such as OMP.



Draw therefore a series of equimomental trajectories aa', bb', cc', &c., to the momentum curves, *i.e.*, trajectories such that the whole momentum impressed on the particle in descending from O to the trajectory is the same whatever momentum curve it describes.

In general a trajectory will cut the curve PMR, but at least one, say bb', will touch it. Let R be the point of contact. Then ORP is the required solution, OR being the momentum curve, and RP the parabola. We have, however, still to ascertain whether this really gives a minimum.

In the first place it is evident that, if R be the point for which the trajectory touches PMR *externally* (and there will obviously be such a point), then ORP is preferable to any neighbouring compound momentum-parabola curve.

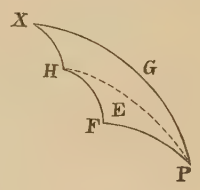
Hence, to compare ORP with *any* neighbouring curve OXP, take the point X where this curve meets PMR and draw the compound momentum-parabola curve which passes through X. If we can prove that, for all positions of X near to R, the momentum-parabola curve gives a better result than any other through X, we shall have verified our solution. If the two curves *touch* at X, it is obvious that the momentum-parabola curve gives the best result, for OX gives a minimum for fixed tangents, and of course XP gives a minimum. If the varied curve does not touch OXP at X, then the first variation under the integral sign in the value of δU obtained by operating on (4) with δ is certainly zero, since O and X are supposed to be fixed points, and the second variation under the integral sign is certainly positive, while the complete variation of the limits is exactly equal to the value of the momentum necessary to alter the direction of the varied motion at X, so as to make it coincide with the direction of motion in the compound curve OXP. Hence it is evident that in order to verify the solution we must solve the following problem.

9. *A point P lies in the free trajectory XGP of a particle starting from X with velocity V; if u and v be the components of a velocity equal to, but differing slightly in direction from, V, find the curve XHP of least momentum from X to P, with the given initial velocities u and v at X.*

If we find that the least momentum is not less than that required to make the particle move along XGP by a blow given at X, then we shall have verified our tentative solution.

Without the restriction on the u, v velocities, this problem would seem more general than the original one, since here we have an initial velocity. But, knowing that we can pass from the varied

motion at X to P with an infinitely small momentum (by giving a blow at X), we see that in the combination of parabolas and momentum curves of which the solution necessarily consists there cannot be a finite length of the latter, for that would give a finite quantity of momentum. But evidently an evanescent length of momentum curve is equivalent to an evanescent impulse at right angles to the direction of motion, so that the curve may be taken to consist of a series of arcs of parabolas, such as XH, HF, FP, u and v being the velocities at X in the motion along XH.



Consider only two arcs, XH and HEP. Let the whole time of passage along $XHEP$ be t, while t' is the time corresponding to HP. Let u' and v' be the velocities communicated by the blow at H, and a and b the coordinates of P relative to X. Then we have the equations

$$a = ut+u't'+\tfrac{1}{2}gt^2, \quad b = vt+v't',$$

$$\{u+g\,(t-t')\}\,u'+vv' = 0,$$

and we have to make $u'^2+v'^2$ a minimum, subject to these conditions, u, v, a, and b being given quantities, and the rest variable. Neglecting the squares of u' and v', we get a cubic equation for t', and after some reductions we obtain the following inequality as the condition that $u'^2+v'^2$ shall not have a minimum value between X and P,

$$u_1u^3+3u_1uv^2+v^4 > v^2u_1^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where u_1 is the u-component of the velocity at P. Hence, if this condition be satisfied, the single parabola XGP is preferable to the two XH and HP. Again, if (5) be satisfied for X and P, it is easily seen to be satisfied for any intermediate point, H with P, whatever be the relative positions of X and P, so that HEP is preferable to HFP, and thus we see that, if (5) be satisfied, the single parabola is preferable to any number of parabolic arcs.

It would not be easy to express generally the velocities u, v, and u_1 in terms of the coordinates of P relative to O, because the result of integrating (2) is complicated, but it is quite easy to satisfy oneself by carefully drawn figures of the curve PMR (Fig. p. 352) that (5) is satisfied when OP makes only a small angle with the horizontal, since in that case u and u_1 are both negative, and u_1 is smaller than u.

And, again, when OP is nearly vertical the condition will not be satisfied, or at least, if it be satisfied, it is evident from a well drawn figure that the solution, though in that case a minimum, would not give the *least* momentum. But doubtless, if the trajectory curves were drawn, we should find that the solution in that case was not even a minimum.

10. The principles on which the solution of this problem depends appear to apply generally; in fact, if the limits be fixed, the only point in which the discussion differs from that of the ordinary discontinuous solution is the mode of ascertaining whether the portion in which u is zero can be altered so as to give a better result, and here the principle laid down seems generally applicable.

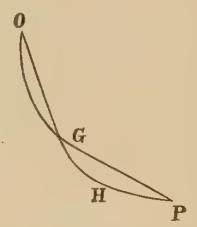
11. If we work the question with Mr. Todhunter's limitation that the curve is to be convex downwards, or, more properly, that it is not to be concave downwards, we may evidently use a straight line anywhere as part of the solution. Then the work will proceed much as in the preceding paragraphs. The parabola being, of course, excluded, the solution consists of momentum curves and straight lines. At P we must have a straight line, just as we had a parabola before. Naturally we try one of each at first. Draw from P a series of lines touching the momentum curves from O. Express the momentum along any one of these compound momentum straight line curves in terms of a parameter. Find the parameter for which the expression is a minimum. Then the corresponding compound curve gives the required minimum. For, representing the left-hand side of (1), p. 350, by θ, we get always

$$\delta U = \frac{\sqrt{2x}}{(1+\dot{y}^2)}\,\delta\dot{y} + \frac{1}{\sqrt{2}}\int_0^P \theta\,\delta y\,dx,$$

it being only assumed that there is no discontinuity in the tangent, and that the limiting coordinates are fixed. Reducing, we obtain

$$\delta U = \frac{\sqrt{2x}}{1+\dot{y}^2}\,\delta\dot{y} - \frac{1}{\sqrt{2}}\int_0^P \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\ \frac{\dot{y}^2}{1+\dot{y}^2}\right)\delta y\,dx.$$

Now, let OGP be the solution, OG being the momentum curve, and



GP being the tangent line, and let OHP be any other consecutive course. First suppose that OHP is *below* OGP, so far as the portion GH is concerned, so that δy is positive from G to P. Then, so far as the limiting term of δU is concerned, δV is positive, for evidently $\delta\dot{y}$ is positive at P, and the integral may be divided into two parts, one from O to G, for which the coefficient of δy is zero, this being the stationary solution, and one from G to P, for which, since $\dot{y}$ is constant, the coefficient becomes

$$\frac{\dot{y}^2}{1+\dot{y}^2}\,\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)dx,$$

which is always negative, dx being necessarily positive. Hence, as the integral has the negative sign, the expression for δU is necessarily positive in both its terms.

If δy should not be positive from G to P, we must draw a consecutive compound curve $OG'P$ above OGP, and so that the δy in passing from this curve to OHP *is* positive along the straight portion $G'P$. Then, by the preceding, the momentum along OHP is greater than that along $OG'P$, and, *a fortiori*, it is greater than that along OGP, which is therefore a true minimum.

13. If we required to find the path of absolutely least momentum from O to P, discontinuity of the tangent being permitted, it is evident that the stationary curve cannot enter into the solution at all, for it is only a minimum when the values of the tangent at its extremities are given. Hence the solution must consist entirely of arcs of parabolas intersecting at points where impulses have been applied. To determine the number and position of these arcs is a problem of the differential calculus.

14. Isoperimetrical problems of this class can be solved by applying the same principle. Suppose it is required to make $\int+\sqrt{u^2dx^2}$ a minimum, subject to $\int v\,dx$ being given. Then

$$\delta U=\int(\delta u+\lambda\delta v)\,dx=\text{limiting terms}+\int(M+\lambda N)\,\delta y\,dx\,;$$

but we must be careful to take the proper signs for δu and M. The

limits being supposed fixed, we should therefore write

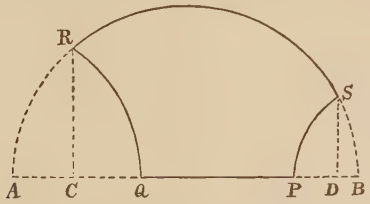
$$\delta U = \int (+\sqrt{M^2\delta y^2 dx^2} + \lambda N \delta y \delta x),$$

and hence the curve $u = 0$ can only be part of the solution when M is numerically greater than λN, for otherwise δU would change sign if the sign of δy were changed. Similar reasoning applies if there are two variables connected by an equation of condition $v = 0$.

The simplest example of such an isoperimetrical problem is to make an area $\int_Q^P + \sqrt{y^2 dx^2}$ a maximum subject to the length of the curve, or $\int_Q^P + \sqrt{(1+\dot{y}^2)\, dx^2}$, being given, P and Q being two points on the axis of x. Here $u = 0$ becomes $y = 0$, which cannot be part of the solution, since

$$\delta \int + \sqrt{y^2 dx^2}$$

is *positive*, when $y = 0$. Hence the solution must consist entirely of stationary curves, which in this case are circles whose radius must necessarily be the same for all the curves (for it is a function of λ alone, and of course λ cannot alter from one portion of the solution to another, for λ is merely Euler's multiplier). But, as has been already stated, when $y\,dx$ is negative, the integral must fulfil the condition for a minimum, not a maximum, and therefore the concavity of the arcs must be turned in opposite directions. It is not hard to show analytically, and is evident geometrically, that, if the given length be greater than the semicircle on PQ, the solution $QRSP$ is obtained,

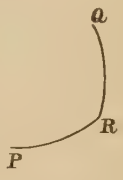


by placing a semicircle $ARSB$ of the given length anywhere on $AQPB$, and drawing arcs QR and PS so as to be exact reflections of AR and BS with regard to verticals through R and S, and the solution is thus partially indeterminate.

15. *Applications of the Foregoing Theory.*—The range of problems in which the integration can be completely effected is so small that the

most interesting application of the theory is to show that in certain cases there is no stationary solution other than that obtained by the ordinary method.

Ex. 1.—Is there any stationary solution for the brachistochrone when angular points such as that at R are allowed?



Evidently a broken solution of this character would not be obtained by the ordinary method, as it requires that we use two different solutions. Now, if we assume that between P and R there is no discontinuity, the ordinary investigation will be applicable to that portion of the integral which must therefore be a part of the brachistochrone curve, and similarly for the portion from R to Q. Hence we have only to examine the terms at the x_1 limits (see § 4). For the sake of variety, however, I will work this problem, using the arc s as the independent variable, and write the time of descent as

$$U = \int_R^P \left\{ \frac{1}{\sqrt{y_p}} + \tfrac{1}{2}\lambda_p (\dot{x}_p^2 + \dot{y}_p^2) \right\} ds + \int_Q^R \left\{ \frac{1}{\sqrt{y_q}} + \tfrac{1}{2}\lambda_q (\dot{x}_q^2 + \dot{y}_q^2) \right\} ds,$$

using the suffixes p and q to denote the two solutions.

The P and Q xy-limits being fixed, the equations we obtain are

$$\frac{1}{\sqrt{y_p}} - \lambda_p = c, \quad \frac{1}{\sqrt{y_q}} - \lambda_q = c',$$

throughout the respective curves, and at R,

$$\Delta s \left\{ \left(\frac{1}{\sqrt{y_p}} - \lambda_p \right) - \left(\frac{1}{\sqrt{y_q}} - \lambda_q \right) \right\} + \Delta x \, (\lambda_p \dot{x}_p - \lambda_q \dot{x}_q) + \Delta y \, (\lambda_p \dot{y}_p - \lambda_q \dot{x}_q) \equiv 0.$$

From the coefficient of Δs we have $c = c'$, wherefore, as $y_p = y_q$ at R, we have $\lambda_p = \lambda_q$ at R; and therefore the coefficients of Δx and Δy give us

$$\dot{x}_p - \dot{x}_q = 0, \quad \dot{y}_p - \dot{y}_q = 0.$$

Hence the two curves y_p and y_q are portions of a single brachistochrone PQ, and there is no point such as R with an abrupt change of direction. The discussion can evidently be applied to show that there cannot be more than one abrupt change of direction.

We are not, however, as yet in a position to assert that there is no maximum or minimum solution *which is not a stationary solution.* For there is a boundary $y = 0$ above which the particle cannot pass without introducing imaginary quantities, and therefore δy must be

positive all along that boundary. It is easy to see that this gives no minimum (but by taking R at infinity on this boundary, so that PR and QR are both horizontal, we do get what may be called a maximum solution, which is, of course, not stationary). Hence there is no minimum solution other than the brachistochrone from P to Q.

I now take the so-called problem of the solid of least resistance.

Ex. 2.—"To find the form of a solid which experiences a minimum resistance when it moves through a fluid in the direction of the axis of revolution." (Todhunter, *Researches*, p. 167.)

The mode of translating this problem into symbols is to say that we require to make

$$U = \int_Q^P \frac{y\dot{y}^3}{1+\dot{y}^2}\,dx$$

a minimum, P and Q being the given limiting points. It is, indeed, well known that if there be angular points the integral ceases in any way to represent the resistance, but, since Legendre pointed out that a zigzag solution rendered the integral indefinitely small, the problem has been discussed as though the integral represented the resistance under all circumstances.

Restricting ourselves to the integral therefore, let us see if there is any compounded solution. We have

$$\delta U = \int \left(\frac{\dot{y}^3 \delta y}{1+\dot{y}^2} - y\dot{y}^2 \frac{3+\dot{y}^2}{(1+\dot{y}^2)^2}\,\delta\dot{y} \right) dx$$

$$= \left| \frac{y\dot{y}^3}{1+\dot{y}^2}\,\Delta x + \right| y\dot{y}^2 \frac{3+\dot{y}^2}{(1+\dot{y}^2} (\Delta y - \dot{y}\Delta x) + \int M \delta y\, dx,$$

and $M = 0$ gives, when integrated,

$$y\dot{y}^3 = c\,(1+\dot{y}^2)^2.$$

Hence, if there be a compound solution where $\dot{y}$ has the value $\dot{y}$ from P to R, and $\dot{z}$ from R to Q, we have, from the coefficients of Δx and Δy at R, the equations

$$\frac{y\dot{y}^3}{(1+\dot{y}^2)^2} = \frac{y\dot{z}^3}{(1+\dot{z}^2)^2}, \quad \frac{y\dot{y}^2\,(3+\dot{y}^2)}{(1+\dot{y}^2)^2} = \frac{y\dot{z}^2\,(3+\dot{z}^2)}{(1+\dot{z}^2)^2} \quad \ldots\ldots\ldots(6).$$

Dividing, we obtain

$$\dot{y} - \dot{z} = 0,$$

which only gives the continuous solution, and the factor

$$\dot{y}\dot{z} - 3 = 0.$$

But, when we substitute from this in (6), we get only

$$(\dot{z}^2-3)^3 = 0,$$

so that $\dot{z} = \sqrt{3}$ and $\dot{y} = \sqrt{3}$, or $\dot{z} = -\sqrt{3}$ and $\dot{y} = -\sqrt{3}$,

and these again are one and the same solution. Hence there is no compounded *stationary* solution, for $y = 0$, which satisfies the equations, is not applicable.

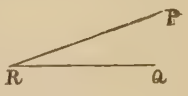
When we examine the sign of $\delta^2 U$, we find that the stationary solution is not a minimum if $\dot{y}^2 > 3$. It is usual to argue that, as there must be a minimum value for the integral in every case, there must be one where the line PQ is so steep that $\dot{y}^2$ must exceed 3 somewhere. But this is incorrect; so long as we confine ourselves to the *integral*, there may be no minimum, because we may use a zigzag solution in which the value of $\dot{y}$ for those parts of the integral which are negative (either because dx is negative, or because y is negative) is greater than for the positive parts, and therefore we may make the integral have a negative, and indeed *any* negative value.

If, however, we restrict the value by taking every term positively, we must have a minimum value. Then we have, by § 7, the boundary given by

$$\frac{y\dot{y}^3}{1+\dot{y}^2}\,dx = 0,$$

which gives only $\dot{y} = 0$, $y = 0$, and $dx = 0$;

but the latter value does not really make the expression vanish, and, as $y = 0$ or $\dot{y} = 0$ is a special case of the continuous solution, it is seen from the above equations that it cannot be joined to any other continuous solution so as to give a stationary value when Δx is changed. Since, therefore, neither $\dot{y} = 0$ nor $y = 0$ can be made to pass through both P and Q, we see that this is a case in which there is no *definite* solution, but by continually shifting the point R further back we can make the integral as small as we choose, until, when R is at infinity, the minimum value of the restricted integral is obtained, or the zigzag may be used instead of the straight line.



Ex. 3.—Mr. Todhunter, in his *Researches*, gives a modification of

this problem in which he introduces the restriction that the volume of the solid of revolution is to be given. He takes a particular case in which the point Q is to lie on the axis of revolution. But, independently of any limiting conditions, we can see from the discussion of the last problem that there cannot be any *definite* solution. For the introduction of the term $\lambda y^2 dx$ under the integral sign is the only difference between this problem and that of the last article, and this change does not in any way affect the coefficient of Δy, nor does it affect the Δx terms, for the only difference is that in deriving the Δx terms at R we have to add and subtract the term $\lambda y^2 \Delta x$. Hence the equations of the last example are applicable to this problem, and there is no compounded solution. Nor do we get any solution other than the zigzag by taking every term positive, for this only requires us to compound with the ordinary solution the curve

$$y\dot{y}^3/(1+\dot{y}^2) = 0,$$

not the whole quantity now under the integral sign, which differs from this by λy^2.

16. With reference to zigzag solutions, it is to be noted that, if they were permitted, no unrestricted integral would have a maximum or minimum value. For, by giving alternate positive and negative signs to dx, we could always arrange that $\delta^2 U$ should have either sign. Thus the integral which gives the shortest distance between two points is

$$U = \int \sqrt{1+\dot{y}^2}\, dx,$$

wherefore

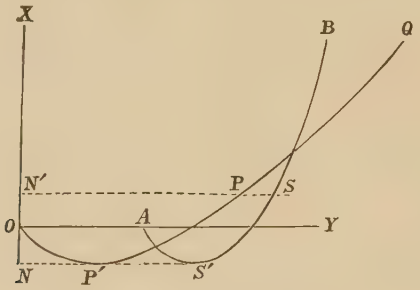
$$\delta^2 U = \int \frac{\delta\dot{y}^2}{(1+\dot{y}^2)^{\frac{3}{2}}}\, dx,$$

and if we take $\delta\dot{y}$ large, where dx is negative, and small where dx is positive, we shall get $\delta^2 U$ negative, wherefore it would appear that the straight line is not the shortest distance between two points. Of course the answer is that the integral does not represent the length unless each element of the integral be taken positively. This is, of course, implied always in problems relating to length, but the curious thing is that the implication is never carried to its logical conclusion, which is that the zero distance between two curves at their point of intersection is a true minimum distance, though not a *stationary* distance. Thus the point of intersection is to be regarded as a solution,

analytically discontinuous, but geometrically continuous, of the problem of finding the shortest distance between two curves.

A similar observation, of course, applies to the brachistochrone from one curve to another, and various other problems.

17. A supposed discontinuous solution has been given* of James Bernoulli's problem to join two given points A and B by a curve ASB of given length, such that, if on any ordinate SN (I take the axis of x vertically upwards) a length PN be taken equal to the arc AS, the area of the curve OPQ so formed shall be a maximum or a minimum (Todhunter, *History*, p. 221).



Analytically the problem is to make $\int s\,dx$ a maximum, subject to

$$\int ds = \text{constant}.$$

Since

$$\int_A^B s\,dx = \Big|_A^B sx - \int_A^B x\,ds,$$

and the limits of s and x are given, the problem is to make $\int x\,ds$ a minimum (or a maximum), s being given. Hence the solution is a catenary, and, examining the problem as we examined the brachistochrone, it is easy to see, what is indeed obvious from physical considerations, that there is no other solution than the continuous catenary which will make $\int x\,ds$ a minimum or a maximum.

* As the mistake made is not uncommon, considering the small number of problems which can be solved at all, it is perhaps worth pointing out. It arises, taking the Jacobian criterion for a maximum or a minimum of $\int u\,dx$ in the incomplete form that $\frac{d^2u}{dy^{(n)2}}$ should not change sign, whereas it is really the product of this quantity and dx which is to remain unchanged (see *Proc. Lond. Math. Soc.*, Vol. XXIII., p. 246, Prop. IV., where the correct statement of the Jacobian criterion is given).

18. There is a class of problem which might fairly be termed discontinuous to which I have not here alluded, *i.e.*, problems in which the variation δU cannot be made to vanish by any *definite* curve, but can be diminished without limit. The best example I can give of this is that of the shortest line of given curvature between two given points, solved in my paper, *Proceedings*, Vol. XXIII., p. 257.

There is another class of discontinuity which seems hardly to have received the attention it deserves, viz., that in which there are too many boundary conditions, or, what is practically the same, where we have to deal with an expression $V + U$, where U is the usual integral and V contains limiting fluxions of a higher order than those for which a continuous solution would be possible if V were absent. It seems to have been generally assumed that in such a case there is no maximum or minimum (see, among other writers, Jellett, pp. 44–47, and 65, 66). But it is evident that there are many cases in which there must be a maximum or a minimum solution. For instance, the problem to draw between two points P and Q the shortest line which shall touch two lines PA and QB at P and Q, respectively, cannot have for solution a length shorter than the straight line PQ, and therefore there must be a minimum, though possibly not a *determinate* one. The solution is evidently the discontinuous line, straight to the eye, which takes, for an infinitesimal length, a sharp turn at P and Q, so as to have the required direction at those points.

A most important example of this class of problem is that in which we are required to give a minimum value to the integral

$$U \equiv \iiint \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2 \right\} dx\, dy\, dz,$$

the solution being of course an electric potential, whatever the boundary conditions may be. Now, if the value of ϕ be given as a function of x, y, and z over two continuous closed surfaces, the problem, so far as the calculus of variations is concerned, presents no discontinuity, although it may, of course, happen that the quantities $\frac{d\phi}{dx}$, &c., appearing in U may, as in any other problem, become discontinuous somewhere within the region of integration. But if, *in addition* to the values being given over *two* surfaces, there are other boundary conditions, as, for instance, if the value over a dozen conductors is given, then we get a necessary discontinuity at the limits quite independently of any discontinuity which may arise in the range of the integration. Physically, of course, this means that

we must have a distribution of electricity at some of the surfaces, whether we have any in the space included in the $dx\,dy\,dz$ integration or not. Analytically, it means that we are no longer to make

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0$$

throughout the entire space of the integration, but only up to an infinitesimal distance from the boundaries where there is electricity, just as in the case of the straight line, we are not to make

$$d^2y/dx^2=0$$

everywhere throughout the integration. but only up to an infinitesimal distance from the limiting points P and Q.

What occurs in the potential problem is probably typical of what happens in multiple integrals generally. For instance, given two continuous closed curves, the solution of the problem to join them by the surface of least area is analytically as well as physically continuous. But, if we replace the two curves by two lines with any number of angular points, we shall evidently have a physically continuous surface giving a solution which has analytical discontinuities at the boundaries. In general, we may expect that the effect of a superabundant number of limiting conditions is merely to introduce discontinuities of some kind *at the boundary*, and to leave the solution continuous within the general extent of the integration, and not by any means to render the function incapable of a maximum or a minimum value.

On those Orthogonal Substitutions that can be Generated by the Repetition of an Infinitesimal Orthogonal Substitution. *By* Henry Taber. Received May 1st, 1895. Read May 9th, 1895.

§ 1.

In the following I show what are the conditions necessary and sufficient that a given orthogonal substitution of n variables may be generated by the repetition of an infinitesimal orthogonal substitution of the same number of variables (that is, by the repetition of an

orthogonal substitution of n variables infinitely near to the identical substitution).

An orthogonal substitution may be designated as of the first or second kind according as it is or is not the second power of an orthogonal substitution. Improper orthogonal substitutions are then of the second kind. All real proper orthogonal substitutions, and all imaginary proper orthogonal substitutions of two or three variables, are of the first kind; but there are imaginary proper orthogonal substitutions of n variables of the second kind for any value of $n \geq 4$. Thus the imaginary proper orthogonal substitution of four variables given on p. 255 of the *Bulletin of the New York Mathematical Society*, for July, 1894, is not the second power of any orthogonal substitution whatever; and, from the existence for four variables of an orthogonal substitution of the second kind, it follows that, for any number of variables greater than four, there are proper orthogonal substitutions of the second kind. In the number of the *Bulletin* referred to above, I have shown that any orthogonal substitution of the first kind can be generated by the repetition of an orthogonal substitution infinitely near to the identical substitution; but that no orthogonal substitution of the second kind can be generated thus. (See § 3.) The conditions necessary and sufficient that a given orthogonal substitution may be generated by the repetition of an orthogonal substitution infinitely near to the identical substitution are, then, the same as the conditions necessary and sufficient that a given orthogonal substitution shall be the second power of an orthogonal substitution, that is, that an orthogonal substitution shall be of the first kind.

In Vol. XVI., p. 130, of the *American Journal of Mathematics*, I have shown that certain conditions, presently to be named, are satisfied by every orthogonal substitution of the first kind, that is, by every orthogonal substitution which is the second power of an orthogonal substitution. I now find that these conditions are sufficient as well as necessary.

That these conditions are sufficient may be most readily shown, if, in accordance with Cayley's "Memoir on the Theory of Matrices," *Philosophical Transactions*, 1858, we regard the operations of *addition* and *subtraction* as capable of extension to linear substitutions or their *matrices*, that is, the square array of their coefficients.* *Multiplication*

* Denoting by $(\phi)_{rs}$ the coefficient of the linear substitution ϕ of n variables in the r^{th} row and s^{th} column of its matrix, the sum or difference of two linear substitutions ϕ and ψ of n variables is defined as follows:—

$$(\phi \pm \psi)_{rs} = (\phi)_{rs} \pm (\psi)_{rs} \quad (r, s = 1, 2, \ldots n).$$

is, of course, taken as equivalent to the composition of linear substitutions, and is associative and distributive. Multiplication is not in general commutative; but, if $f(\phi)$ and $F(\phi)$ are two polynomials in the linear substitution ϕ, we have

$$f(\phi) \cdot F(\phi) = F(\phi) \cdot f(\phi).$$

In what follows the *identical substitution* will be denoted by δ; the linear substitution which, multiplied by or into ϕ, gives the identical substitution will be denoted by ϕ^{-1}; and the linear substitution *transverse* or *conjugate* to ϕ will be denoted by $\breve{\phi}$.* We then have

$$(\phi\psi)^{-1} = \psi^{-1}\phi^{-1}, \quad \breve{\phi+\psi} = \breve{\phi}+\breve{\psi}, \quad \breve{(\phi\psi)} = \breve{\psi}\,\breve{\phi},$$

and

$$\breve{(\phi^{-1})} = (\breve{\phi})^{-1}.$$

The linear substitution ϕ is *symmetric*, if $\breve{\phi} = \phi$; is *skew symmetric*, if $\breve{\phi} = -\phi$; and is *orthogonal*, if $\breve{\phi} = \phi^{-1}$. Finally, the *determinant* of the linear substitution ϕ will be denoted by $|\phi|$. The characteristic equation of ϕ is then

$$|\phi - z\delta| = 0.$$

Further, following Sylvester, I shall employ the term *nullity* to denote the complement relative to n, the number of variables, of the order of the non-evanescent minor formed from the rows and columns of the determinant or matrix of a linear substitution. Thus, the nullity of the linear substitution ϕ of n variables is m, if the $(m-1)^{\text{th}}$ minors of $|\phi|$ (the minors of order $n-m+1$) all vanish, but not all the m^{th} minors (the minors of order $n-m$). In particular, if

$$|\phi| \neq 0,$$

the nullity of ϕ is zero. If g is a root of multiplicity m of the characteristic equation of ϕ, the nullity of $\phi - g\delta$ is at least 1, and the nullity of successive integer powers of $\phi - g\delta$ increases until a power of index $\mu \leqq m$ is attained, whose nullity is m. The nullity

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* With the notation of the preceding note, we have r and s taking all integer values from 1 to n,

$$(\delta)_{rr} = 1, \quad (\delta)_{rs} = 0 \quad (r \neq s),$$

$$(\breve{\phi})_{rs} = (\phi)_{sr},$$

and if, as in what follows, we denote the determinant of ϕ by $|\phi|$,

$$(\phi^{-1})_{rs} = \frac{1}{|\phi|} \frac{d|\phi|}{d(\phi)_{sr}}.$$

of the $(\mu+1)^{\text{th}}$ and higher powers of $\phi - g\delta$ is then also m. And if we designate respectively by

$$, m_1,\ m_2,\ \ldots\ m_{\mu-1},\ m_\mu = m,$$

the nullities of $(\phi - g\delta)$, $(\phi - g\delta)^2$, ... $(\phi - g\delta)^{\mu-1}$, $(\phi - g\delta)^\mu$,

we have $$m_2 - m_1 \geqq m_3 - m_2 \geqq \ldots \geqq m_\mu - m_{\mu-1} \geqq 1.$$

The numbers m_1, m_2, &c., may be termed the numbers *belonging* to the root g of the characteristic equation of ϕ. If g is not a root of the characteristic equation of ϕ, that is, if the multiplicity of g is zero, the nullity of $\phi - g\delta$ [and of $(\phi - g\delta)^2$, &c.] is zero, and we may say that the number belonging to g is zero. Again, by the "corollary of the law of nullity," if g_1 and g_2 are distinct roots of the characteristic equation of ϕ, the nullity of $(\phi - g_1)^{p_1}(\phi - g_2)^{p_2}$ is the sum of the nullities of the two factors.

Let, now, ϕ be an orthogonal substitution which is the second power of an orthogonal substitution ψ; that is, let $\phi = \psi^2$, ψ being orthogonal. The roots of the characteristic equation of ϕ are the squares of the roots of the characteristic equation of ψ. Therefore, if -1 is a root of the characteristic equation of ϕ, $\sqrt{-1}$ is a root of the characteristic equation of ψ; that is, the determinant of $\psi - \sqrt{-1}\,\delta$ is zero. But then the determinant of the transverse of $\psi - \sqrt{-1}\,\delta$, namely, $\breve{\psi} - \sqrt{-1}\,\delta$, obtained from $\psi - \sqrt{-1}\,\delta$ by interchanging the rows and columns of its matrix, is also zero; and, since

$$\breve{\psi} - \sqrt{-1}\,\delta = \psi^{-1} - \sqrt{-1}\,\delta = -\sqrt{-1}\,\psi^{-1}(\psi + \sqrt{-1}\,\delta),$$

therefore, because $$|\ \psi^{-1}\ | \neq 0,$$

$$|\ \psi + \sqrt{-1}\,\delta\ | = 0,$$

that is, $-\sqrt{-1}$ is also a root of the characteristic equation of ψ.

If the nullity of $\psi - \sqrt{-1}\,\delta$ is m_1, the nullity of its transverse, namely, $\breve{\psi} - \sqrt{-1}\,\delta = \psi^{-1} - \sqrt{-1}\,\delta) = -\sqrt{-1}\,\psi^{-1}(\psi + \sqrt{-1}\,\delta)$ is also m_1; and, since the nullity of ψ^{-1} is zero, the nullity of $\psi + \sqrt{-1}\,\delta$ is m_1. Therefore, the nullity of

$$(\phi + \delta) = (\psi - \sqrt{-1}\,\delta)(\psi + \sqrt{-1}\,\delta)$$

is $2m_1$.

Similarly, if the nullity of $(\psi-\sqrt{-1})^p$ is m_p, the nullity of its transverse

$$(\psi-\sqrt{-1}\,\delta)^p=(\psi^{-1}-\sqrt{-1}\,\delta)^p=(-\sqrt{-1}\,\psi^{-1})^p\,(\psi+\sqrt{-1}\,\delta)^p$$

is also m_p. Therefore, the nullity of

$$(\phi+\delta)^p=(\psi-\sqrt{-1})^p\,(\psi+\sqrt{-1})^p$$

is $2m_p$.

§ 2.

Conversely, if ϕ is orthogonal, and if for any positive integer p the nullity of $(\phi+\delta)^p$ is even, ϕ is the second power of an orthogonal substitution.

In Vol. LXXXIV. of *Crelle's Journal*, Frobenius has given substantially the following theorem, namely, that an orthogonal substitution ψ can always be formed of whose characteristic equation any given quantities (other than zero) are roots of any given multiplicities, provided that, if $g\neq\pm1$ is a root of the characteristic equation, g^{-1} is also a root of the same multiplicity as g; moreover, that we may take any set of numbers $m_1, m_2, \ldots m_\mu$, subject to the conditions

$$m_2-m_1\geqq m_3-m_2\geqq\ldots\geqq m_\mu-m_{\mu-1}\geqq1,$$

as the numbers belonging to the root $g\neq\pm1$, provided that the same set of numbers belongs to g^{-1}. Further, ψ may have $+1$ as a root of its characteristic equation, and the numbers belonging to $+1$ may be taken the same as the numbers belonging to the root $+1$ of the characteristic equation of any other orthogonal substitution.

Let, now, the roots of the characteristic equation of ϕ be $+1$ of multiplicity m, -1 of multiplicity $2m$, and g_r, g_r^{-1} each of multiplicity $m^{(r)}$, r taking all integer values from 1 to ν. Let the numbers belonging respectively to $+1$ and -1 be

$$(\mathrm{m}_1, \mathrm{m}_2, \ldots \mathrm{m}_{\mu_0})\,(2m_1, 2m_2, \ldots 2m_\mu),$$

and the numbers belonging to g_r, g_r^{-1}, for $r=1, 2, \ldots, \nu$, be

$$(m_1^{(r)}, m_2^{(r)}, \ldots m_{\mu_r}^{(r)}).$$

Let us now form an orthogonal substitution ψ whose characteristic equation shall have as roots $+1$ of multiplicity m, $\pm\sqrt{-1}$ each of multiplicity m, and for

$$r=1, 2, \ldots \nu, \quad h_r=\sqrt{g_r}, \quad h_r^{-1}=\frac{1}{\sqrt{g_r}},$$

each of multiplicity $m^{(r)}$. Further, let the numbers belonging to $+1$ be

$$(\mathrm{m}_1,\ \mathrm{m}_2,\ \ldots\ \mathrm{m}_{\mu_0});$$

let the numbers belonging to $+\sqrt{-1}$ and to $-\sqrt{-1}$ be

$$(m_1,\ m_2,\ \ldots\ m_\mu);$$

and, for $r = 1, 2, \ldots \nu$, let the numbers belonging to h_r and h_r^{-1} be

$$(m_1^{(r)},\ m_2^{(r)},\ \ldots\ m_{\mu_r}^{(r)}).$$

The roots of the characteristic equation of ψ^2 are then $+1$ of multiplicity m, -1 of multiplicity $2m$, and (for $r = 1, 2, \ldots \nu$) g_r and g_r^{-1} each of multiplicity $m^{(r)}$. Further, since -1 is not a root of the characteristic equation of ψ, the nullity of $\psi+\delta$ is zero; therefore, the nullity of

$$(\psi^2-\delta)^p = (\psi+\delta)^p\,(\psi-\delta)^p$$

is equal to the nullity of $(\psi-\delta)^p$. Consequently, the numbers belonging to the root $+1$ of the characteristic equation of ψ^2 are

$$(\mathrm{m}_1,\ \mathrm{m}_2,\ \ldots\ \mathrm{m}_{\mu_0}).$$

Again, the nullity of

$$(\psi^2+\delta)^p = (\psi-\sqrt{-1}\,\delta)^p\,(\psi+\sqrt{-1}\,\delta)^p$$

is equal to the sum of the nullities of

$$(\psi-\sqrt{-1}\,\delta)^p \quad \text{and} \quad (\psi+\sqrt{-1}\,\delta)^p,$$

since both $\pm\sqrt{-1}$ are roots of the characteristic equation of ψ. Therefore, the numbers belonging to the root -1 of the characteristic equation of ψ^2 are

$$(2m_1,\ 2m_2,\ \ldots\ 2m_\mu).$$

Finally, the nullity of

$$(\psi^2+g_r\delta)^p = (\psi+h_r\delta)^p\,(\psi-h_r\delta)^p$$

is equal to the nullity of $(\psi-h_r\delta)^p$, since $-h_r$ is not a root of the characteristic equation of ψ; and, therefore, the nullity of $(\psi+h_r\delta)^p$ is zero. Similarly, the nullity

$$(\psi^2-g_r^{-1}\delta)^p = (\psi+h_r^{-1}\delta)^p\,(\psi-h_r^{-1}\delta)^p$$

is equal to the nullity of $(\psi-h_r^{-1}\delta)^p$, since $-h_r^{-1}$ is not a root of the characteristic equation of ψ. Whence it follows that, for $r = 1, 2, \ldots \nu$,

the numbers belonging to the root g_r, g_r^{-1} of the characteristic equation of ψ^2 are

$$(m_1^{(r)},\ m_2^{(r)},\ \ldots\ m_{\mu_r}^{(r)}).$$

Since ϕ and ψ^2 are similar, that is, the roots of the characteristic equations of ϕ and ψ and the numbers belonging to these roots are the same, a linear substitution ϖ of non-zero determinant can be found such that

$$\phi = \varpi\,\psi^2\,\varpi^{-1}.$$

Since both ϕ and ψ are orthogonal, we can always so choose ϖ that it shall be orthogonal. For we have

$$\delta = \breve{\phi}\,\phi = \breve{\varpi}^{-1}\,\breve{\psi}^2\,\breve{\varpi}\,\varpi\,\psi^2\,\varpi^{-1}.$$

That is, denoting $\breve{\varpi}\,\varpi$ by ω,

$$\breve{\psi}^2\,\omega\,\psi^2 = \omega;$$

or, since ψ^2 is orthogonal, $\quad \omega\,\psi^2 = \psi^2\,\omega.$

The linear substitution ω is of non-zero determinant; there are, therefore, one or more polynomials in ω whose second power is equal to ω. Let $\omega^{\frac{1}{2}}$ denote any one of these polynomials. Then, since ω is symmetric, $\omega^{\frac{1}{2}}$ is also symmetric; and, moreover, since ψ^2 is commutative with ω, it is also commutative with $\omega^{\frac{1}{2}}$, that is,

$$\omega^{\frac{1}{2}}\,\psi^2 = \psi^2\,\omega^{\frac{1}{2}}.$$

Any linear substitution ϖ satisfying the equation

$$\breve{\varpi}\,\varpi = \omega$$

is given by the expression $\omega^{\frac{1}{2}}\chi$, in which χ is an orthogonal substitution. For the last equation may be written

$$(\omega^{\frac{1}{2}})^{-1}\,\breve{\varpi}\,\varpi\,(\omega^{\frac{1}{2}})^{-1} = \delta;$$

and, if we put $\qquad \chi = \varpi\,(\omega^{\frac{1}{2}})^{-1},$

it becomes $\qquad \breve{\chi}\,\chi = \delta.$

Conversely, if χ is orthogonal, and

$$\varpi = \chi\,\omega^{\frac{1}{2}},$$

we have $\qquad \breve{\varpi}\,\varpi = \omega^{\frac{1}{2}}\,\breve{\chi}\,\chi\,\omega^{\frac{1}{2}} = \omega.$

We therefore have

$$\phi = \varpi\psi^2\varpi^{-1} = \chi\,\omega^{\frac{1}{2}}\psi^2\omega^{-\frac{1}{2}}\chi^{-1} = \chi\psi^2\chi^{-1}.$$

And, if we put $\Psi = \chi\psi\chi^{-1}$,

Ψ is orthogonal, since both ψ and χ are orthogonal; and

$$\phi = \chi\psi^2\chi^{-1} = \Psi^2;$$

that is, ϕ is the second power of an orthogonal substitution.

§ 3.

We may show as follows that any orthogonal substitution of the first kind can be generated by the repetition of an orthogonal substitution infinitely near to the identical substitution, but that no orthogonal substitution of the second kind can be generated thus. Let e^{ϑ} denote the infinite series

$$\delta + \vartheta + \frac{1}{2!}\vartheta^2 + \frac{1}{3!}\vartheta^3 + \ldots + \frac{1}{m!}\vartheta^m + \ldots,$$

convergent for any linear substitution ϑ. We then have

$$(e^{\vartheta})^{-1} = e^{-\vartheta},$$

$$(\breve{e^{\vartheta}}) = e^{\breve{\vartheta}};$$

and, if m is any positive integer,

$$(e^{\vartheta})^m = e^{m\vartheta}.$$

Moreover, if ϑ and ϑ' are commutative,

$$e^{\vartheta}\,e^{\vartheta'} = e^{\vartheta+\vartheta'}.$$

Finally, for any linear substitution ϕ of non-zero determinant, we can always find a polynomial in ϕ,

$$\vartheta = f(\phi),$$

such that $\phi = e^{\vartheta}$.

If, now, ϕ is orthogonal, we have

$$\breve{\vartheta} = f(\breve{\phi}) = f(\phi^{-1}),$$

that is, $\breve{\vartheta}$ is also a polynomial in ϕ;* and, consequently, $\breve{\vartheta}$ is commutative with ϑ.

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* The reciprocal of a linear substitution ϕ is expressible as a polynomial in ϕ.

Therefore, $$e^{\breve{\vartheta}+\vartheta} = e^{\breve{\vartheta}} e^{\vartheta} = \breve{\phi}\,\phi = \delta.$$

Let $$2\theta_0 = \vartheta + \breve{\vartheta}, \quad 2\theta = \vartheta - \breve{\vartheta}.$$

Since ϑ and $\breve{\vartheta}$ are commutative, θ_0 and θ, their half sum and difference, are commutative; and, consequently,

$$\phi = e^{\vartheta} = e^{\theta_0+\theta} = e^{\theta_0} e^{\theta} = e^{\theta} e^{\theta_0}.$$

Therefore, $$\phi^2 = (e^{\theta_0})^2 (e^{\theta})^2 = e^{2\theta_0} e^{2\theta} = e^{2\theta},$$

since $$e^{2\theta_0} = e^{\vartheta+\breve{\vartheta}} = \delta.$$

Let m be any positive integer, and let

$$\psi = e^{(2/m)\theta};$$

then, since θ is skew symmetric,

$$\breve{\psi}\,\psi = e^{(2/m)\breve{\theta}} e^{(2/m)\theta} = e^{-(2/m)\theta} e^{(2/m)\theta} = e^{-(2/m)\theta+(2/m)\theta} = \delta;$$

moreover, $$\psi^m = (e^{(2/m)\theta})^m = e^{2\theta} = \phi^2.$$

By taking m sufficiently great, we can make the coefficients of $\frac{2}{m}\theta$ as small as we please, and, consequently, we can make

$$\psi = e^{(2/m)\theta}$$

as nearly as we please equal to the identical substitution. But, however great m may be, we have

$$\breve{\psi}\,\psi = \delta \quad \text{and} \quad \psi^m = \phi^2.$$

Therefore, any orthogonal substitution, as ϕ^2, which is the second power of an orthogonal substitution can be generated by the repetition of an infinitesimal orthogonal substitution.

Every orthogonal substitution given by Cayley's expression is of the first kind. For, if

$$\phi = (\delta - \Upsilon)(\delta + \Upsilon)^{-1},$$

in which Υ is skew symmetric, and such that

$$|\, \delta + \Upsilon \,| \neq 0,$$

we can find a polynomial in Υ,

$$\vartheta = f(\Upsilon),$$

such that $$(\delta + \Upsilon) = e^{\vartheta}.$$

Equating the transverse of either side, we have

$$\delta - \Upsilon = e^{\breve{\vartheta}}.$$

And, since $$\breve{\vartheta} = f(\breve{\Upsilon}) = f(-\Upsilon)$$

is also a polynomial in Υ, ϑ and $\breve{\vartheta}$ are commutative. Therefore, if we put

$$\theta = \vartheta - \breve{\vartheta} = f(-\Upsilon) - f(\Upsilon),$$

θ is skew symmetric, as it is a polynomial in odd powers of the skew symmetric matrix Υ, and

$$\phi = (\delta - \Upsilon)(\delta + \Upsilon) = e^{\breve{\vartheta}} e^{-\vartheta} = e^{\theta}.$$

If, now, $$\psi = e^{\frac{1}{2}\theta},$$

since θ is skew symmetric,

$$\breve{\psi}\psi = e^{\frac{1}{2}\breve{\theta}} e^{\frac{1}{2}\theta} = e^{-\frac{1}{2}\theta} e^{\frac{1}{2}\theta} = e^{-\frac{1}{2}\theta + \frac{1}{2}\theta} = \delta;$$

moreover, $$\psi^2 = (e^{\frac{1}{2}\theta})^2 = e^{\theta} = \phi.$$

That is, ϕ is an orthogonal substitution of the first kind.

If we take the orthogonal substitution ϕ sufficiently near to the identical substitution, -1 cannot be a root of the characteristic equation of ϕ; and ϕ is therefore given by Cayley's expression, and is consequently of the first kind. But the repetition of an orthogonal substitution of the first kind gives an orthogonal substitution of that kind. Whence it follows that no orthogonal substitution of the second kind can be generated by the repetition of an infinitesimal orthogonal substitution. Nevertheless, we can approximate as near as we please to any proper orthogonal substitution of the second kind by the repetition of an infinitesimal orthogonal substitution properly chosen. For we can obtain an orthogonal substitution of the first kind which shall be as nearly as we please equal to any proper orthogonal substitution of the second kind,* and the former

* In particular, if ϕ is any proper orthogonal substitution of the second kind, we can find an orthogonal substitution ϕ_ρ of the first kind whose coefficients are rational functions of a parameter ρ such that, by taking ρ sufficiently small, the several coefficients of ϕ_ρ can be made as nearly as we please equal to the corresponding coefficients of ϕ. Consequently, if the rational functions are properly chosen, we shall have $(\phi_\rho)_{\rho=0} = \phi$. So long as $\rho \neq 0$, there exists an orthogonal substitution ψ_ρ whose coefficients are algebraic functions of ρ such that $\psi_\rho^2 = \phi_\rho$; and thus, by taking ρ sufficiently small, we may make ψ_ρ^2 as nearly as we please equal to ϕ. We thus have $\phi = \lim_{\rho=0}(\psi_\rho^2)$. But, for $\rho = 0$, ψ_ρ becomes illusory, as its coefficients are then infinite. (See *Bulletin of the New York Mathematical Society*, for July, 1894, p. 255.)

can be generated by the repetition of an infinitesimal orthogonal substitution.

Since an orthogonal substitution of the first kind, and only an orthogonal substitution of the first kind, can be generated by the repetition of an orthogonal substitution infinitely near to the identical substitution, we have, by § 1 and § 2, the following theorem:

The necessary and sufficient condition that a given orthogonal substitution may be generated by the repetition of an infinitesimal orthogonal substitution is that either -1 *shall not be a root of the characteristic equation of the substitution, or, if* -1 *is a root of this equation, that the numbers belonging to* -1 *shall all be even.*

§ 4.

The preceding division of the substitutions of the orthogonal group gives, of course, a corresponding division of the group of linear substitutions which transform automorphically a symmetric bilinear form with cogredient variables. Thus we may designate a substitution of this group as of the first or second kind according as it is or is not the second power of a substitution of the group; and then any substitution of the first kind may be generated by the repetition of an infinitesimal substitution of the group, but no substitution of the second kind can be generated thus.

For let the variables of the symmetric bilinear form

$$(\Omega \mathfrak{x}_1, x_2, \ldots x_n \mathfrak{x}_1, y_2, \ldots y_n)$$

be cogredient. The necessary and sufficient condition that the linear substitution ϕ shall transform the form automorphically is that ϕ shall satisfy the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega.$$

It is assumed that the determinant of the form is not zero, that is, that

$$|\,\Omega\,| \neq 0.$$

There are therefore one or more polynomials in Ω whose second power is equal to Ω. Let $\Omega^{\frac{1}{2}}$ denote any such polynomial in Ω. We may then write the preceding equation as

$$(\Omega^{\frac{1}{2}})^{-1}\,\breve{\phi}\,\Omega^{\frac{1}{2}}\,.\,\Omega^{\frac{1}{2}}\phi\,(\Omega^{\frac{1}{2}})^{-1} = \delta;$$

and, if $$\psi = \Omega^{\frac{1}{2}} \phi \, (\Omega^{\frac{1}{2}})^{-1},$$

it becomes $$\breve{\psi}\, \psi = \delta,$$

since $\Omega^{\frac{1}{2}}$ is symmetric, as it is a polynomial in the symmetric matrix Ω. Whence it follows that the most general expression for the linear substitution ϕ is

$$(\Omega^{\frac{1}{2}})^{-1} \psi \, \Omega^{\frac{1}{2}},$$

in which $\Omega^{\frac{1}{2}}$ is a symmetric square root of Ω, and ψ is an arbitrary orthogonal linear substitution or matrix.

If, now, ψ is an orthogonal substitution of the first kind, ϕ is also of the first kind, and conversely. For, if ψ_0 is orthogonal, and

$$\psi_0^2 = \psi,$$

then, if $$\phi_0 = (\Omega^{\frac{1}{2}})^{-1} \psi_0 \Omega^{\frac{1}{2}},$$

we have $$\breve{\phi}_0 \Omega \, \phi_0 = \Omega,$$

and $$\phi_0^2 = (\Omega^{\frac{1}{2}})^{-1} \psi_0^2 \, \Omega^{\frac{1}{2}} = (\Omega^{\frac{1}{2}})^{-1} \psi \Omega^{\frac{1}{2}} = \phi.$$

Conversely, if $$\phi_0^2 = \phi,$$

and $$\breve{\phi}_0 \Omega \, \phi_0 = \Omega,$$

then, if $$\psi_0 = \Omega^{\frac{1}{2}} \phi_0 \, (\Omega^{\frac{1}{2}})^{-1},$$

we have $$\breve{\psi}_0 \psi_0 = \delta,$$

and $$\psi_0^2 = \psi.$$

As stated above, the orthogonal substitutions of the second kind are all imaginary. But the linear substitutions of the second kind which transform automorphically certain real symmetric bilinear forms are not all imaginary. Thus the bilinear form

$$\mathfrak{f} \equiv a x_1 y_1 + 2b x_3 y_3 - 2b \, (x_2 y_4 + x_4 y_2) + b \, (x_3 y_4 + x_4 y_3),$$

is transformed automorphically, if we put

$$x_1 = -\xi_1, \quad x_2 = -\xi_2 + \xi_3, \quad x_3 = -\xi_3 + \xi_4, \quad x_4 = -\xi_4,$$

$$y_1 = -\eta_1, \quad y_2 = -\eta_2 + \eta_3, \quad y_3 = -\eta_3 + \eta_4, \quad y_4 = -\eta_4;$$

and this substitution, which is real, is of the second kind.

If a and b are both positive, three of the roots of the equation

$$\Gamma(z) \equiv \begin{vmatrix} a-z, & 0, & 0, & 0 \\ 0, & -z, & 0, & -2b \\ 0, & 0, & 2b-z, & b \\ 0, & -2b, & b, & -z \end{vmatrix} = 0$$

are positive and one negative. If a and b are of different sign, two of the roots of this equation are positive and two negative. If both a and b are negative, all but one of the roots of this equation are negative. But any real symmetric bilinear form

$$(\Omega \mathfrak{X} x_1, x_2, x_3, x_4 \mathfrak{X} y_1, y_2, y_3, y_4)$$

with cogredient variables can be transformed into the form $\mathfrak{F}$ by a real linear substitution ϖ, if the number of positive roots of the equation

$$| \Omega - z\delta | = 0$$

is equal to the number of positive roots of the equation

$$\Gamma(z) = 0.$$

If this condition is satisfied, and if φ denotes the linear substitution given above, the real linear substitution $\varpi \phi \varpi^{-1}$ transforms

$$(\Omega \mathfrak{X} x_1, x_2, x_3, x_4 \mathfrak{X} y_1, y_2, y_3, y_4)$$

automorphically, and is of the second kind. Whence it follows that any real symmetric bilinear form

$$(\Omega \mathfrak{X} x_1, x_2, x_3, x_4 \mathfrak{X} y_1, y_2, y_3, y_4),$$

with two sets of four cogredient variables the roots of whose characteristic equation

$$| \Omega - z\delta | = 0$$

are not all of the same sign, is transformed automorphically by a real inear substitution of the second kind.

Thursday, June 13th, 1895.

Major P. A. MACMAHON, R.A., F.R.S., President, in the Chair.

Mr. Gilbert Thomas Walker, M.A., Fellow of Trinity College, Cambridge, was elected a member.

Mr. Bryan communicated a note "On an Extension of Boltzmann's Minimum Theorem," by Mr. S. H. Burbury, F.R.S.

Dr. Larmor gave a sketch of a paper by Mr. J. Brill, entitled "On the Form of the Energy Integral in the Varying Motion of a Viscous Incompressible Fluid for the case in which the Motion is Two-Dimensional, and the case in which the Motion is Symmetrical about an Axis."

A paper by Dr. Routh, "On an Expansion of the Potential Function $1/R^{s-1}$ in Legendre's Functions," was taken as read.

Mr. Macaulay read a paper entitled "Groups of Points on Curves treated by the Method of Residuation."

The President informed the meeting of the death of Prof. A. M. Nash, of the Presidency College, Calcutta, which took place on the voyage home, for a two years' furlough, after twenty years' service in India.

The following presents were made to the Library :—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XIX., St. 5 ; Leipzig, 1895.

"Proceedings of the Royal Society," Vol. LVII., No. 345.

"Journal of the Institute of Actuaries," Vol. XXXII., Pt. 1 ; April, 1895.

"Berichte über die Verhandlungen der Königl. Sachsischen Gesells. der Wissenschaften zu Leipzig," 1895, I.

"Wiskundige Opgaven met de Oplossingen door de Leden van het Wiskundig Genootschap," Deel 6, St. 5 ; Amsterdam, 1895.

Mantel, W.—"Gewone Lineaire Differentiaalvergelijkengen," pamphlet, 8vo, 1894.

D'Ocagne, M.—"Sur la Composition des Lois de Probabilité des Erreurs de Situation d'un Point sur un Plan," pamphlet.

D'Ocagne, M.—"Abaque en Points Isoplèthes de l'Equation de Képler," pamphlet.

"Proceedings of the Physical Society of London," Vol. XIII., Pt. 7, No. 57 ; June, 1895.

"Nyt Tidsskrift for Mathematik," Aargang Sjette, A., Nr. 1, 2 ; B., Nr. 1 ; Copenhagen, 1895.

"Nieuw Archief voor Wiskunde," Reeks 2, Deel 1, 2; Amsterdam, 1895.

Schouten, Dr. G.—"Theorie der Functies naar Weierstrass," pamphlet, 8vo; 1895.

"Bulletin of the American Mathematical Society," Series 2, Vol. I., No. 8; May, 1895.

"Bulletin de la Société Mathématique de France," Tome XXIII., Nos. 2 and 3; Paris, 1895.

"Bulletin des Sciences Mathématiques," Tome XIX., mai et juin, 1895; Paris.

Lamb, H.—"Hydrodynamics," 8vo; Cambridge, 1895. From the Author.

Schwarz, H. A.—"Uber die analytische Darstellung elliptischer Functionen mittelst rationaler Functionen einer Exponentialfunction," pamphlet.

Schwarz, H. A.—"Zur Theorie der Minimalflächen, deren Begrenzung aus geradlinigen Strecken besteht," pamphlet.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Serie 3, Vol. I., Fasc. 4; 1895.

"Journal of the Japan College of Science," Vol. VII., Pt. 4; Tokyo, 1895.

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," 1895, 1–25.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. IV., Fasc. 8, 9, 10; Roma, 1895.

"Annali di Matematica," Ser. 2, Tomo XXIII., Fasc. 2; Milano, 1895.

"Educational Times," June, 1895.

D'Ocagne, M.—"Sur une Application de la Théorie de la Probabilité des Erreurs aux Nivellements de Haute Précision," pamphlet.

Weierstrass, K.—"Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen," herausgegeben von H. A. Schwartz, Zweite Ausgabe, Abt. 1, 8vo; Berlin, 1893.

"Annales de la Faculté des Sciences de Toulouse," Tome IX., Fasc. 2; Paris, 1895.

"Journal für die reine und angewandte Mathematik," Bd. CXV., Heft 1; Berlin, 1895.

"Indian Engineering," Vol. XVII., Nos. 16–20.

"Application de la Géométrie à la Résolution d'une Classe de Problèmes relatifs au Calcul des Probabilités," by Rev. T. C. Simmons. (Offprint of the Association Française, Congrès de Caen, 1894.) From the Author.

On Elliptic and Hyper-Elliptic Systems of Differential Equations and their Rational and Integral Algebraic Integrals, with a Discussion of the Periodicity of Elliptic and Hyper-Elliptic Functions.

By W. R. Westropp Roberts, M.A. Received and communicated 4th April, 1895. Received, in revised form, 28th August, 1895.

I do not by any means claim originality or novelty for some of the results given in this paper, but only for the method by means of which they are obtained. The subject of elliptic and the higher Abelian integrals still receives such attention from, and occupies so largely the time of, our distinguished mathematicians that I do not hesitate to put forward a method which enables us immediately to write down all the *rational* and *integral* algebraic integrals of the system of differential equations known as Abelian, and embraces them in the unity of the larger theory of covariants. In what follows in these pages, it will appear that, being given any one of these rational and integral algebraic integrals, all the remaining ones may be obtained by an operative process alone.

Further, it will be shown, by integration of what I term the "fundamental equation," an irrational algebraic integral will be obtained, which will yield us a whole series of equations of a similar nature by the application of the same operative process.

Finally, I treat of Abelian functions, defining them, and giving a complete proof of the nature of their periodicity.

I now give a short summary of the contents of my paper.

Articles 1–4 treat of general theorems connected with functions of the differences of various sets of variables, and discuss two operators δ and Δ.

5 contains a definition of $I(z)$.

6. Investigation of the conditions which must be fulfilled in order that a binary quantic of degree $2n$ may be a perfect square. Derivation of all these conditions from a "square-matrix."

7. Abelian system of differential equations.

8–9. Development of our theory and definition of the function $F(z)$, which introduces $m-1$ arbitrary constants.

10–16. Further development and determination of all *rational* and

integral algebraic integrals by showing that $F(z)$ must be a perfect square.

17–18. Integral in the case of $m = 2$. Cayley's form.

19–20. Deduction of well-known equation

$$\frac{dz_1}{\sqrt{A_0 f(z_1)}} + \frac{dz_2}{\sqrt{A_0 f(z_2)}} = \frac{-d\lambda}{\sqrt{-4\lambda^3 + I\lambda - J}}.$$

21–22. Value of $\dfrac{z_1^i dz_1}{\sqrt{A_0 f(z_1)}} + \dfrac{z_2^i dz_2}{\sqrt{A_0 f(z_2)}}$.

23. Application to the case $m = 3$, and deduction of the rational algebraic integrals, which are, as far as I am aware, now given for the first time.

24. Case of $m = 4$.

25. General theory and integration of the "fundamental equation."

26–27. The T-function.

28. Transformation theory.

29–30. Determination of p_m in terms of $t - t^0$.

31–33. Determination of $\Sigma I_r(z) - \Sigma I_r(\zeta)$.

34. Determination of $\Sigma I_{-r}(z) - \Sigma I_{-r}(\zeta)$.

35. Discussion of $\Sigma L(z, n) - \Sigma L(\zeta, n)$.

36. General theory.

37. Proof that $\Sigma I_r(z) - \Sigma_r I(\zeta)$ can be found by an operative process alone.

38–41. Definition of Abelian functions and their periodicity.

42. The number of periods.

43. Interesting algebraic equation, when $z_m = \alpha_1$ and $\zeta_m = \alpha_2$.

1. Before entering on the discussion of the theory of the transcendents known as Abelian, we shall first investigate some algebraic theorems having an important bearing on the subject.

Let the roots of the equation

$$\phi(z) \equiv z^m + p_1 z^{m-1} + p_2 z^{m-2} + \ldots + p_m = 0$$

be denoted by $z_1, z_2, \ldots z_m$, and let us suppose we have a relation such as the following

$$c_1 z^{m-1} + c_2 z^{m-2} + \ldots + c_m = 0,$$

connecting m quantities $c_1, c_2, \ldots c_m$ with z, where z is a root of

$$\phi(z) = 0;$$

then it is plain that, if this relation is satisfied for each root of

$$\phi(z) = 0,$$

it being understood that $\phi(z) = 0$

has no equal roots, we must have

$$c_1 = 0, \quad c_2 = 0, \ \ldots \ c_m = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now, since $z^m = -\{p_1 z^{m-1} + p_2 z^{m-2} + \ldots + p_m\}$,

when z is a root of $\phi(z) = 0$,

we can express z^r, r being greater than $m-1$, in terms of $z^{m-1}, z^{m-2}, \ldots z$ and the quantities $p_1, p_2, \ldots p_m$; so that, if we had a relation of the form

$$D_0 z^r + D_1 z^{r-1} + \ldots + D_r = 0$$

satisfied for each root of the equation

$$\phi(z) = 0,$$

we could reduce this case to the one first discussed, by substituting for all powers of z higher than the $m-1^{\text{th}}$ their values in terms of $z^{m-1}, z^{m-2}, \ldots z$ and the quantities $p_1, p_2, \ldots p_m$, and then, equating the coefficients of $z^{m-1}, z^{m-2}, \ldots z, z^0$ each to zero, arrive at m equations connecting the quantities $D_0, D_1, \ldots D_r$ with the quantities $p_1, p_2, \ldots p_m$.

As an example, suppose

$$\phi(z) \equiv z^2 + p_1 z + p_2,$$

and that we had $D_0 z^3 + D_1 z^2 + D_2 z + D_3 = 0$

for both roots of $\phi(z) = 0$.

We have $z^2 = -(p_1 z + p_2)$;

consequently $z^3 = p_1 (p_1 z + p_2) - p_2 z$

$$= (p_1^2 - p_2) z + p_1 p_2;$$

introducing then these values of z^3 and z^2 into the expression

$$D_0 z^3 + D_1 z^2 + D_2 z + D_3,$$

we find $\{D_0(p_1^2-p_2)-D_1p_1+D_2\}z+D_0p_1p_2-D_1p_2+D_3$,

and this being, by hypothesis, zero for both roots of

$$\phi(z)=0,$$

we must have

$$\left.\begin{aligned} D_0(p_1^2-p_2)-D_1p_1+D_2&=0\\ D_0p_1p_2-D_1p_2+D_3&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

2. Again, being given that an expression such as

$$V\equiv B_1z^{m-1}+B_2z^{m-2}+\ldots B_m$$

is a function of the differences of the quantities

$$z_1, z_2, \ldots z_m;\quad \alpha_1, \alpha_2, \ldots \alpha_{2m};\quad \zeta_1, \zeta_2, \ldots \zeta_m,$$

where z stands for any root of

$$\phi(z)=0,$$

and the coefficients B_1, B_2, ... B_m are *symmetric* functions of the above quantities, then it is clear that, when V is operated on by

$$\Sigma\frac{d}{dz}+\Sigma\frac{d}{d\alpha}+\Sigma\frac{d}{d\zeta},$$

it must vanish identically where

$$\Sigma\frac{d}{dz}\equiv\frac{d}{dz_1}+\frac{d}{dz_2}+\ldots+\frac{d}{dz_m},$$

$$\Sigma\frac{d}{d\alpha}=\frac{d}{d\alpha_1}+\frac{d}{d\alpha_2}+\ldots+\frac{d}{d\alpha_{2m}},$$

$$\Sigma\frac{d}{d\zeta}=\frac{d}{d\zeta_1}+\frac{d}{d\zeta_2}+\ldots+\frac{d}{d\zeta_m}.$$

Professor Cayley has applied the name *facients* to the quantities to which the expression "degree" refers, or in regard to which a function is considered a quantic, the term quantic being used to denote the entire subject of rational and integral functions.

Now, although most of the algebraic expressions we shall have occasion to deal with are neither rational nor integral, we shall

retain the name "facients" to denote the various sets of quantities which enter into them.

If we now call the operation

$$\Sigma \frac{d}{dz} + \Sigma \frac{d}{da} + \Sigma \frac{d}{d\zeta} \equiv -\delta,$$

and operate on the expression V with δ, we shall find

$$\delta V \equiv z^{m-1}\,\delta B_1 + z^{m-2}\{\delta B_2 - (m-1)\,B_1\} + \ldots$$
$$\ldots + z\{\delta B_{m-1} - 2B_{m-2}\} + \{\delta B_m - B_{m-1}\} = 0,$$

and, since δV must vanish for each root of

$$\phi(z) = 0,$$

the coefficient of each power of z in the above expression must vanish by Art. 1, giving us

$$\left.\begin{aligned} B_{m-1} - \delta B_m &= 0, \\ 2B_{m-2} - \delta B_{m-1} &= 0, \\ 3B_{m-3} - \delta B_{m-2} &= 0, \\ \ldots \quad \ldots \quad \ldots \quad \ldots & \\ \delta B_1 &= 0, \end{aligned}\right\}$$

or

$$\left.\begin{aligned} B_{m-1} &= \delta B_m \\ B_{m-2} &= \frac{\delta^2 B_m}{1.2} \\ B_{m-3} &= \frac{\delta^3 B_m}{1.2.3} \\ \ldots \quad \ldots \quad \ldots & \\ \delta B_1 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

From these results we learn that, if an expression such as

$$V \equiv B_1 z^{m-1} + B_2 z^{m-2} + \ldots + B_m,$$

where z is a root of $\phi(z) = 0$,

is a function of the differences of the facients

$$z_1, z_2, \ldots z_m; \quad a_1, a_2, \ldots a_{2m}; \quad \zeta_1, \zeta_2, \ldots \zeta_m,$$

the coefficient of z^{m-1} is a function of the differences of these facients,

and all the coefficients B_1, B_2, ... B_{m-1}, can be derived by successive operations of δ on B_m, to which function we shall apply the term *source*, a name which has already been applied to that term from which covariants of a quantic are derived.

The function V must then be capable of being written in the form

$$V \equiv B_m + z\delta B_m + \frac{z^2}{1.2}\delta^2 B_m + \ldots + \frac{z^{m-1}\delta^{m-1}B_m}{(m-1)!} \quad \ldots\ldots\ldots(2),$$

or, symbolically, $$V \equiv (1+z\delta)^{m-1}B_m \quad \ldots\ldots\ldots(3).$$

To illustrate our theory, let us take $m = 3$, and consider the expression

$$z^2(H_1+H_2+H_3) - z\{H_1(z_2+z_3)+H_2(z_3+z_1)+H_3(z_1+z_2)\} + H_1z_2z_3+H_2z_3z_1+H_3z_1z_2,$$

in which H_1, H_2, H_3 are functions of the differences of the facients, and consequently vanish when operated on by δ.

Now this expression is evidently a function of the differences of the facients, whenever z is a root of

$$\phi(z) = 0,$$

but it is also clearly a function of the differences of the facients and a *new quantity* z, no matter what z may be.

And this is easily seen to be true in general, that, if any expression

$$V \equiv B_1z^{m-1} + B_2z^{m-2} + \ldots + B_m$$

is a function of the differences of the facients z_1, z_2, ... z_m and other sets of facients in each case, that z is a root of

$$\phi(z) = 0,$$

then the same expression is a function of the differences of the same facients and a new quantity z, no matter what z may be. For, if the above expression is a function of the differences of z and the various facients z_1, z_2, ... z_m; α_1, α_2, ... α_{2m}; ζ_1, ζ_2, ... ζ_m it must, when operated on by $-\frac{d}{dz}+\delta$, vanish identically, and we shall be led to the very same equations connecting the functions B_1, B_2, ... B_m and their derivation from B_m as a source, by equating to zero the coefficients of

$$z^{m-1},\ z^{m-2},\ \ldots\ z',\ z^0,$$

in the result which is to vanish identically.

In general, then, V can be written in the form

$$V \equiv (1+z\delta)^{m-1} B_m.$$

3. It will be found convenient in what follows to introduce new sets of facients in order to render homogeneous certain algebraic expressions we shall have to deal with.

We shall write

$$z_1 = \frac{x_1}{y_1}, \quad z_2 = \frac{x_2}{y_2}, \quad \ldots \quad z_m = \frac{x_m}{y_m};$$

$$a_1 = \frac{a_1}{b_1}, \quad a_2 = \frac{a_2}{b_2}, \quad \ldots \quad a_{2m} = \frac{a_{2m}}{b_{2m}};$$

$$\zeta_1 = \frac{\xi_1}{\eta_1}, \quad \zeta_2 = \frac{\xi_2}{\eta_2}, \quad \ldots \quad \zeta_m = \frac{\xi_m}{\eta_m};$$

the introduced facients or symbols being ultimately replaced by unity in each case. With this change in our notation, the operator we have called δ becomes altered as follows:

$$-\delta \equiv \Sigma y \frac{d}{dx} + \Sigma b \frac{d}{da} + \Sigma \eta \frac{d}{d\xi}.$$

It is clear we have also an operator which affects the introduced facients, which we shall denote by Δ, and write

$$-\Delta \equiv \Sigma x \frac{d}{dy} + \Sigma a \frac{d}{db} + \Sigma \xi \frac{d}{d\eta}.$$

Any function of the differences of the old facients will vanish when operated on by δ, and any invariant will vanish when operated on by δ or Δ.

4. Let us now write

$$f(z) \equiv z^{2m} + P_1 z^{2m-1} + P_2 z^{2m-2} + \ldots + P_{2m},$$
$$\equiv (z-\alpha_1)(z-\alpha_2) \ldots (z-\alpha_{2m}),$$

calling $\alpha_1, \alpha_2, \ldots \alpha_{2m}$, the $2m$ roots of

$$f(z) = 0.$$

If we now put $z = \frac{x}{y}$ and $a_r = \frac{a_r}{b_r}$,

r being an integer which may have any value from $2m$ to unity, and introduce these values into the above identity, we find

$$f(z)\, y^{2m} b_1 b_2 \ldots b_{2m} \equiv (xb_1 - ya_1)(xb_2 - ya_2) \ldots (xb_{2m} - ya_{2m}),$$
$$\equiv U, \text{ say,}$$

and we write U with binomial coefficients, so that

$$U \equiv A_0 x^{2m} + 2mA_1 x^{2m-1} y + \ldots A_{2m} y^{2m};$$

we have finally, then, $b_1 b_2 \ldots b_{2m}$ being equal to A_0,

$$f(z) \equiv \frac{U}{A_0 y^{2m}},$$

and also, by differentiation,

$$dz = \frac{y\,dx - x\,dy}{y^2}.$$

5. Let us now write

$$I_r(z) \equiv \int^z \frac{z^r\, dz}{\sqrt{A_0 f(z)}} \equiv \int^{x/y} \frac{x^r y^{m-r-2}\,(y\,dx - x\,dy)}{\sqrt{U}},$$

r being any positive or negative integer. We find easily, by operating with δ,

$$\delta I_r(z) = -\, rI_{r-1}(z),$$

and again, by operating with Δ,

$$\Delta I_r(z) = -\,(m-2-r)\, I_{r+1}(z).$$

We shall also write

$$L(z, n) \equiv \int^z \frac{dz}{(z-n)\sqrt{A_0 f(z)}},$$

and we readily find

$$\delta L(z, n) = \int^z \frac{dz}{(z-n)^2 \sqrt{A_0 f(z)}}.$$

6. We shall now investigate the conditions which must be fulfilled in order that a binary quantic of degree $2n$ may be a perfect square. These conditions are of the utmost importance to us, as they enable

us to write down the rational and integral algebraic integrals of the Abelian system of differential equations, which we shall treat of presently.

If a binary quantic contains a square factor, it is clear one condition among its coefficients is necessary, and if it contain two square factors, two conditions are necessary, and if the quantic of degree $2n$ has n square factors, or, what is the same thing, becomes a perfect square, n conditions must be fulfilled.

Let U be our quantic, which we shall suppose written with binomial coefficients, and which, by hypothesis, is the square of a quantic u, of the degree n, which we also write with binomial coefficients.

We write then

$$U \equiv A_0 z^{2n} + 2nA_1 z^{2n-1} + \ldots + A_{2n} \equiv u^2 \equiv (a_0 z^n + na_1 z^{n-1} + \ldots + a_n)^2.$$

Now we have, of course,

$$A_{2n} = a_n^2,$$

and it is easily seen that the same operator δ which converts A_r, r being an integer which may have any value from zero to $2n$, into A_{r-1}, so that

$$\delta A_r = rA_{r-1},$$

will also convert a_s into a_{s-1}, agreeably to the same law of derivation, so that

$$\delta a_s = sa_{s-1},$$

s being an integer having any value from zero to n.

Now the source of the Hessian of U is

$$A_{2n-2}A_{2n} - A_{2n-1}^2,$$

and that of the Hessian of u is

$$a_{n-2}a_n - a_{n-1}^2,$$

and, as we easily find that

$$A_{2n} = a_n^2,$$

$$A_{2n-1} = a_n a_{n-1},$$

$$(2n-1)A_{2n-2} = na_{n-1}^2 + (n-1)a_n a_{n-2},$$

it follows that

$$(2n-1)\,A_{2n-2}A_{2n}-A_{2n-1}^2=(n-1)\,a_n^2\left\{a_{n-2}a_n-a_{n-1}^2\right\},$$

$$(2n-1)\,H=(n-1)\,A_{2n}h,$$

where we have called the source of $H(U)$, H, and that of $h(u)$, h; $H(U)$ and $h(u)$ denoting the Hessians of the quantics U and u respectively.

We can now operate $4n-4$ times on the equation

$$(2n-1)\,H=(n-1)\,A_{2n}h,$$

thus obtaining $4n-4$ equations which, added to the one above written, give us in all $4n-3$ equations connecting the $2n-3$ unknown quantities h, δh, $\delta^2 h$, ... $\delta^{2n-4}h$.

The various results of elimination of these quantities may be most clearly exhibited in the following manner.

Form the matrix written below, which I call the "square-matrix" for the quantic U, consisting of $2n-2$ columns, and $4n-3$ rows, the law of its formation being obvious on inspection,

$$\left\|\begin{array}{cccccccc}
A_{2n} & 0 & 0 & 0 & \ldots\ldots & H & \\
\delta A_{2n} & A_{2n} & 0 & 0 & \ldots\ldots & \delta H & \\
\dfrac{\delta^2 A_{2n}}{1.2} & \delta A_{2n} & A_{2n} & 0 & \ldots\ldots & \dfrac{\delta^2 H}{1.2} & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots\ldots & \dfrac{\delta^{4n-4}H}{(4n-4)!} &
\end{array}\right\|;$$

then the conditions that U may be a perfect square will be obtained by equating to zero all the determinants which can be derived from this matrix, and consisting of $2n-2$ columns of $2n-2$ elements taken in the order they stand in any $2n-2$ of the $4n-3$ rows of this matrix.

This "square-matrix" is full of interest, and an investigation of it leads us to many curious theorems which, however, I cannot here examine, but will only mention one of its most striking properties, namely, that, if *any one* of the algebraic equations resulting from it be given, all the remaining equations can be found by repeated operation of δ on the given equation, and on that which results from it by substituting for the suffixes their complementary values.

7. We now enter on the discussion of the Abelian system of differential equations, which system consists of $m-1$ differential equations connecting m quantities $z_1, z_2, \ldots z_m$, and which we shall write as follows:

$$\left.\begin{aligned}
&\frac{dz_1}{\sqrt{A_0 f(z_1)}} + \frac{dz_2}{\sqrt{A_0 f(z_2)}} + \ldots + \frac{dz_m}{\sqrt{A_0 f(z_m)}} = 0 \\
&\frac{z_1 dz_1}{\sqrt{A_0 f(z_1)}} + \frac{z_2 dz_2}{\sqrt{A_0 f(z_2)}} + \ldots + \frac{z_m dz_m}{\sqrt{A_0 f(z_m)}} = 0 \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&\frac{z_1^{m-2} dz_1}{\sqrt{A_0 f(z_1)}} + \frac{z_2^{m-2} dz_2}{\sqrt{A_0 f(z_2)}} + \ldots + \frac{z_m^{m-2} dz_m}{\sqrt{A_0 f(z_m)}} = 0
\end{aligned}\right\} \quad \ldots\ldots\ldots(1).$$

Any one of these $m-1$ equations may be written in the more compact form

$$\Sigma \frac{z^i dz}{\sqrt{A_0 f(z)}} = 0,$$

Σ denoting summation with regard to the m quantities $z_1, z_2, \ldots z_m$, which in future we shall call the z's; and the complete Abelian system can thus be obtained by giving to i successively all the integer values it can assume from zero to $m-2$ in the equation which is typical of all, viz.,

$$\Sigma \frac{z^i dz}{\sqrt{A_0 f(z)}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If we now assume, with Jacobi,

$$\Sigma \frac{z^{m-1} dz}{\sqrt{A_0 f(z)}} = dt,$$

thus introducing a new variable t, it is clear that the z's are functions of t, and that consequently the quantities $p_1, p_2, \ldots p_m$, which, for brevity, we shall call the p's, are functions of t also. Now, from the system of equations (1) joined to the equation (3), there results, as is well known,

$$\frac{dz_1}{dt} = \frac{\sqrt{A_0 f(z_1)}}{(z_1 - z_2)(z_1 - z_3) \ldots (z_1 - z_m)} = \frac{\sqrt{A_0 f(z_1)}}{\phi'(z_1)},$$

and generally, z representing any root of

$$\phi(z) = 0,$$

we have

$$\frac{dz}{dt} = \frac{\sqrt{A_0 f(z)}}{\phi'(z)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

8. So far we have followed Jacobi in the introduction of a new variable t, leading to the well known equation written above, but we now proceed to obtain the algebraic integrals of the differential system in a totally different manner. If we differentiate the equation

$$\phi(z) = 0,$$

we obtain

$$\frac{d\phi(z)}{dz}\left(\frac{dz}{dt}\right) + \frac{d\phi(z)}{dt} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where

$$\frac{d\phi(z)}{dz} = \phi'(z),$$

and

$$\frac{d\phi(z)}{dt} = \dot{p}_1 z^{m-1} + \dot{p}_2 z^{m-2} + \ldots \dot{p}_m,$$

the dot indicating differentiation with regard to t. Introducing into equation (1) the value of $\frac{dz}{dt}$ given in the last article, we find

$$\frac{d\phi(z)}{dt} + \sqrt{A_0 f(z)} = 0,$$

from which we obtain

$$\left\{\frac{d\phi(z)}{dt}\right\}^2 = A_0 f(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

an equation which is true for all values of z which make $\phi(z)$ become zero.

9. Let us now take $m-1$ new quantities λ_m, λ_{m-1}, ... λ_2, and let us determine them so that the equation

$$\left\{\frac{d\phi(z)}{dt}\right\}^2 = A_0\left\{f(z) + [\phi(z)]^2 - 2\phi(z)\, L(z)\right\} \equiv F(z)$$

may be *identically satisfied* for all possible values of z, where

$$L(z) \equiv z^m + \frac{P_1}{z} z^{m-1} + \lambda_2 z^{m-2} + \lambda_3 z^{m-3} + \ldots + \lambda_m.$$

Now the left-hand side of the above identity is obviously of the degree $2m-2$ in z, and it will be found that the right-hand side is likewise of the degree $2m-2$, since the coefficients of both z^{2m} and z^{2m-1} vanish; we have consequently $2m-1$ equations connecting the p's, $\dot{p}$'s and λ's with the quantities P_1, P_2, ... P_{2m}. But it is obvious

that, if we eliminate, from these $2m-1$ equations, the λ's, we arrive at the typical equation of the last article

$$\left\{\frac{d\varphi(z)}{dt}\right\}^2 = A_0 f(z),$$

where z is now a root of $\qquad \phi(z) = 0,$

which determines the z's and consequently the p's in terms of t, their values being agreeable to this system of differential equations, contained in the equations

$$\Sigma \frac{z^i dz}{\sqrt{A_0 f(z)}} = 0, \quad \Sigma \frac{z^{m-1} dz}{\sqrt{A_0 f(z)}} = dt;$$

we have therefore $m-1$ equations to determine the $m-1$ λ's in terms of the p's and $\dot{p}$'s; and finally in terms of t.

The problem is then a possible one.

10. If we now consider the equation

$$\frac{d\phi(z)}{dt} + \sqrt{A_0 f(z)} = 0,$$

and operate on it with δ, we obtain, since $\delta A_0 f(z)$ is zero,

$$\delta \frac{d\phi(z)}{dt} = 0,$$

or $\quad z^{m-1}\delta\dot{p}_1 + z^{m-2}\{\delta\dot{p}_2 - (m-1)\dot{p}_1\} + z^{m-3}\{\delta\dot{p}_3 - (m-2)\dot{p}_2\} + \ldots$

$$\ldots + \{\delta\dot{p}_m - \dot{p}_{m-1}\} = 0,$$

whenever z is a root of $\qquad \phi(z) = 0;$

hence by Article 1, the coefficients of the several powers of z must vanish, and we find

$$\left.\begin{aligned} \delta\dot{p}_m &= \dot{p}_{m-1} \\ \delta\dot{p}_{m-1} &= 2\dot{p}_{m-2} \\ \delta\dot{p}_{m-2} &= 3\dot{p}_{m-3} \\ \ldots \quad &\ldots \quad \ldots \quad \ldots \\ \delta\dot{p}_2 &= (m-1)\dot{p}_1 \\ \delta\dot{p}_1 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, it follows from Article 2 that, since

$$z^{m-1}\dot{p}_1 + z^{m-2}\dot{p}_2 + \ldots + \dot{p}_m$$

is a function of the differences of the facients when z is a root of

$$\phi(z) = 0,$$

it must also be a function of the differences of the various facients and a new quantity z, no matter what z may be, and moreover derivable from a source $\dot{p}_m$. Hence it follows that, if we write

$$\left\{\frac{d\phi(z)}{dt}\right\}^2 \equiv F(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$F(z)$ must vanish when operated on by δ, and must be itself derivable from a source which is easily seen to be

$$A_0(p_m^2 - 2\lambda_m p_m + P_{2m}),$$

from which it follows that the λ's are obedient to the same law of derivation from a source λ_m. This being the case, we must have

$$\left.\begin{aligned}\delta\lambda_r &= (m+1-r)\lambda_{r-1}\\ \delta\lambda_2 &= (m-1)P_1\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

r being an integer which may have any value from 3 to m.

We now see that the $2m-1$ equations obtained by equating the coefficients of like powers of z in the identity

$$\left\{\frac{d\phi(z)}{dt}\right\}^2 \equiv F(z)$$

can all be obtained by successive applications of the operator δ *on the equation*

$$\dot{p}_m^2 = A_0(p_m^2 - 2\lambda_m p_m + P_{2m}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

This equation I call the *fundamental equation* in the theory of elliptic or hyper-elliptic functions, since from it flow all the Protean forms which the algebraic equivalents of the transcendental system are capable of assuming. We also find easily

$$\dot{p}_1^2 = A_0\{(p_1 - P_1)^2 + P_2 + P_1^2 - 2\lambda_2\},$$

an equation we shall make use of presently.

11. We shall now write the identity

$$\dot{\phi}(z)^2 \equiv F(z),$$

in the form $\dot{\phi}(z)^2 = A_0 V(z) + A_0\{\phi(z) - L(z)\}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$

where $V(z) \equiv f(z) - L(z)^2 \equiv V_0 z^{2m-2} + V_1 z^{2m-1} + \ldots + V_{2m-2}$

$$\equiv V_0 (z^{m-1} + R_1 z^{m-1} + \ldots + R_{m-1})(z^{m-1} + s_1 z^{m-1} + \ldots s_{m-1})$$

$$\equiv V_0 R(z) S(z),$$

and we shall denote the roots of

$$R(z) = 0 \quad \text{and} \quad S(z) = 0$$

by $r_1, r_2, \ldots r_{m-1}$; $s_1, s_2, \ldots s_{m-1}$, respectively.

If we now differentiate the identity

$$\dot{\phi}(z)^2 = A_0 \{ f(z) - 2L(z)\phi(z) + \phi(z)^2 \}$$

with regard to t, we readily obtain

$$A_0 L(\dot{z})\phi(z) \equiv \{ A_0 [\phi(z) - L(z)] - \ddot{\phi}(z) \} \dot{\phi}(z).$$

Now the left-hand side of the identity vanishes for

$$z = z_1, \quad z = z_2, \quad \ldots \quad z = z_m,$$

while $\dot{\phi}(z)$ does not vanish for such values of z. Consequently, if $L(\dot{z})$ be not zero, we should have

$$A_0 [\phi(z) - L(z)] - \ddot{\phi}(z),$$

a function of the $m-1^{\text{th}}$ degree vanishing for m values of z, which is impossible; hence it follows that $L(\dot{z})$ must vanish identically, that is to say, that $\lambda_2, \lambda_3, \ldots \lambda_{m-2}$ must be regarded as constants, and independent of t. We must then have

$$\ddot{\phi}(z) \equiv A_0 [\phi(z) - L(z)] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

from which we infer $\ddot{p}_r = A_0 (p_r - \lambda_r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$

r being an integer which may have any value from $r = 1$ to $r = m$.

12. Being given, then, the system of differential equations contained in the following typical equation

$$\Sigma \frac{z^i dz}{\sqrt{A_0 f(z)}} = 0,$$

holding for all integer values of i from $i = 0$ to $i = m-2$, we see that, if we determine m quantities $\lambda_2, \lambda_3, \ldots \lambda_m$ by the $m-1$ conditions that must be satisfied in order that $F'(z)$, as previously defined, may be a perfect square, these quantities will be constants, and that con-

sequently these $m-1$ conditions constitute $m-1$ rational and integral algebraic integrals, involving $m-1$ arbitrary constants, of the above system of differential equations. Hence we have the following canon for the determination of the *rational* and *integral* forms of the differential system

$$\Sigma \frac{z^i dz}{\sqrt{A_0 f(z)}} = 0.$$

First form the function $F(z)$, thus introducing $m-1$ arbitrary constants $\lambda_2, \lambda_3, \ldots \lambda_m$; then construct the square-matrix of $F(z)$, considered as a binary quantic of the $2m-2^{\text{th}}$ degree, this matrix consisting of $2m-4$ columns and $4m-7$ rows; then every determinant which can be derived from this matrix, and consisting of any $2m-4$ of the $4m-7$ rows of the matrix of $2m-4$ elements, will, when equated to zero, be an algebraic integral of the system of differential equations.

13. If we now write

$$\dot{\phi}(z)^2 - A_0 \{\phi(z) - L(z)\}^2 \equiv A_0 V_0 R(z) S(z),$$

we may write

$$\dot{\phi}(z) + \sqrt{A_0}\{\phi(z) - L(z)\} \equiv \theta \sqrt{A_0} R(z) \quad \ldots\ldots\ldots\ldots\ldots (1),$$

$$\phi(\dot{z}) - \sqrt{A_0}\{\phi(z) - L(z)\} \equiv \frac{V_0 \sqrt{A_0}}{\theta} S(z) \ldots\ldots\ldots\ldots\ldots (2),$$

θ being some function of t.

To determine θ we shall differentiate (1) with regard to t, when we shall find, since the λ's and consequently $r_1, r_2, \ldots r_{m-1}$ are constants,

$$\ddot{\phi}(z) + \sqrt{A_0}\dot{\phi}(z) = \dot{\theta}\sqrt{A_0} R(z).$$

But from (2) of the last article, we have

$$\ddot{\phi}(z) = A_0 \{\phi(z) - L(z)\},$$

and, introducing this value into the above equation, we obtain

$$\dot{\theta} = \sqrt{A_0}\,\theta,$$

or, by integration, $$\theta = \theta_0 e^{\sqrt{A_0}(t - t_0)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where θ_0 is the value of θ when $t = t_3$, we have also, by equating the coefficients of z^{m-1} in (1),

$$\theta \sqrt{A_0} = \dot{p}_1 + \sqrt{A_0}(p_1 - P_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

14. From (1) and (2) of Article 13, we readily deduce

$$2\sqrt{A_0}\{\phi(z)-L(z)\} = \theta\sqrt{A_0}\,R(z) - \frac{V_0\sqrt{A_0}\,S(z)}{\theta},$$

or
$$2\theta\phi(z) \equiv \theta^2 R(z) + 2\theta L(z) - V_0 S(z) \quad \ldots\ldots\ldots\ldots\ldots(1),$$

and, if we multiply each side of this identity by $R(z)$, we obtain, calling

$$R(z)\,\phi(z) \equiv \Psi(z),$$

$$2\theta\Psi(z) \equiv \{\theta R(z) + L(z)\}^2 - L(z)^2 - V_0 R(z) S(z)$$

$$\equiv \{\theta R(z) + L(z)\}^2 - f(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

We see then that the roots of the expression

$$\{\theta R(z) + L(z)\}^2 - f(z),$$

when equated to zero, are

$$z_1,\ z_2,\ \ldots\ z_m;\quad r_1,\ r_2,\ \ldots\ r_{m-1};$$

and, if we differentiate

$$\{\theta R(z) + L(z)\}^2 - f(z) = 0,$$

on the hypothesis that the z's and the r's as well as θ are variable, we obtain

$$2\theta\Psi'(z)\,dz + 2\{\theta R(z) + L(z)\}\,(\theta dR + R d\theta) = 0,$$

z being a root of
$$\phi(z)\,R(z) = 0.$$

Now, when
$$\Psi(z) = 0,$$

we must have
$$\theta R(z) + L(z) = h\sqrt{f(z)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where
$$h^2 = 1;$$

introducing this value into the above equation, we easily obtain

$$\frac{dz}{\sqrt{f(z)}} + h\left\{\frac{\theta\,dR(z) + R(z)\,d\theta}{\theta\,\Psi'(z)}\right\} = 0 \quad \ldots\ldots\ldots\ldots\ldots(4),$$

the symbol d affecting only the coefficients in $R(z)$.

Now, the above equation holds for each root of

$$\Psi(z) = 0;$$

consequently, if we sum for the z's and the r's, that is, for the roots of

$$\Psi(z) = 0,$$

we obtain, by the well-known theory of partial fractions, $\Psi'(z)$ being of the $2m-2^{\text{th}}$ degree in z, $R(z)$ of the $m-1^{\text{th}}$, and $dR(z)$ of the $m-2^{\text{th}}$,

$$\Sigma \frac{dz}{\sqrt{f(z)}} \pm \Sigma \frac{dr}{\sqrt{f(r)}} = 0,$$

and if we multiply (4) by z^r, where r has any integer value from $r=1$ to $r=m-1$, and sum, we obtain results which are contained in the typical equation

$$\Sigma \frac{z^i dz}{\sqrt{A_0 f(z)}} \pm \Sigma \frac{r^i dr}{\sqrt{A_0 f(r)}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where i has any integer value from $i=0$ to $i=m-2$; and the following

$$\Sigma \frac{z^{m-1} dz}{\sqrt{A_0 f(z)}} \pm \Sigma \frac{r^{m-1} dr}{\sqrt{A_0 f(r)}} + \frac{h\, d\theta}{\sqrt{A_0}\,\theta} = 0 \ldots\ldots\ldots\ldots\ldots(6),$$

where we must take $h = -1$ to give results in accordance with what precedes.

15. We shall now investigate the relations connecting the $2m-2$ roots of

$$V(z) = 0,$$

which are independent of any values the λ's may assume.

We have $$V(z) \equiv f(z) - L(z)^2;$$

and if we differentiate, regarding the roots of

$$V(z) = 0$$

and the λ's as variable, we find

$$V'(z)\, dz - 2\, L(z)\, d\, L(z) = 0;$$

and, replacing $L(z)$ by its value $\sqrt{f(z)}$, which it possesses when z is a root of

$$V(z) = 0,$$

we readily obtain $$\frac{dz}{\sqrt{A_0 f(z)}} - 2 \frac{d\, L(z)}{V'(z)\sqrt{A_0}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and, multiplying this typical equation by z^k, where k has any integer value from $k=0$ to $k=m-1$, and summing, we obtain

$$\Sigma \frac{r^k dr}{\sqrt{A_0 f(r)}} + \Sigma \frac{s^k ds}{\sqrt{A_0 f(s)}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

holding for all integer values of k from $k = 0$ to $k = m-2$; also

$$\pm \Sigma \frac{r^{m-1}\,dr}{\sqrt{A_0 f(r)}} \pm \delta \Sigma \frac{s^{m-1}\,ds}{\sqrt{A_0 f(s)}} = \frac{2d\lambda_2}{V_0\sqrt{A_0}} \quad \ldots\ldots\ldots\ldots(3).$$

16. If we were to connect the z's with the s's by a process of reasoning precisely similar to that employed in Article 14, we must have

$$\Sigma \frac{z^i\,dz}{\sqrt{A_0 f(z)}} \pm \Sigma \frac{s^i\,ds}{\sqrt{A_0 f(s)}} = 0 \quad \ldots\ldots\ldots\ldots(1),$$

holding for all integer values of i from $i = 0$ to $i = m-2$, also

$$\Sigma \frac{z^{m-1}dz}{\sqrt{A_0 f(z)}} \pm \Sigma \frac{s^{m-1}ds}{\sqrt{A_0 f(s)}} = \frac{2d\lambda_2}{V_0\sqrt{A_0}} + \frac{d\theta}{\theta\sqrt{A_0}} \quad \ldots\ldots\ldots(2),$$

which gives results which are agreeable to the results obtained in the last two articles.

17. If we now write

$$\left.\begin{aligned} \pi_m &= p_m - \lambda_m \\ \pi_{m-1} &= p_{m-1} - \lambda_{m-1} \\ \cdots \quad &\cdots \quad \cdots \quad \cdots \\ \pi_1 &= p_1 - P_{\frac{1}{2}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1),$$

it is clear that $\dot{\pi}_i = \dot{p}_i$, i being any integer from $i = 1$ to $i = m$; the fundamental equation becomes

$$\dot{\pi}_m^2 = A_0(\pi_m^2 + V_{2m-2}),$$

and from this equation, as before indicated, we can obtain, by successive operations of δ, the systems

$$\left.\begin{aligned} \dot{\pi}_m^2 &= A_0(\pi m^2 + V_{2m-2}), \\ 2\dot{\pi}_m\dot{\pi}_{m-1} &= A_0(2\pi_m\pi_{m-1} + V_{2m-3}), \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad &\cdots \quad \cdots \quad \cdots \\ \dot{\pi}_1^2 &= A_0(\pi_1^2 + V_0). \end{aligned}\right\}$$

18. I now discuss the case in which $m = 2$, or the case of elliptic integrals. Here we have

$$f(z) \equiv z^4 + P_1 z^3 + P_2 z^2 + P_3 z + P_4,$$

and there are two variables z_1 and z_2 connected by the equation

$$\frac{dz_1}{\sqrt{A_0 f(z_1)}} + \frac{dz_2}{\sqrt{A_0 f(z_2)}} = 0.$$

The source of $F(z)$ being $A_0(\pi_2^2 + V_2)$, the function itself is evidently

$$F(z) \equiv A_0\{(\pi_2^2 + V_2) + z(2\pi_1\pi_2 + V_1) + (\pi_1^2 + V_0)\},$$

and, since $F(z)$ must be a square, the rational and integral algebraic integral is at once seen to be

$$(2\pi_1\pi_2 + V_1)^2 = 4(\pi_2^2 + V_2)(\pi_1^2 + V_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

or

$$V_0\pi_2^2 - V_1\pi_1\pi_2 + V_2\pi_1^2 = \frac{V_1^2 - 4V_0V_2}{4} = \Lambda_2, \text{ say} \quad \ldots\ldots\ldots (2),$$

where

$$V_2 = P_4 - \lambda_2^2,$$

$$V_1 = P_3 - \lambda_2 P_1,$$

$$V_0 = P_2 - P_1^2 - 2\lambda_2,$$

it being remembered that $\delta\lambda_2$, by Article 10, is $\frac{m-1}{2}P_1$; or in this case $\frac{1}{2}P_1$.

This result will be found identical with that given by Cayley in his *Elementary Treatise on Elliptic Functions*, provided we write

$$2\lambda_2 = P_2 - c.$$

The reader will perceive that the above integral (2) is obtained directly by the application of our theory, and not derived from an irrational form of the integral which, when rationalized, contains the irrelevant factor $(z_1 - z_2)^2$, of which it must be cleared before the result given above can be obtained.

19. We have also, by Article 14,

$$\frac{dz_1}{\sqrt{A_0 f(z_1)}} + \frac{dz_2}{\sqrt{A_0 f(z_2)}} - \frac{dr}{\sqrt{A_0 f(r)}} = 0 \quad \ldots\ldots\ldots\ldots (1),$$

where we consider λ_2 variable, r being a root of the equation

$$V_0 r^2 + V_1 r + V_2 = 0,$$

from which we find

$$(V_0 r + V_1)^2 = \frac{V_1^2 - 4V_0V_2}{4} = \Lambda_2,$$

or, taking the square root,

$$V_0 r + V_1 = \sqrt{\Lambda_2}.$$

We have also $$V(z) \equiv f(z) - [L(z)]^2 \quad \ldots\ldots(2);$$

hence, differentiating this identity with regard to λ_2, we find

$$\frac{dV(z)}{d\lambda_2} \equiv -2L(z)\frac{dL(z)}{d\lambda_2} = -2L(z),$$

since $$\frac{dL(z)}{d\lambda_2} = 1.$$

If we now differentiate the equation

$$V_0 r^2 + V_1 r + V_2 = 0,$$

we obtain $$2(V_0 r + V_1)\,dr + \frac{dV(r)}{d\lambda_2}\,d\lambda_2 = 0,$$

and consequently, by what we have proved above, we find

$$2\sqrt{\Lambda_2}\,dr - 2L(r)\,d\lambda_2 = 0;$$

but, since $$L(r) = \sqrt{f(r)},$$

we have $$\frac{dr}{\sqrt{f(r)}} = \frac{d\lambda_2}{\sqrt{\Lambda_2}} \quad \ldots\ldots(3),$$

and, finally, $$\frac{dz_1}{\sqrt{A_0 f(z_1)}} + \frac{dz_2}{\sqrt{A_0 f(z_2)}} = -\frac{d\lambda_2}{\sqrt{A_0 \Lambda_2}} \quad \ldots\ldots(4),$$

20. If we now introduce binomial coefficients into the equations discussed in the last two articles, and write

$$A_0\lambda_2 = A_2 + 2\lambda,$$

we find

$$\left.\begin{aligned} A_0^2 V_2 &= \{A_0 A_4 - A_2^2 - 4\lambda^3 - 4\lambda A_2\} \\ A_0^2 V_1 &= 4\{A_0 A_3 - A_1 A_2 - 2\lambda A_1\} \\ A_0^2 V_0 &= 4\{A_0 A_2 - A_1^2 - \lambda A_0\} \\ \delta\lambda &= 0 \end{aligned}\right\} \quad \ldots\ldots(1).$$

Again, if we write

$$p_2 = \frac{a_2}{a_0}, \quad p_1 = \frac{2a_1}{a_0},$$

we find

$$\left.\begin{aligned} A_0 a_0 \pi_2 &= A_0 a_2 - A_2 a_0 - 2\lambda a_0 \\ A_0 a_0 \pi_1 &= 2(A_0 a_1 - A_1 a_0) \end{aligned}\right\} \quad \ldots\ldots(2).$$

and also we have

$$A_0^4\Lambda_2 = 4A_0\left\{-4\lambda^3+I\lambda-J\right\} \equiv 4A_0\Lambda, \text{ say,}$$

whence

$$\Lambda_2 = \frac{4\Lambda}{A_0^3},$$

I and J being the invariants of the quartic.

The algebraic equation of Article 18 then becomes, when arranged in powers of λ,

$$\begin{aligned}4\lambda^2(a_1^2-a_0a_2)+\lambda\left\{A_0a_2^2-4A_1a_1a_2+2A_2(2a_1^2+a_0a_2)-4A_3a_0a_1+a_0^2A_4\right\}\\ +a_2^2(A_1^2-A_0A_2)+a_1^2(A_2^2-A_0A_4)+a_0^2(A_3^2-A_2A_4)\\ +2a_0a_2(A_2^2-A_1A_3)+2a_0a_1(A_1A_4-A_2A_3)=0 \quad \text{.....(3),}\end{aligned}$$

an equation which vanishes identically when operated on by δ. At the same time the transcendental equation assumes the form

$$\frac{y_1dx_1-x_1dy_1}{\sqrt{U(x_1,\,y_1)}}+\frac{y_2dx_2-x_2dy_2}{\sqrt{U(x_2,\,y_2)}}=\frac{-d\lambda}{\sqrt{-4\lambda^3+I\lambda-J}} \quad \text{.....(4),}$$

where we have put

$$z_1=\frac{x_1}{y_1}, \quad z_2=\frac{x_2}{y_2}.$$

21. Again, from Article 14, we have

$$\frac{z_1dz_1}{\sqrt{A_0f(z_1)}}+\frac{z_2dz_2}{\sqrt{A_0f(z_2)}}\pm\frac{r\,dr}{\sqrt{A_0f(r)}}-\frac{d\theta}{\sqrt{A_0}\theta}=0;$$

now

$$\frac{dr}{\sqrt{A_0f(r)}}=\frac{d\lambda}{\sqrt{\Lambda}}$$

by what precedes, and

$$V_0r+V_1=\sqrt{\Lambda_2}=\frac{2\sqrt{\Lambda}}{A_0\sqrt{A_0}};$$

consequently

$$\frac{r\,dr}{\sqrt{A_0f(r)}}=\left\{\frac{V_0r+V_1-V_1}{V_0}\right\}\frac{d\lambda}{\sqrt{\Lambda}} \quad \text{.....(1),}$$

which, after a few reductions, we find to equal

$$-\frac{1}{\sqrt{A_0}}d\log(H-A_0\lambda)-\frac{V_1}{2V_0}\frac{d\lambda}{\sqrt{\Lambda}},$$

where

$$H=A_0A_2-A_1^2.$$

We have also

$$\sqrt{A_0}\theta=p_1+\sqrt{A_0}\pi_1=\sqrt{A_0}(\pi_1+\sqrt{\pi_1^2+V_0}),$$

so that we easily arrive at the formula

$$\frac{x_1}{y_1}\left\{\frac{y_1dx_1-x_1dy_1}{\sqrt{U(x_1,y_1)}}\right\}+\frac{x_2}{y_2}\left\{\frac{y_2dx_2-x_2dy_2}{\sqrt{U(x_2,y_2)}}\right\}$$

$$\pm\frac{1}{\sqrt{A_0}}d\log\left\{\frac{(A_0a_1-A_1a_0)+\sqrt{(A_0a_1-A_1a_0)^2+a_0^2(H-A_0\lambda)}}{a_0\sqrt{H-A_0\lambda}}\right\}$$

$$-\tfrac{1}{2}\left\{\frac{A_0A_3-A_1A_2-2\lambda A_1}{H-A_0\lambda}\right\}\frac{d\lambda}{\sqrt{\Lambda}}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

22. From the form of the equation (4) in Article 20, and from other considerations, it is easy to see that λ is an absolute invariant of the facients; and that consequently $\Delta\lambda$ vanishes identically.

If we now operate on equation (2) of the last article, bearing in mind that

$$\Delta I_r(z)=rI_{r+1}(z),$$

we obtain a formula which gives us the value of

$$\frac{x_1}{y_1^2}\left\{\frac{y_1dx_1-x_1dy_1}{\sqrt{U(x_1,y_1)}}\right\}+\frac{x_2^2}{y_2^2}\left\{\frac{y_2dx_2-x_2dy_2}{\sqrt{U(x_2,y_2)}}\right\},$$

in terms of a_0, a_1, a_2, A_0, A_1, A_2, A_3, A_4 and λ, and from which, by a second application of Δ, we can obtain a formula for $\Sigma dI_3(z)$.

Again, in equation (2) above referred to, if we interchange x and y and (replace the suffixes of the A's and a's by their complementary values) λ remains unaltered, and we find

$$\frac{y_1}{x_1}\left\{\frac{y_1dx_1-x_1dy_1}{\sqrt{U(x_1,y_1)}}\right\}+\frac{y_2}{x_2}\left\{\frac{y_2dx_2-x_2dy_2}{\sqrt{U(x_2,y_2)}}\right\}$$

$$=\frac{1}{\sqrt{A_4}}d\log\left\{\frac{(A_4a_1-A_3a_2)+\sqrt{(A_4a_1-A_3a_2)^2+a_2^2(A_4A_2-A_3^2-A_4\lambda)}}{a_2\sqrt{A_4A_2-A_3^2-A_4\lambda}}\right\}$$

$$-\tfrac{1}{2}\left\{\frac{A_4A_1-A_3A_2-2\lambda A_3}{A_4A_2-A_3^2-A_4\lambda}\right\}\frac{d\lambda}{\sqrt{\Lambda}}.$$

If we wish to obtain $\Sigma dI_{-2}(z)$, we can operate on the above formula with δ, and so, having obtained $\Sigma dI_{-3}(z)$, we can, by repeated applications of the operator δ, obtain $\Sigma dI_{-r}(z)$, r being any positive integer.

23. We shall now discuss the case in which $m = 3$. Here we have three quantities z_1, z_2, z_3, and two equations, which we may write

$$\left.\begin{aligned} \Sigma dI_0(z) &= 0 \\ \Sigma dI_1(z) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and two arbitrary constants λ_3 and λ_2 connected as follows:

$$\delta\lambda_3 = \lambda_2, \quad \delta\lambda_2 = P_1.$$

The source of the function $F(z)$ is

$$A_0(\pi_3^2 + V_4),$$

and the coefficients of $F(z)$, supposing it written in the form

$$F(z) \equiv F_4 + zF_3 + z^2F_2 + \ldots z^4F_0,$$

are found, by application of δ, to be

$$\left.\begin{aligned} F_4 &= A_0(\pi_3^2 + V_4) \\ F_3 &= A_0(2\pi_3\pi_2 + V_3) \\ F_2 &= A_0(\pi_2^2 + 2\pi_3\pi_1 + V_2) \\ F_1 &= A_0(2\pi_2\pi_1 + V_1) \\ F_0 &= A_0(\pi_1^2 + V_0) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

it being remembered that

$$\delta\pi_3 = \pi_2, \quad \delta\pi_2 = 2\pi_1, \quad \delta\pi_1 = 0.$$

The square-matrix is then at once seen to be

$$\left\|\begin{matrix} F_4 & 8F_4F_2 - 3F_3^2 \\ F_3 & 4(6F_4F_1 - F_3F_2) \\ F_2 & 2(3F_3F_1 + 24F_4F_0 - 2F_2^2) \\ F_1 & 4(6F_3F_0 - F_2F_1) \\ F_0 & 8F_2F_0 - 3F_1^2 \end{matrix}\right\| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

giving us the well known conditions that a binary quartic should be a square.

From these we deduce

$$\frac{F_4}{F_0} = \frac{F_3^2}{F_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and, if we now operate on this equation with δ, we obtain

$$F_3F_1^2+8F_4F_1F_0-4F_0F_3F_2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

We have consequently the two following rational and integral algebraic equations as equivalents of the transcendental system (1) of this article:

$$\left.\begin{aligned} \frac{\pi_3^2+V_4}{\pi_1^2+V_0} &= \left\{\frac{2\pi_3\pi_2+V_3}{2\pi_2\pi_1+V_1}\right\}^2, \\ (2\pi_3\pi_2+V_3)(2\pi_2\pi_1+V_1)^2+8\,(\pi_3^2+V_4)(\pi_1^2+V_0)(2\pi_2\pi_1+V_1) & \\ -4\,(\pi_1^2+V_0)(2\pi_3\pi_2+V_3)(\pi_2^2+2\pi_3\pi_1+V_2) &= 0 \end{aligned}\right\} \ldots (6).$$

I merely give these two equations as an example of the method, an immense number being easily obtained by combining the various results flowing from the square-matrix.

We have also the transcendental forms

$$\left.\begin{aligned} \Sigma dI_0(z)+\frac{dr_1}{\sqrt{A_0f(r_1)}}+\frac{dr_2}{\sqrt{A_0f(r_2)}} &= 0 \\ \Sigma dI_1(z)+\frac{r_1dr_1}{\sqrt{A_0f(r_1)}}+\frac{r_2dr_2}{\sqrt{A_0f(r_2)}} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (7),$$

r_1 and r_2 being a pair of roots of the equation

$$V(z)=0.$$

In this case I have not succeeded in expressing in terms of λ_3 and λ_2 the sums

$$\frac{dr_1}{\sqrt{A_0f(r_1)}}+\frac{dr_2}{\sqrt{A_0f(r_2)}},$$

$$\frac{r_1dr_1}{\sqrt{A_0f(r_1)}}+\frac{r_2dr_2}{\sqrt{A_0f(r_2)}}.$$

By a process of reasoning similar to that employed in Articles 21 and 22, we can obtain $\Sigma dI_r(z)$ in terms of r_1 and r_2, the p's, and λ_3 and λ_2, r being any positive or negative integer.

24. For the case in which $m=4$, we have four quantities z_1, z_2, z_3, z_4, and three equations, viz.,

$$\Sigma dI_0(z)=0,$$

$$\Sigma dI_1(z)=0,$$

$$\Sigma dI_3(z)=0,$$

and three arbitrary constants λ_4, λ_3, λ_2 connected as follows :

$$\delta\lambda_4 = \lambda_3, \quad \delta\lambda_3 = 2\lambda_2, \quad \delta\lambda_2 = \frac{3P_1}{2}.$$

Now the source of $F(z)$ is

$$A_0(\pi_4^2 + V_6),$$

and the coefficients of $F(z)$ found by repeated applications of the operator δ are found as follows :—

$$F_6 = A_0(\pi_1^2 + V_6),$$

$$F_5 = A_0(2\pi_4\pi_3 + V_5),$$

$$F_4 = A_0(2\pi_4\pi_2 + \pi_3^2 + V_4),$$

$$F_3 = A_0(2\pi_2\pi_3 + 2\pi_4\pi_1 + V_3),$$

$$F_2 = A_0(2\pi_3\pi_1 + \pi_2^2 + V_2),$$

$$F_1 = A_0(2\pi_2\pi_1 + V_1),$$

$$F_0 = A_0(\pi_1^2 + V_0).$$

If we now form, by the general rule, the square-matrix for $F(z)$, we find it to be

$$\begin{Vmatrix} F_6 & 0 & 0 & H_8 \\ F_5 & F_6 & 0 & H_7 \\ F_4 & F_5 & F_6 & H_6 \\ F_3 & F_4 & F_5 & H_5 \\ F_2 & F_3 & F_4 & H_4 \\ F_1 & F_2 & F_3 & H_3 \\ F_0 & F_1 & F_2 & H_2 \\ 0 & F_0 & F_1 & H_1 \\ 0 & 0 & F_1 & H_0 \end{Vmatrix},$$

where $H_0, H_1, \ldots H_8$ are the coefficients of the Hessian of $F(z)$ written without binomial coefficients.

We thus find for the equation the weight of which is the greatest

the following form :

$$\begin{vmatrix} F_6 & 0 & 0 & 9F_6F_4-5F_5^2 \\ F_5 & F_6 & 0 & 27F_6F_3-11F_5F_4 \\ F_4 & F_5 & F_6 & 54F_6F_2-11F_4^2-3F_5F_3 \\ F_3 & F_4 & F_5 & 90F_6F_1+12F_5F_2-24F_4F_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots(1),$$

and from this equation by successive operations of δ all the rest can be obtained by the rules already indicated in Article 6.

A simpler form can, however, easily be found by forming a determinant from the first two and last two rows of the square-matrix, giving us

$$F_6^2(F_1H_0-F_0H_1) = F_0^2(F_5H_8-F_6H_7),$$

or

$$\frac{F_6^2}{F_0^2} = \frac{20F_6F_5F_4-5F_5^3-27F_6^2F_3}{20F_2F_1F_0-5F_1^3-27F_0^2F_3} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

We have also

$$\pm\Sigma dI_0(z)+\frac{dr_1}{\sqrt{A_0f(r_1)}}+\frac{dr_2}{\sqrt{A_0f(r_2)}}+\frac{dr_3}{\sqrt{A_0f(r_3)}}=0,$$

$$\pm\Sigma dI_1(z)+\frac{r_1dr_1}{\sqrt{A_0f(r_1)}}+\frac{r_2dr_2}{\sqrt{A_0f(r_2)}}+\frac{r_3dr_3}{\sqrt{A_0f(r_3)}}=0,$$

$$\pm\Sigma dI_2(z)+\frac{r_1^2dr_1}{\sqrt{A_0f(r_1)}}+\frac{r_2^2dr_2}{\sqrt{A_0f(r_2)}}+\frac{r_3^2dr_3}{\sqrt{A_0f(r_3)}}=0,$$

r_1, r_2, r_3 being three roots of $V(z)=0$.

25. We now proceed to investigate some forms and properties of the algebraic integrals when the arbitrary constants are simultaneous values of the z's, when

$$t-t^0=0.$$

Let $\zeta_1, \zeta_2, \ldots \zeta_m$ be the corresponding values of $z_1, z_2, \ldots z_m$, when $t-t_0$ is zero ; we shall have then

$$\Sigma I_i(z)-\Sigma I_i(\zeta)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

for all integer values of i from $i=0$ to $i=m-2$, and in addition we have the equation

$$\Sigma I_{m-1}(z)-\Sigma I_{m-1}(\zeta)=t-t^0 \quad \ldots\ldots\ldots\ldots\ldots(2).$$

It easily follows from these equations that $t-t^0$ is a function of the differences of the facients involved in these formulæ, namely the z's, the ζ's, and the α's, for we find

$$\delta(t-t_0) = 0.$$

We shall now integrate the fundamental equation, writing it in the form

$$\dot{\pi}_m^2 = A_0(\pi_m^2 + V_{2m-2});$$

we have then

$$\frac{d\pi_m}{\sqrt{\pi_m^2 + V_{2m-2}}} = \sqrt{A_0}\,dt,$$

and by integration find

$$\sqrt{A_0}(t-t^0) = \log\left\{\frac{\pi_m + \sqrt{\pi_m^2 + V_{2m-2}}}{c}\right\} \quad \ldots\ldots\ldots\ldots\ldots(3),$$

c being a constant.

If we now suppose $\zeta_1, \zeta_2, \ldots \zeta_m$ to be the roots of the equation

$$\zeta^m + q_1\zeta^{m-1} + q_2\zeta_{m-2} + \ldots q_m = 0,$$

it is clear that, when $t-t^0 = 0$, $p_i = q_i$,

and also $\dot{p}_i = \dot{q}_i$,

the q's being differentiated with regard to t^0.

Again, if we write $\kappa_i = q_i - \lambda_i$,

we must have $\pi_i = \kappa_i$,

when $t-t^0 = 0$,

as well as $\dot{\pi}_i = \dot{\kappa}_i$,

from which it follows that the constant

$$c = \kappa_m + \sqrt{\kappa_m^2 + V_{2m-2}};$$

introducing this value of c into equation (3), we obtain

$$\sqrt{A_0}(t-t_0) = \log\left\{\frac{\pi_m + \sqrt{\pi_m^2 + V_{2m-2}}}{\kappa_m + \sqrt{\kappa_m^2 + V_{2m-2}}}\right\} \quad \ldots\ldots\ldots\ldots\ldots(4),$$

a formula we may also write in the more convenient form

$$\sqrt{A_0}(t-t_0) = \log\left\{\frac{\sqrt{A_0}\,\pi_m + \dot{\pi}_m}{\sqrt{A_0}\,\kappa_m + \dot{\kappa}_m}\right\} \quad \ldots\ldots\ldots\ldots\ldots(5).$$

26. From the value of $t-t^0$ given in the last article, we find

$$e^{\sqrt{A_0}(t-t^0)} = \frac{\sqrt{A_0}\,\pi_m + \dot{\pi}_m}{\sqrt{A_0}\,\kappa_m + \dot{\kappa}_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

from which equation we obtain

$$\frac{e^{\sqrt{A_0}(t-t^0)}-1}{e^{\sqrt{A_0}(t-t^0)}+1} = T, \text{ say, } = \frac{\sqrt{A_0}\,(\pi_m-\kappa_m)+\dot{\pi}_m-\dot{\kappa}_m}{\sqrt{A_0}\,(\pi_m+\kappa_m)+\dot{\pi}_m+\dot{\kappa}_m}.$$

Now, since the κ's are simultaneous values of the π's, when $t-t^0$ is zero, we have

$$\dot{\kappa}_m^2 = A_0\,(\kappa_m^2 + V_{2m-2}),$$

and consequently $\quad \dot{\pi}_m^2 - \dot{\kappa}_m^2 = A_0\,(\pi_m^2 - \kappa_m^2),$

or $$\frac{\dot{\pi}_m - \dot{\kappa}_m}{\sqrt{A_0}\,(\pi_m - \kappa_m)} = \frac{\sqrt{A_0}\,(\pi_m+\kappa_m)}{\dot{\pi}_m + \dot{\kappa}_m},$$

from which it easily follows that

$$T = \frac{\sqrt{A_0}\,(\pi_m-\kappa_m)}{\dot{\pi}_m+\dot{\kappa}_m} = \frac{\dot{\pi}_m - \dot{\kappa}_m}{\sqrt{A_0}\,(\pi_m+\kappa_m)}.$$

This function of $t-t^0$ which we have called T plays an important part in the theory of Abelian functions and integrals, and obviously vanishes when

$$t-t^0 = 0,$$

and when $$t-t^0 = \frac{2i\pi}{\sqrt{A_0}},$$

becoming infinite for $$t-t^0 = \frac{i\pi}{\sqrt{A_0}},$$

but for no real values of $t-t^0$.

27. By Article 7, we have

$$\frac{dz}{dt} = \frac{\sqrt{A_0 f(z)}}{\phi'(z)};$$

consequently $\dot{p}_i$ and $\dot{q}_i$ are expressible in terms of the z's and the ζ's respectively, and therefore we are enabled to express T in terms of the z's and ζ's, by means of the formula

$$T = \sqrt{A_0}\left\{\frac{p_m - q_m}{\dot{p}_m + \dot{q}_m}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which the reader will observe does not contain the λ's. By operating on this formula with δ, bearing in mind that

$$\delta T = 0,$$

since T is a function of the differences of the facients $z_1, z_2, \ldots z_m$; $\zeta_1, \zeta_2, \ldots \zeta_m$; $\alpha_1, \alpha_2, \ldots \alpha_{2m}$, we arrive at the important system of equations

$$T = \frac{\surd A_0 (p_m - q_m)}{\dot{p}_m + \dot{q}_m} = \frac{\surd A_0 (p_{m-1} - q_{m-1})}{\dot{p}_{m-1} + \dot{q}_{m-1}} = \ldots = \frac{\surd A_0 (p_1 - q_1)}{\dot{p}_1 + \dot{q}_1} \ldots (2),$$

which affords us a ready means of obtaining a number of algebraic integrals involving simultaneous values of the variables $z_1, z_2, \ldots z_m$, which are algebraic equivalents of the transcendental system

$$\Sigma I_i (z) - \Sigma I_i (\zeta) = 0.$$

We have also the system

$$T = \frac{\dot{p}_m - \dot{q}_m}{\surd A_0 (p_m + q^m - 2\lambda_m)} = \frac{\dot{p}_{m-1} - \dot{q}_{m-1}}{\surd A_0 (p_{m-1} + q_{m-1} - 2\lambda_{m-1})} = \ldots$$

$$= \frac{\dot{p}_1 - \dot{q}_1}{\surd A_0 (p_1 + q_1 - P_1)} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

This latter system enables us to express the λ's linearly in terms of the z's and ζ's.

28. We shall now derive a new and important system of equations from the value of T given in the last article, viz.,

$$T = \frac{\surd A_0 (p_m - q_m)}{\dot{p}_m + \dot{q}_m}.$$

If we diminish all the facients in the above expression by ζ_1, say, ζ_1 being the value of z_1 when

$$t - t^0 = 0,$$

and call $$(z_1 - \zeta_1)(z_2 - \zeta_1) \ldots (z_m - \zeta_1) = u (\zeta_1),$$

or, simply u if we have no occasion to put in evidence the element ζ_1, it is clear that

$$\left.\begin{array}{lll} p_m & \text{becomes} & (-1)^m u (\zeta_1) \\ \dot{p}_m & ,, & (-1)^m \dfrac{du (\zeta_1)}{dt} \\ q_m & ,, & \text{zero} \\ \dot{q}_m & ,, & -\surd A_0 f (\zeta_1) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

while T remains unaltered; we then find, introducing these values into the above formula, having first multiplied by $\dot{p}_m + \dot{q}_m$,

$$T\left\{\frac{du(\zeta_1)}{dt} + (-1)^{m-1}\sqrt{A_0 f(\zeta_1)}\right\} = \sqrt{A_0}\, u(\zeta_1) \quad \ldots\ldots\ldots(2),$$

it being understood, of course, that differentiation with respect to t does not affect ζ_1.

By substituting in succession for ζ_1 the remaining $m-1$ values of ζ, we obtain $m-1$ other equations, giving us, in all, m equations of which the following is typical:

$$T\left\{d\frac{u(\zeta)}{dt} + (-1)^{m-1}\sqrt{A_0 f(\zeta)}\right\} = \sqrt{A_0}\, u(\zeta) \ldots\ldots\ldots\ldots(3).$$

Now, if we call

$$(\zeta_1 - z)(\zeta_2 - z) \ldots (\zeta_m - z) \equiv v(z),$$

z standing for any root of $\quad \phi(z) = 0,$

and interchange corresponding z's and ζ's, it is clear that $t - t^0$ changes sign, and consequently T, while $u(\zeta)$ becomes $v(z)$, and therefore from the above equation we are led to the following:

$$T\left\{\frac{dv(z)}{dt_0} + (-1)^{m-1}\sqrt{A_0 f(z)}\right\} = -\sqrt{A_0}\, v(z) \quad \ldots\ldots\ldots(4),$$

from which it appears that, if one of the z's becomes equal to one of the ζ's, we must have $T = 0$, and consequently $p_i = q_i$.

29. These equations are of considerable value, as we shall show by what follows; in the first place, they enable us to obtain readily an algebraic equation which contains only *two* of the quantities $z_1, z_2, \ldots z_m$. This equation we obtain by eliminating T between two equations of the type (4) in the last article, and find

$$\left\{\frac{dv(z_1)}{dt_0} + (-1)^{m-1}\sqrt{A_0 f(z_1)}\right\} v(z_2)$$

$$= \left\{\frac{dv(z_2)}{dt_0} + (-1)^{m-1}\sqrt{A_0 f(z_2)}\right\} v(z_1),$$

into which equation only enter z_1, z_2, and the ζ's.

30. By comparison of the coefficients of z^{m-1} in (1) and (2) of Article 13, we find

$$\dot{p}_m + \sqrt{A_0}\,(p_m - \lambda_m) = \theta \sqrt{A_0}\, R_{m-} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\dot{p}_m - \sqrt{A_0}\,(p_m - \lambda_m) = \frac{V_0 \sqrt{A_0}\, S_{m-1}}{\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

from which we obtain

$$2\,(p_m - \lambda_m) = \theta R_{m-1} - \frac{V_0 S_{m-1}}{\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$2\,\frac{\dot{p}_m}{\sqrt{A_0}} = \theta\, R_{m-1} + \frac{V_0 S_{m-1}}{\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

We have also, from (3) of Article 11,

$$\frac{\ddot{p}_m}{A_0} = p_m - \lambda_m \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

If we now put $t = t^0$, θ becomes θ^0, and p_m, $\dot{p}_m$, and $\ddot{p}_m$ go into q_m, $\dot{q}_m$, and $\ddot{q}_m$, respectively, and we have

$$\left.\begin{aligned} 2\,(q_m - \lambda_m) &= \theta^0 R_{m-1} - \frac{V_0 S_{m-1}}{\theta^0} \\ 2\,\frac{\dot{q}_m}{\sqrt{A_0}} &= \theta^0 R_{m-1} + \frac{V_0 S_{m-1}}{\theta^0} \\ 2\,\frac{\ddot{q}_m}{A_0} &= \theta^0 R_{m-1} - \frac{V_0 S_{m-1}}{\theta^0} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

from which we find

$$\left.\begin{aligned} R_{m-1} &= \left(\frac{\dot{q}_m}{\sqrt{A_0}} + \frac{\ddot{q}_m}{A_0}\right)\frac{1}{\theta^0} \\ V_0 S_{m-1} &= \left(\frac{\dot{q}_m}{\sqrt{A_0}} - \frac{\ddot{q}_m}{A_0}\right)\theta^0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

and, introducing these values into the equation

$$2\,(p_m - q_m) = (\theta - \theta^0)\, R_{m-1} + \frac{(\theta - \theta^0)}{\theta\theta^0}\, V^0 S_{m-1},$$

we find easily

$$2\,(p_m - q_m) = \left\{\frac{\theta^2 - \theta^{02}}{\theta\theta^0}\right\}\frac{\dot{q}_m}{\sqrt{A_0}} + \frac{(\theta - \theta^0)^2}{\theta\theta^0}\,\frac{\ddot{q}_m}{A_0} \quad \ldots\ldots\ldots\ldots (8),$$

or, since $$T = \frac{\theta - \theta^0}{\theta + \theta^0},$$

$$(p_m - q_m)(1 - T^2) = 2T\frac{\dot{q}_m}{\sqrt{A_0}} + 2T^2\frac{\ddot{q}_m}{A_0} \quad \ldots\ldots\ldots\ldots\ldots(9).$$

We can operate on this equation with δ and readily deduce

$$(p_i - q_i)(1 - T^2) = 2T\frac{\dot{q}_i}{\sqrt{A_0}} + 2T^2\frac{\ddot{q}_i}{A_0} \quad \ldots\ldots\ldots\ldots\ldots(10),$$

which holds for any integer value of i from $i = 1$ to $i = m$.

The form of the typical equation (10) shows us that we can, by elimination of $1 - T^2$, T and T^2, obtain $m - 2$ independent linear equations connecting the p's with the constants $\zeta_1, \zeta_2, \ldots \zeta_m$; the $m - 1^{\text{th}}$ equation will involve the p's in the second degree, as has been shown by Jacobi.

31. We shall next investigate the value of $\Delta\dot{p}_i$, Δ being the operator referred to in Article 3. We have, from Article 8,

$$\dot{p}_m + z\dot{p}_{m-1} + z^2\dot{p}_{m-2} + \ldots + z^{m-1}\dot{p}_1 = -\sqrt{A_0 f(z)},$$

for all values of z which make $\phi(z)$ vanish. If we now write

$$z = \frac{x}{y}, \quad a_r = \frac{a_r}{b_r},$$

where a_r is a root of $$f(z) = 0,$$

and r can have any integer value from $r = 1$ to $r = 2m$, we find, introducing these values,

$$y^m\dot{p}_m + y^{m-1}x\dot{p}_{m-1} + y^{m-2}x^2\dot{p}_{m-2} + \ldots + yx^{m-1}\dot{p}_1 = -\sqrt{U(x, y)}.$$

If we now operate on this equation with Δ, we readily obtain

$$y^m\Delta\dot{p}_m + y^{m-1}x\Delta\dot{p}_{m-1} + y^{m-2}x^2\Delta\dot{p}_{m-2} + \ldots + yx^{m-1}\Delta\dot{p}_1$$
$$- \{my^{m-1}x\dot{p}_m + (m-1)\,y^{m-2}x^2\dot{p}_{m-1} + \ldots + x^m\dot{p}_1\} = 0.$$

We must now substitute for x^m its value derived from the equation

$$x^m + p_1x^{m-1}y + p_2x^{m-2}y^2 + \ldots + p_my^m = 0,$$

and then equate to zero the coefficients of $x^{m-1}y$, $x^{m-2}y^2$, ..., as explained in the rule laid down in Article 1.

We then arrive at the values of $\Delta\dot{p}_m$, $\Delta\dot{p}_{m-1}$, ... $\Delta\dot{p}_1$, which are given by the following formulæ:

$$\left.\begin{array}{l} \Delta\dot{p}_m+\dot{p}_1 p_m = 0 \\ \Delta\dot{p}_{m-1}-m\dot{p}_m+\dot{p}_1 p_{m-1} = 0 \\ \Delta\dot{p}_{m-2}-(m-1)\,\dot{p}_{m-1}+\dot{p}_1 p_{m-2} = 0 \\ \quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots \\ \Delta\dot{p}_1-2\dot{p}_2+\dot{p}_1 p_1 = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots(1).$$

We have also obviously

$$\Delta p_i = (i+1)\,p_{i+1}-p_1 p_i$$

for all integer values of i from $i=1$ to $i=m-1$.

32. If we now express $t-t^0$ in terms of T, we find

$$\surd A_0\,(t-t^0) = \log\left\{\frac{1+T}{1-T}\right\},$$

and consequently we learn that

$$\Sigma I_{m-1}(z)-\Sigma I_{m-1}(\zeta) = \frac{1}{\surd A_0}\log\left\{\frac{1+T}{1-T}\right\} = t-t^0 \ldots\ldots\ldots(1).$$

We shall now find the value of

$$\Sigma I_m(z)-\Sigma I_m(\zeta).$$

To do so we have but to operate on the equation above given with Δ, and we find, since by Article 5,

$$\Delta I_r(z) = -(m-r-2)\,I_{r+1}(z),$$

$$\Sigma I_m(z)-\Sigma I_m(\zeta) = \Delta\,(t-t^0) \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and the problem is reduced to finding $\Delta\,(t-t^0)$, or, what is the same thing, ΔT, the value of which we now proceed to investigate.

We have $$T = \frac{\surd A_0\,(p_1-q_1)}{\dot{p}_1+\dot{q}_1} = \frac{\dot{p}_1-\dot{q}_1}{\surd A_0\,(p_1+q_1-P_1)};$$

hence

$$\Delta T = \surd A_0\left\{(\dot{p}_1+\dot{q}_1)\,\Delta\,(p_1-q_1)-(p_1-q_1)\,\Delta\,(\dot{p}_1+\dot{q}_1)\right\}/(\dot{p}_1+\dot{q}_1)^2$$
$$+\tfrac{1}{2}\,(p_1-q_1)\surd A_0 P_1/(\dot{p}_1+\dot{q}_1),$$

which, after a few reductions, we find can be expressed in the form

$$\Delta T = -\frac{T}{2}\,(1-T^2)(p_1+q_1-P_1) = -\frac{T}{2}\,(1-T^2)(\pi_1+\kappa_1)\ldots(3);$$

operating now with Δ on the equation

$$\surd A_0 (t-t^0) = \log \left\{ \frac{1+T}{1-T} \right\},$$

we find $$\surd A_0 \Delta (t-t^0) + \surd A_0 (t-t^0) \frac{P_1}{2} = \frac{2\Delta T}{1-T^2} \quad \ldots\ldots\ldots\ldots\ldots (4),$$

and consequently,

$$\Delta (t-t^0) = \Sigma I_m (z) - \Sigma I_m (\zeta) = -\frac{P_1}{2} (t-t^0) - \frac{T(\pi_1+\kappa_1)}{\surd A_0}$$

$$= -\frac{P_1}{2} (t-t^0) - \frac{(\dot{\pi}_1+\dot{\kappa})}{A_0} \quad \ldots\ldots (5)$$

We can now, by a second operation of Δ on the above equation, determine

$$\Sigma I_{m+1} (z) - \Sigma I_{m+1} (\zeta),$$

since we know the value of $\Delta (t-t^0)$, whence it follows that we can obtain the value of

$$\Sigma I_r (z) - \Sigma I_r (\zeta),$$

r being any positive integer, by continued application of the operator Δ.

33. We now proceed to find the value of

$$\Sigma I_{-1} (z) - \Sigma I_{-1} (\zeta),$$

which we shall denote by $\tau - \tau^0$, remarking that when we interchange each x^2 and its corresponding y, and each a and its corresponding b, $t-t^0$ passes into $-(\tau-\tau^0)$, and T into $-\mathfrak{T}$, $\mathfrak{T}$ corresponding to $\tau-\tau^0$ in the same way that T does to $t-t^0$; we have, z standing for any root of

$$\phi (z) = 0,$$

$$\frac{dz}{dt} = \frac{\surd A_0 f(z)}{\phi' (z)} = \frac{\surd U (x, y)}{\Phi' (x, y)} \left\{ \frac{y_1 y_2 \ldots y_n}{y^2} \right\},$$

where $$\Phi' (x, y) \equiv \phi' (z) (y_1 y_2 \ldots y_m) y^{m-2}.$$

If we now interchange x and y, a and b, $U (x, y)$ remains unaltered,

and $\Phi'(x, y)$ becomes $(-1)^{m-1}\Phi'(x, y)$; hence $\frac{dz}{dt}$ becomes

$$(-1)^{m-1}\frac{\sqrt{U(x, y)}}{\Phi'(x, y)}\left\{\frac{x_1x_2 \dots x_m}{x^2}\right\}$$

$$= (-1)^{m-1}\frac{\sqrt{U(x, y)}}{\Phi'(x, y)}\left\{\frac{y_1y_2 \dots y_m}{y^2}\right\}\left\{\frac{x_1x_2 \dots x_m}{y_1y_2 \dots y_m}\right\}\frac{y^3}{x^2}$$

$$= (-1)^{m-1}\frac{dz}{dt}(-1)^m\frac{p_m}{z^2} = -\left(\frac{dz}{dt}\right)\frac{p_m}{z^2} = \frac{d\left(\frac{1}{2}\right)}{dt}p_m,$$

Consequently dt becomes $\frac{dt}{p_m}$, and $I_r(z)$ becomes

$$-I_{m-r-2}(z).$$

Hence we have the following scheme of transformation :

$$\left.\begin{array}{llll} x \text{ into } y, & & dt \text{ becomes } & \frac{dt}{p_m} \\ a \quad ,, \quad b, & & \dot{p}_m \quad ,, & -\frac{\dot{p}_m}{p_m} \\ z \quad ,, \quad \frac{1}{z}, & & T \quad ,, & -\mathfrak{T} \\ a \quad ,, \quad \frac{1}{a}, & & \lambda_m \quad ,, & \frac{\lambda_m}{P_{2m}} \\ t-t^0 \quad ,, \quad -(\tau-\tau^0), & & \dot{p}_r = \frac{p_m\dot{p}_{m-r}-p_{m-r}\dot{p}_m}{p_m} & \end{array}\right\} \quad \dots\dots\dots(1),$$

where $$\mathfrak{T} = \frac{e^{\sqrt{A_{2m}}(\tau-\tau^0)}-1}{e^{\sqrt{A_{2m}}(\tau-\tau^0)}+1}.$$

Now $\mathfrak{T}$ is known, for T, which is equal to

$$\frac{\sqrt{A_0}(p_m-q_m)}{\dot{p}_m+\dot{q}_m},$$

becomes, in consequence of the interchanges above spoken of,

$$-\frac{\sqrt{A_{2m}}(p_m-q_m)}{\dot{p}_mq_m+p_m\dot{q}_m},$$

from which we learn that

$$\mathfrak{T} = \frac{\sqrt{A_{2m}}(p_m-q_m)}{\dot{p}_mq_m+p_m\dot{q}_m} \quad \dots\dots\dots(2).$$

Consequently $\tau - \tau^0$ is known from the formula

$$\surd A_{2m}\,(\tau - \tau^0) = \log\left\{\frac{1+\mathfrak{T}}{1-\mathfrak{T}}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and, finally, $$\Sigma I_{-1}(z) - \Sigma I_{-1}(\zeta) = \frac{1}{\surd A_{2m}} \log\left\{\frac{1+\mathfrak{T}}{1-\mathfrak{T}}\right\} \quad \ldots\ldots\ldots\ldots (4).$$

34. We shall now proceed to find the value of

$$\Sigma I_{-2}(z) - \Sigma I_{-2}(\zeta),$$

and to do so we shall express $\mathfrak{T}$ in terms of T, p_m, and q_m. From the fundamental equation, we have

$$\left.\begin{aligned} \dot{p}_m^2 &= A_0\,(p_m^2 - 2\lambda_m p_m + P_{2m}) \\ \dot{q}_m^2 &= A_0\,(q_m^2 - 2\lambda_m q_m + P_{2m}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now, if we multiply the first of these equations by q_m^2, and the second by p_m^2, and then add, we find

$$\dot{p}_m^2 q_m^2 + p_m^2 \dot{q}_m^2 = A_0\left\{2p_m^2 q_m^2 - 2\lambda_m p_m q_m (p_m + q_m) + P_{2m}(p_m^2 + q_m^2)\right\} \quad \ldots\ldots\ldots\ldots (2),$$

and, if we square the equation

$$\dot{p}_m + \dot{q}_m = \surd A_0\, T^{-1}(p_m - q_m),$$

we obtain $$2\dot{p}_m \dot{q}_m = A_0 T^{-2}(p_m - q_m)^2 - \dot{p}_m^2 - \dot{q}_m^2,$$

and substituting the values of $\dot{p}_m^2$ and $\dot{q}_m^2$ from (1), we have

$$2\dot{p}_m \dot{q}_m = A_0\left\{T^{-2}(p_m - q_m)^2 - (p_m^2 + q_m^2) + 2\lambda_m (p_m + q_m) - 2P_{2m}\right\} \quad \ldots\ldots\ldots\ldots (3).$$

If we now multiply (3) by $p_m q_m$, and add to (1), we obtain

$$(p_m \dot{q}_m + \dot{p}_m q_m)^2 = A_0\left\{(T^{-2} - 1)\, p_m q_m + P_{2m}\right\}(p_m - q_m)^2 \ldots\ldots (4),$$

from which we find easily

$$\mathfrak{T}^{-2} - 1 = (T^{-2} - 1)\frac{A_0 p_m q_m}{A_{2m}} \quad \ldots\ldots\ldots\ldots\ldots (5).$$

If we now operate on this equation (5) with δ, we obtain

$$-\frac{2\delta\mathfrak{T}}{\mathfrak{T}^3} = \frac{A_0 (T^{-2} - 1)}{A_{2m}^2}\left\{A_{2m}(p_m q_{m-1} + p_{m-1} q_m) - 2_m A_{2m-1} p_m q_m\right\};$$

or $$2\delta\mathfrak{T} = -\mathfrak{T}(1 - \mathfrak{T}^2)\left\{\frac{p_{m-1}}{p_m} + \frac{q_{m-1}}{q_m} - \frac{P_{2m-1}}{P_{2m}}\right\} \quad \ldots\ldots\ldots\ldots (6).$$

Now, $$(\tau-\tau^0)\surd A_{2m} = \log\left\{\frac{1+\mathfrak{X}}{1-\mathfrak{X}}\right\}.$$

Operating on this equation with δ, we find

$$\surd A_{2m}\,\delta(\tau-\tau^0)+(\tau-\tau^0)\frac{2mA_{2m-1}}{2\surd A_m} = \frac{2\delta\mathfrak{X}}{1-\mathfrak{X}^2},$$

and, substituting the above value of δT, we obtain

$$\surd A_{2m}\,\delta(\tau-\tau^0) = \mathfrak{X}\left\{\frac{P_{2m-1}}{P_{2m}} - \frac{p_{m-1}}{p_m} - \frac{q_{m-1}}{q_m}\right\}\frac{-(\tau-\tau^0)}{2}\surd A_{2m}\times\frac{P_{2m-1}}{P_{2m}} \quad \ldots\ldots\ldots\ldots(7),$$

and, finally,

$$\Sigma I_{-2}(z) - \Sigma I_{-2}(\zeta) = \frac{\mathfrak{X}}{\surd A_{2m}}\left\{\frac{P_{2m-1}}{P_{2m}} - \frac{p_{m-1}}{p_m} - \frac{q_{m-1}}{q_m}\right\}\frac{-(\tau-\tau^0)}{2}\,\frac{P_{2m-1}}{P_{2m}} \quad \ldots\ldots\ldots\ldots(8).$$

Having now obtained $\tau-\tau^0$ and $\delta(\tau-\tau^0)$, we can find $\delta^2(\tau-\tau^0)$, and so, by repeated operations of δ, arrive at the value of $\delta^r(\tau-\tau^0)$. But the reader will perceive that this work is unnecessary if we have already arrived at the value of $\Delta^r(t-t^0)$; for the values of $\delta^r(\tau-\tau^0)$ and $\Delta^r(t-t^0)$ are not independent, but can be derived, one from the other, by means of the interchange of x and y, a and b, as noticed in Article 33.

If, then, we have calculated the ascending series

$$\Sigma I_m(z)-\Sigma I_m(\zeta),\quad \Sigma I_{m+1}(z)-\Sigma I_{m+1}(\zeta),\quad \Sigma I_{m+r}(z)-\Sigma I_{m+r}(\zeta),$$

the descending series

$$\Sigma I_{-2}(z)-\Sigma I_{-2}(\zeta),\ \ldots\ \Sigma I_{-r-2}(z)-\Sigma I_{-r-2}(\zeta)$$

can be found from the rules laid down for transformation in Article 33.

35. It remains to show how to find the value of

$$\Sigma L(z, n)-\Sigma L(\zeta, n),$$

where $L(z, n)$ is, as before defined,

$$\int^z \frac{dz}{(z-n)\surd A_0 f(z)}.$$

Now, it is clear that $I_{-1}(z)$ passes into $L(z, n)$ when z and each

of the a's is diminished by the same quantity n; when this is the case,

$$\begin{aligned}
A_0 &\text{ becomes } A_0 = N_0, \text{ say,}\\
A_1 &\quad ,, \quad A_1 + nA_0 = N_1,\\
A_2 &\quad ,, \quad A_2 + 2nA_1 + n^2 A_0 = N_2,\\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
A_{2m} &\quad ,, \quad A_{2m} + 2mnA_{2m-1} + \ldots n^{2m} A_0 = N_{2m},
\end{aligned}$$

and, adopting the notation of Article 28, p_m becomes $(-1)^m u(n)$, and q_m, in like manner, becomes $(-1)^m v(n)$.

If, now, in the formula of Article 33, viz.,

$$\tau - \tau^0 = \Sigma I_{-1}(z) - \Sigma I_{-1}(\zeta) = \frac{1}{\sqrt{A_{2m}}} \log\left\{\frac{1+\mathfrak{X}}{1-\mathfrak{X}}\right\},$$

we suppose the z's, ζ's, and a's all diminished by n, we find

$$\Sigma L(z, n) - \Sigma L(\zeta, n) = \frac{1}{\sqrt{N_{2m}}} \log\left\{\frac{1+\mathfrak{X}_n}{1-\mathfrak{X}_n}\right\} \quad \ldots\ldots\ldots\ldots (1),$$

where $\mathfrak{X}_n$ denotes the value of $\mathfrak{X}$ when the facients are diminished by n, and is easily seen to assume the form

$$\mathfrak{X}_n = \frac{(-1)^m \sqrt{N_{2m}} \{u(n) - v(n)\}}{u(n)\frac{dv(n)}{dt^0} + v(n)\frac{du(n)}{dt}}.$$

36. We shall now examine the value of

$$\Sigma L(z, a) - \Sigma L(\zeta, a),$$

a particular case of the formula discussed in the last article, in which $n = a$, a root of

$$f(z) = 0.$$

It is evident that N_{2m} vanishes in this case, since $n = a$, and therefore, in order to obtain the value of

$$\Sigma L(z, a) - \Sigma L(\zeta, a),$$

we must evaluate $\dfrac{1}{\sqrt{N_{2m}}} \log\left\{\dfrac{1+\sqrt{N_{2m}}\, D_n}{1-\sqrt{N_{2m}}\, D_n}\right\}$,

when $\sqrt{N_{2m}} = 0$,

where we have written for the sake of brevity

$$D_n = \frac{(-1)^m \{u(n) - v(n)\}}{u(n)\frac{dv(n)}{dt^0} + v(n)\frac{du(n)}{dt}},$$

and we readily perceive that the value we seek is $2D_n$, when $n = a$; hence we have

$$\Sigma L(z, a) - \Sigma L(\zeta, a) = 2D_a \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If one of the ζ's happened to be equal to a, then $v(a)$ vanishes, and D_a assumes the very simple form

$$D_a = \frac{(-1)^m}{\frac{dv(a)}{dt^0}}.$$

37. If we wish to determine the value of

$$\Sigma \int^z \frac{dz}{(z-u)^r \sqrt{A_0 f(z)}} - \Sigma \int^\zeta \frac{d\zeta}{(\zeta - u)^r \sqrt{A_0 f(\zeta)}},$$

we have only to diminish the facients by n in the formula for

$$\Sigma I_{-r}(z) - \Sigma I_{-r}(\zeta),$$

which we have shown how to find in Article 34, or we can differentiate the formula

$$\Sigma L(z, n) - \Sigma L(\zeta, n) = \frac{1}{\sqrt{N_{2m}}} \log \left\{ \frac{1 + \mathfrak{T}_n}{1 - \mathfrak{T}_n} \right\}$$

$n-1$ times with respect to n.

We have now shown that, if we have

$$\Sigma I_i(z) - \Sigma I_i(\zeta) = 0$$

for all integer values of i from $i = 0$ to $i = m-2$, we have

$$\Sigma I_{m-1}(z) - \Sigma I_{m-1}(\zeta) = \frac{1}{\sqrt{A_0}} \log \left\{ \frac{1+T}{1-T} \right\},$$

and have laid before the reader the proof that

$$\Sigma I_r(z) - \Sigma I_r(\zeta),$$

r being any positive or negative integer, can be obtained by an *operative process alone* on the equation

$$\Sigma I_{m-1}(z) - \Sigma I_{m-1}(\zeta) = \frac{1}{\sqrt{A_0}} \log\left\{\frac{1+T}{1-T}\right\}.$$

So that we have, r being a *positive* integer,

$$\Sigma I_{m+r}(z) - \Sigma I_{m+r}(\zeta) = \frac{\Delta^{r+1}}{(r+1)!}\left\{\frac{1}{\sqrt{A_0}} \log\left(\frac{1+T}{1-T}\right)\right\},$$

$$\Sigma I_{-r-2}(z) - \Sigma I_{-r-2}(\zeta) = \frac{\delta^{r+1}}{(r+1)!}\left\{\frac{1}{\sqrt{A_{2m}}} \log\left(\frac{1+\mathfrak{T}}{1-\mathfrak{T}}\right)\right\}.$$

38. We now turn to the discussion of Abelian or hyper-elliptic functions, of which we propose to give a definition and an investigation of their periodicity.

We shall consider the general case in which

$$I_r(z) \equiv \int^z \frac{z^r\, dz}{\sqrt{A_0 f(z)}},$$

and $f(z)$ is of the degree $2m$ in z, and we shall take $m-1$ quantities $z_1, z_2, \ldots z_{m-1}$, instead of m quantities previously considered in connexion with the above integral, and we shall suppose the $m-1$ quantities to be the roots of the equation

$$z^{m-1} + \mathfrak{p}_1 z^{m-2} + \mathfrak{p}_2 z^{m-3} + \ldots \mathfrak{p}_{m-1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If we write

$$\left.\begin{array}{lll} \Sigma I_0(z) = J_0(z), & \text{or} & J_0 \text{ if we have no} \\ \Sigma I_1(z) = J_1(z), & & \text{occasion to put the} \\ \ldots \quad \ldots \quad \ldots \quad \ldots & & z \text{ system in evidence} \\ \Sigma I_{m-2}(z) = J_{m-2}(z), & & \end{array}\right\} \ldots\ldots\ldots (2).$$

Σ denoting summation with regard to the $m-1$ quantities $z_1, z_2, \ldots z_{m-1}$, it is clear that, if we consider the $m-1$ coefficients which occur in equation (1) as known quantities, the $m-1$ sums $J_0, J_1, \ldots J_{m-2}$, are known; conversely, if we are given $J_0, J_1, \ldots J_{m-2}$, the $m-1$ quantities $\mathfrak{p}_1, \mathfrak{p}_2, \ldots \mathfrak{p}_{m-2}$ could be determined.

These latter quantities we shall call *primary* Abelian functions, and any rational functions of them Abelian functions.

We shall write, then,

$$\left.\begin{aligned} \mathfrak{p}_1 &= \mathfrak{f}_1\{J_0, J_1, \ldots J_{m-2}\} \\ \mathfrak{p}_2 &= \mathfrak{f}_2\{J_0, J_1, \ldots J_{m-2}\} \\ \cdots &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \mathfrak{p}_{m-1} &= \mathfrak{f}_{m-1}\{J_0, J_1, \ldots J_{m-2}\} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

39. We now turn to the general case discussed in this paper, namely,

$$\Sigma I_i(z) - \Sigma I_i(\zeta) = 0,$$

where we have m quantities $z_1, z_2, \ldots z_m$, and also m simultaneous values of these quantities $\zeta_1, \zeta_2, \ldots \zeta_m$. Let us now take $z_m = \alpha_1$, a root of

$$f(z) = 0,$$

and also take $\zeta_m = \alpha_2$, another root of

$$f(z) = 0,$$

and we shall have, adopting the notation introduced in this article,

$$J_i(z) + I_i(\alpha) = J_i(\zeta) + I_i(\alpha_2),$$

or

$$J_i(z) + \{I_i(\alpha_1) - I_i(\alpha_2)\} = J_i(\zeta) \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

when the above equation holds for all integer values of i from $i = 0$ to $i = m-2$.

Now, T for this system is found from the equation

$$T = \frac{\surd A_0 (p_m - q_m)}{\dot{p}_m + \dot{q}_m},$$

or

$$\dot{p}_m + \dot{q}_m = \surd A_0 T^{-1} p_m - \surd A_0 T^{-1} q_m.$$

Now, from Article 28, we learn that

$$\left.\begin{aligned} T\left\{\frac{du(\zeta)}{dt} + (-1)^{m-1}\surd\overline{A_0 f(\zeta)}\right\} &= \surd A_0\, u(\zeta) \\ T\left\{\frac{dv(z)}{dt^0} + (-1)^{m-1}\surd\overline{A_0 f(z)}\right\} &= -\surd A_0\, v(z) \end{aligned}\right\} \ldots\ldots\ldots(2),$$

and, if in the first of these equations we put $\zeta = \alpha_2$, and in the

second $z = a_1$, we obtain

$$\left.\begin{aligned} T\frac{du(a_2)}{dt} &= \sqrt{A_0}\,u(a_2) \\ T\frac{dv(a_1)}{dt^0} &= -\sqrt{A_0}\,v(a_1) \end{aligned}\right\} \quad \text{.....(3).}$$

The first of these equations gives us

$$\sqrt{A_0}T^{-1} = \frac{du(a_2)}{u(a_2)} = \frac{\Sigma\dfrac{dz}{dt}}{z-a} = \frac{\Sigma\sqrt{A_0 f(z)}}{(z-a_1)(z-a_2)\theta'(z)},$$

where $\theta(z) \equiv (z-z_1)(z-z_2)\ldots(z-z_{m-1})$,

or, if we write $Q(z) \equiv \dfrac{\sqrt{A_0 f(z)}}{\theta'(z)}$,

we have $$\sqrt{A_0}\,T^{-1} = \frac{\Sigma Q(z)}{(z-a_1)(z-a_2)} \quad \text{.....(4),}$$

and from the second equation we obtain in like manner

$$\sqrt{A_0}T^{-1} = -\frac{\Sigma Q(\zeta)}{(\zeta-a_1)(\zeta-a_2)} \quad \text{.....(5).}$$

40. We now investigate the values of $\dot{p}_m$ and $\dot{q}_m$; we have

$$p_m = (-1)^m a_1 z_1 z_2 \ldots z_{m-1},$$

$$q_m = (-1)^m a_2 \zeta_1 \zeta_2 \ldots \zeta_{m-1};$$

differentiating, then, the first of these equations with respect to t, and the second with respect to t^0, we obtain, remembering that $\dfrac{dz}{dt}$ vanishes when $z = a$, a root of

$$f(z) = 0,$$

$$\dot{p}_m = (-1)^m a_1 \left\{ z_2 z_3 \ldots z_{m-1} \frac{\sqrt{A_0 f(z_1)}}{(z_1-a_1)\theta'(z_1)} + \ldots \right\},$$

or $$\dot{p}_m = (-1)^m a_1 \Sigma \frac{Q(z_1)\, z_2 z_3 \ldots z_{m-1}}{(z_1-a_1)} \quad \text{.....(1),}$$

and, similarly, $\dot{q}_m = (-1)^m a_2 \Sigma \dfrac{Q(\zeta_1)\,\zeta_2\zeta_3\ldots\zeta_{m-1}}{(\zeta_1-a_2)}$ .....(2).

Introducing now these values of p_m, q_m, $\dot{p}_m$, and $\dot{q}_m$ into the equation

$$\dot{p}_m + \dot{q}_m = \surd A_0 T^{-1} p_m - \surd A_0 T^{-1} q_m,$$

we obtain

$$\alpha_1 \Sigma \frac{Q(z_1)\, z_2 z_3 \dots z_{m-1}}{(z_1 - \alpha_1)} + \alpha_2 \Sigma \frac{Q(\zeta_1)\, \zeta_2 \zeta_3 \dots \zeta_{m-1}}{\zeta_1 - \alpha_2}$$

$$= \alpha_1 z_1 z_2 \dots z_{m-1} \surd A_0 T^{-1} - \alpha_2 \zeta_1 \zeta_2 \dots \zeta_{m-1} \surd A_0 T^{-1} \dots\dots (3).$$

Now, from the values of $\surd A_0 T^{-1}$ given in (4) and (5) of the last article, we can write

$$\alpha_1 z_1 z_2 \dots z_{m-1} \surd A_0 T^{-1} = \alpha_1 z_1 z_2 \dots z_{m-1} \Sigma \frac{Q(z_1)}{(z_1 - \alpha_1)(z_1 - \alpha_2)},$$

$$\alpha_2 \zeta_1 \zeta_2 \dots \zeta_{m-1} \surd A_0 T^{-1} = -\alpha_2 \zeta_1 \zeta_2 \dots \zeta_{m-1} \Sigma \frac{Q(\zeta_1)}{(\zeta_1 - \alpha_1)(\zeta_1 - \alpha_2)},$$

and, introducing these values into the equation (3), it assumes the form

$$\alpha_1 \Sigma \frac{Q(z_1)\, z_2 z_3 \dots z_{m-1}}{(z_1 - \alpha_1)} + \alpha_2 \Sigma \frac{Q(\zeta_1)\, \zeta_2 \zeta_3 \dots \zeta_{m-1}}{(\zeta_1 - \alpha_2)}$$

$$= \alpha_1 \Sigma \frac{Q(z_1)\, z_1 z_2 z_3 \dots z_{m-1}}{(z_1 - \alpha_1)(z_1 - \alpha_2)} + \alpha_2 \Sigma \frac{Q(\zeta_1)\, \zeta_1 \zeta_2 \zeta_3 \dots \zeta_{m-1}}{(\zeta_1 - \alpha_1)(\zeta_1 - \alpha_2)},$$

or $\alpha_1 \Sigma \frac{Q(z_1)\, z_2 z_3 \dots z_m}{(z_1 - \alpha_1)(z_1 - \alpha_2)} (z_1 - \alpha_2 - z_1) + \alpha_2 \Sigma \frac{Q(\zeta_1)\, \zeta_2 \zeta_3 \dots \zeta_{m-1}}{(\zeta_1 - \alpha_1)(\zeta_1 - \alpha_2)} (\zeta_1 - \alpha_1 - \zeta_1) = 0,$

and dividing by $\alpha_1 \alpha_2$, we obtain, finally,

$$\Sigma \frac{Q(z)\, z_2 z_3 \dots z_{m-1}}{(z_1 - \alpha_1)(z_1 - \alpha_2)} + \Sigma \frac{Q(\zeta_1)\, \zeta_2 \zeta_3 \dots \zeta_{m-1}}{(\zeta_1 - \alpha_1)(\zeta_1 - \alpha_2)} = 0 \dots\dots (4).$$

This remarkable result furnishes us by successive operations of δ with $m-2$ other equations of a similar nature, for it must be observed that

$$\delta Q(z) = 0,$$

since $Q(z)$ is a function of the difference of the facients which enter into it. We remark further that this equation, and all those which flow from it by operations of δ, thus constituting a system of algebraic equivalents of the transcendental system, remain unaltered when we interchange α_1 and α_2, so that the systems

$$J_i(z) + \{I_i(\alpha_1) - I_i(\alpha_2)\} = J_i(\zeta),$$

$$J_i(z) + \{I_i(\alpha_2) - I_i(\alpha_1)\} = J_i(\zeta)$$

give identical algebraic equivalents, so that, as far as algebraic equivalents are concerned, we get the same algebraic equations connecting the $m-1$ z's, and the $m-1$ ζ's, whether we write

$$J_i(z)+I_i\begin{pmatrix}a_1\\a_2\end{pmatrix}=J_i(\zeta)$$

or

$$J_i(z)-I_i\begin{pmatrix}a_1\\a_2\end{pmatrix}=J_i(\zeta),$$

where we have written $I_i(a_1)-I_i(a_2)$

in the form $I_i\begin{pmatrix}a_1\\a_2\end{pmatrix}$.

41. We shall now write

$$J_i(z)+I_i\begin{pmatrix}a_1\\a_2\end{pmatrix}=J_i(\zeta) \quad \ldots\ldots\ldots\ldots (1),$$

and then take a new set of $m-1$ quantities $\eta_1, \eta_2, \ldots \eta_{m-1}$ connected with the ζ's by the system of equations contained in the typical equation

$$J_i(\zeta)+I_i\begin{pmatrix}a_1\\a_2\end{pmatrix}=J_i(\eta) \quad \ldots\ldots\ldots\ldots (2),$$

and seek the algebraic relations connecting the z's and the η's, or, what is the same thing, the relations connecting the $\mathfrak{p}$'s and the $\mathfrak{r}$'s, the z's and η's being given respectively by the following equations:

$$z^{m-1}+\mathfrak{p}_1 z^{m-2}+\mathfrak{p}_2 z^{m-3}+\ldots+\mathfrak{p}_{m-1}=0,$$

$$\eta^{m-1}+\mathfrak{r}_1\eta^{m-2}+\mathfrak{r}_2\eta^{m-3}+\ldots+\mathfrak{r}_{m-1}=0.$$

We seek, in fact, the algebraic relations connecting the $\mathfrak{p}$'s and the $\mathfrak{r}$'s which constitute the algebraic equivalents of the transcendental system

$$J_i(z)+2I_i\begin{pmatrix}a_1\\a_2\end{pmatrix}=J_i(\eta) \quad \ldots\ldots\ldots\ldots (3),$$

which results from (1) and (2).

Now the required algebraic equivalents of (3) are plainly to be found by writing down the algebraic equivalents of (1) connecting the z's and ζ's, and the algebraic equivalents of (2) connecting the ζ's and the η's, and from these $2m-2$ equations eliminating the $m-1$ ζ's, thus obtaining $m-1$ equations between the $m-1$ z's and $m-1$ η's.

But the algebraic equivalents of (2) are precisely the same as those of

$$J_i(\zeta) - I_i\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = J_i(\eta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and this equation would, in conjunction with (1), lead us to the transcendental system

$$J_i(z) = J_i(\eta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

it follows then that the algebraic equivalents of (3) and (5) must be identical.

Now, with regard to the system

$$J_i(z) = J_i(\eta),$$

we observe that it results from the general case of

$$\Sigma I_i(z) = \Sigma I_i(\eta),$$

in which one of the z's has become equal to one of the η's, and has thus disappeared from the system of equations, leaving us

$$J_i(z) = J_i(\eta).$$

Now, let $z_m = \eta_m$ be the common value which has disappeared from the system, and adding to both sides of the typical equation

$$J_i(z) = J_i(\eta),$$

to the left-hand side $I_i(z_m)$, and to the right-hand side $I_i(\eta_m)$, we have

$$\Sigma I_i(z) = \Sigma I_i(\eta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

in which $z_m = \eta_m$.

From Article 28, we have T for system (6) given by the equation

$$T\left\{\frac{du(\eta)}{dt} + (-1)^{m-1}\sqrt{A_0 f(\eta)}\right\} = \sqrt{A_0}\, u(\eta),$$

and if we take $\eta = \eta_m = z_m$, it is clear that $u(\eta_m)$ vanishes, and as the expression within the brackets does not become zero, it follows that T must vanish for the system (6), and that we must have

$$p_s = r_i,$$

s being an integer which may have any value from $s = 1$ to $s = m$. Now, since

$$p_m = -z_m \mathfrak{p}_{m-1},$$

z_m being equal to η_m, $\qquad r_m = -\zeta_m \mathfrak{r}_{m-1},$

it readily follows that we must have

$$\mathfrak{p}_s = \mathfrak{r}_s$$

for all integer values of s from $s = 1$ to $s = m-1$; consequently the $m-1$ algebraic equivalents of system (5) are contained in the typical equation

$$\mathfrak{p}_s = \mathfrak{r}_s,$$

which must also be expressive of the algebraic equivalents of the system

$$J_i(z) + 2I\begin{pmatrix}a_1\\a_2\end{pmatrix} = J_i(\eta),$$

from which it follows that $\mathfrak{p}_1, \mathfrak{p}_2, \ldots \mathfrak{p}_{m-1}$

are periodic functions of the quantities

$$J_0(z),\ J_1(z),\ \ldots\ J_{m-1}(z);$$

the periods being $2I\begin{pmatrix}a_1\\a_2\end{pmatrix}$, a_1 and a_2 being any two roots of the equation

$$f(z) = 0.$$

We shall not here enter on a discussion touching the reality of the roots of

$$f(z) = 0,$$

and the question of the real and imaginary portions of $I_i\begin{pmatrix}a_1\\a_2\end{pmatrix}$, but shall rest content with the statement that

$$\mathfrak{p}_s = \mathfrak{f}_s\{J_0(z), J_1(z), \ldots J_{m-1}(z)\}$$

is a periodic function of the J's, and remains unaltered when $J_i(z)$ is increased or diminished by one or more of the various quantities $2I_i\begin{pmatrix}a_1\\a_2\end{pmatrix}$.

42. If we now consider the periods, it would at first sight appear that we had $2m-1$ independent periods of the type $I_i\begin{pmatrix}a_1\\a_2\end{pmatrix}$; but such is not the case, as we shall now show. We wish to prove that a relation of the following kind exists between the $2m-1$ integrals

$$I_i\begin{pmatrix}a_1\\a\end{pmatrix},\ I_i\begin{pmatrix}a_2\\a\end{pmatrix},\ \ldots\ I_i\begin{pmatrix}a_{2m}\\a\end{pmatrix};$$

a being some one root which we take as the inferior limit of the integral

$$\int_a^z \frac{z^i dz}{\sqrt{A_0 f(z)}},$$

in each case, viz.,

$$\pm I_i\begin{pmatrix}a_1\\ a\end{pmatrix} \pm I_i\begin{pmatrix}a_2\\ a\end{pmatrix} \pm \dots \pm I_i\begin{pmatrix}a_m\\ a\end{pmatrix} = \pm I_i\begin{pmatrix}a_{m+1}\\ a\end{pmatrix} \pm \dots \pm I_i\begin{pmatrix}a_{2m}\\ a\end{pmatrix} \quad \dots(1)$$

holding for integer values of i from $i=0$ to $i=m-1$, some one of the integrals vanishing when the superior limit becomes a.

If we take as a set of z's $a_1, a_2, \dots a_m$, say, then, if we can show that a set of ζ's can have the values $a_{m+1}, a_{m+2}, \dots a_{2m}$, the relations implied in (1) are proved to exist.

Now, we have in general

$$T = \frac{\sqrt{A_0}(p_m - q_m)}{\dot{p}_m + \dot{q}_m} = \frac{\sqrt{A_0}(p_{m-1} - q_{m-1})}{\dot{p}_{m-1} + \dot{q}_{m-1}} = \dots = \frac{\sqrt{A_0}(p_1 - q_1)}{\dot{p}_1 + \dot{q}_1} \quad \dots(2),$$

and, since $\dot{p}_m, \dot{p}_{m-1}, \dots \dot{p}_1$; $\dot{q}_m, \dot{q}_{m-1}, \dots \dot{q}_1$ all vanish when the z's and ζ's have the values assigned above, it follows that we must have a corresponding transcendental set of relations which are those contained in (1). There are consequently but $2m-2$ independent periods of the type $2I_i\begin{pmatrix}a_1\\ a_2\end{pmatrix}$.

43. There is an interesting algebraic integral of the system

$$J_i(z) + I_i\begin{pmatrix}a_1\\ a_2\end{pmatrix} = J_i(\zeta),$$

which, I think, is worthy of notice.

We have $T\left\{\frac{du(\zeta)}{dt} + (-1)^{m-1}\sqrt{A_0 f(\zeta)}\right\} = \sqrt{A_0}\, u(\zeta)$.

Now let us take $\zeta_m = a_2$, and the above equation becomes

$$T\frac{du(a_2)}{dt} = \sqrt{A_0}\, u(a_2),$$

an equation which, being integrated with respect to t, gives us

$$u(a_2) = c\left\{e^{\sqrt{A_0}(t-t^0)} + e^{-\sqrt{A_0}(t-t^0)} - 2\right\},$$

c being a constant depending on $\zeta_1, \zeta_2, \dots \zeta_{m-1}$.

If we now take

$$u_1(z) = (z_1 - z)(z_2 - z) \dots (z_{m-1} - z)' \dots\dots\dots\dots\dots\dots (1),$$

z being any quantity, we have

$$u(a_2) = (z_m - a_2)\, u_1(a_2).$$

In order to determine c we must determine the value of $\frac{d^2 u(a_2)}{dt^2}$, when

$$t - t^0 = 0.$$

Now $\frac{d^2 u(a_2)}{dt^2} = (z_m - a_2)\frac{d^2 u_1(a_2)}{dt^2} + 2\frac{dz_m}{dt}\frac{du_1(a_2)}{dt} + u_1(a_2)\frac{d^2 z_m}{dt^2}$,

but, since the first two terms of the right-hand side of the equation vanish when

$$t - t^0 = 0,$$

for then

$$z_m - a_2 = 0,$$

and

$$\frac{dz_m}{dt} = 0,$$

when z is a root of $f(z)$, we are only concerned with the value of

$$u_1(a_2)\frac{d^2 z_m}{dt^2},$$

when

$$t - t^0 = 0.$$

Now,

$$\frac{dz_m}{dt} = \frac{\sqrt{A_0 f(z_m)}}{\phi'(z_m)};$$

consequently, differentiating with respect to t, we find

$$\frac{d^2 z_m}{dt^2} = \frac{1}{\phi'(z_m)}\frac{\sqrt{A_0 f'(z_m)}}{2\sqrt{f(z_m)}}\frac{dz_m}{dt} + \sqrt{A_0 f(z_m)}\,\frac{d}{dt}\left\{\frac{1}{\phi'(z_m)}\right\},$$

or

$$\frac{d^2 z_m}{dt^2} = \frac{A_0 f'(z_m)}{2\{\phi'(z_m)\}^2} + \sqrt{A_0 f(z_m)}\,\frac{d}{dt}\left\{\frac{1}{\phi'(z_m)}\right\} \dots\dots\dots\dots (2).$$

If we now let

$$t - t^0 = 0,$$

z_m becomes a_2, and we obtain

$$\frac{d^2 z_m}{dt^2}(\text{when } z_m = a_2) = \frac{A_0 f'(a_2)}{2\{v_1(a_2)\}^2} \dots\dots\dots\dots\dots (3),$$

where $$v_1(z) \equiv (\zeta_1 - z)(\zeta_2 - z) \dots (\zeta_{m-1} - z),$$

and consequently

$$\frac{A_0 f(a_2)}{2v_1(a_2)} = c\frac{d}{dt^2}\left\{e^{\sqrt{A_0}(t-t^0)} + e^{-\sqrt{A}(t-t^0)} - 2\right\} = 2cA_0;$$

hence $$c = \frac{f'(a_2)}{4v_1(a_2)},$$

and, finally, $$(a_1 - a_2)\, u_1(a_2)\, v_1(a_2) = \frac{f'(a_2)}{4}\chi^2 \quad \dots\dots\dots\dots\dots\dots(4),$$

where $$\chi = e^{\sqrt{A_0}\frac{1}{2}(t-t^0)} - e^{-\sqrt{A_0}\frac{1}{2}(t-t^0)},$$

or $$u_1(a_2)\, v_1(a_2) = -\frac{\chi^2}{4}(a_2 - a_3)(a_2 - a_4) \dots (a_2 - a_{2m}) \quad \dots\dots(5),$$

and, by parity of reasoning, we have also

$$u_1(a_1)\, v_1(a_1) = -\frac{\chi^2}{4}(a_1 - a_3)(a_1 - a_4) \dots (a_1 - a_{2m}) \dots\dots\dots(6).$$

If we now divide the first equation (5) by (6), we obtain an *unique* result—

$$\frac{u_1(a_2)\, v_1(a_2)}{u_1(a_1)\, v_1(a_1)} = \frac{(a_2 - a_3)(a_2 - a_4) \dots (a_2 - a_{2m})}{(a_1 - a_3)(a_1 - a_4) \dots (a_1 - a_{2m})} \quad \dots\dots\dots(7),$$

or $$\frac{(a_2^{m-1} + \mathfrak{p}_1 a_2^{m-2} + \mathfrak{p}_2 a_2^{m-3} + \dots + \mathfrak{p}_{m-1})(a_2^{m-1} + \mathfrak{q}_1 a_2^{m-2} + \mathfrak{q}_2 a_2^{m-3} + \dots + \mathfrak{q}_{m-1})}{(a_1^{m-1} + \mathfrak{p}_1 a_1^{m-2} + \mathfrak{p}_2 a_1^{m-3} + \dots + \mathfrak{p}_{m-1})(a_1^{m-1} + \mathfrak{q}_1 a_1^{m-2} + \mathfrak{q}_2 a_1^{m-3} + \dots + \mathfrak{q}_{m-1})}$$

$$= \frac{(a_2 - a_3)(a_2 - a_4) \dots (a_2 - a_{2m})}{(a_1 - a_3)(a_1 - a_4) \dots (a_1 - a_{2m})},$$

from which it follows that the above relation between the $\mathfrak{p}$'s and $\mathfrak{q}$'s is linear in both sets of quantities.

We can determine no other equation or equations from the above, by operation of δ, since it consists of functions of the differences of the various facients which enter into it, and is easily seen to be an absolute invariant, remaining unaltered when we write for the facients their respective reciprocals.

The results of this article when applied to the cases $m = 2$ and $m = 3$ are many, and interesting, and enable us to obtain the laws of the periodicity of elliptic functions and hyper-elliptic functions for the case $m = 3$, with ease, without the application of the general theorem of Article 41.

44. For the case $m = 2$, we may write

$$I_0(z) + I_0\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = I_0(\zeta) \quad \text{.....(1),}$$

$$I_0(\zeta) + I_0\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = I_0(\eta) \quad \text{.....(2),}$$

from which we infer $I_0(z) + 2I\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = I_0(\eta)$ .....(3).

Now, from (3) of the last article, we obtain

$$\frac{(a_2-z)(a_2-\zeta)}{(a_1-z)(a_1-\zeta)} = \frac{(a_2-a_3)(a_2-a_4)}{(a_1-a_3)(a_1-a_4)} \quad \text{.....(4),}$$

$$\frac{(a_2-\zeta)(a_2-\eta)}{(a_1-\zeta)(a_1-\eta)} = \frac{(a_2-a_3)(a_2-a_4)}{(a_1-a_3)(a_1-a_4)} \quad \text{.....(5),}$$

from which it follows that $z = \eta$ is the algebraic equivalent of (3).

We must regard, then, z as a periodic function of $I_0(z)$, and its periods are, say,

$$2I_0\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad 2I\begin{pmatrix} a_3 \\ a_2 \end{pmatrix}, \quad 2I\begin{pmatrix} a_4 \\ a_2 \end{pmatrix},$$

and these three are easily seen to be connected, since $z = a_3$ and $\zeta = a_4$ are values which satisfy (4), and which correspond to the transcendental equation

$$\pm I_0\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \pm I\begin{pmatrix} a_3 \\ a_2 \end{pmatrix} \pm I\begin{pmatrix} a_4 \\ a_2 \end{pmatrix} = 0.$$

45. For the case $m = 3$, we have

$$\left.\begin{aligned} J_0(z) + I_0\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= J_0(\zeta) \\ J_1(z) + I_1\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= J_1(\zeta) \end{aligned}\right\} \quad \text{.....(1),}$$

$$\left.\begin{aligned} J_0(\zeta) + I_0\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= J_0(\eta) \\ J_1(\zeta) + I_1\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= J_1(\eta) \end{aligned}\right\} \quad \text{.....(2).}$$

Now, from (5) and (6) of Article 43, we have, for the system (1),

$$\left.\begin{aligned} u_1(a_2)\, v_1(a_2) &= -\frac{\chi^2}{4}(a_2-a_3)(a_2-a_4)(a_2-a_5)(a_2-a_6) \\ u_1(a_1)\, v_1(a_1) &= -\frac{\chi^2}{4}(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_1-a_6) \end{aligned}\right\} \quad \text{.....(3).}$$

Now, I say that $t-t^0$ is the *same* for both the systems (1) and (2), as appears from the value of T^{-1} given in Article 39, and consequently χ will have the same value for both these systems.

We shall have, then, two corresponding equations to those given above in (3), which we may write

$$\left.\begin{aligned} v_1(a_2)\,w_1(a_2) &\equiv -\frac{\chi^2}{4}(a_2-a_3)(a_2-a_4)(a_2-a_5)(a_2-a_6) \\ v_1(a_1)\,w_1(a_1) &= -\frac{\chi^2}{4}(a_1-a_3)(a_1-a_4)(a_1-a_5)(a_1-a_6) \end{aligned}\right\} \quad \ldots\ldots(4),$$

where $w_1(z) \equiv (z-\eta_1)(z-\eta_2)$,

from which we infer $u_1(a_1) = w_1(a_1)$,

$u_1(a_2) = w_1(a_1)$,

or $\mathfrak{p}_1 = \mathfrak{r}_1, \quad \mathfrak{p}_2 = \mathfrak{r}_2$,

from which it follows that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are periodic functions of $J_0(z)$ and $J_1(z)$, the corresponding periods being of the types

$$I_0\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad I_1\begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

a_1 and a_2 being two roots of $f(z) = 0$.

In a manner similar to that adopted in the discussion of the periods in the last article, we learn that there are but four independent periods of each class, the five integrals

$$I_0\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad I_0\begin{pmatrix} a_3 \\ a_2 \end{pmatrix}, \ \ldots \ I_0\begin{pmatrix} a_6 \\ a_2 \end{pmatrix},$$

as well as $I_1\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad I_1\begin{pmatrix} a_3 \\ a_2 \end{pmatrix}, \ \ldots \ I_1\begin{pmatrix} a_6 \\ a_2 \end{pmatrix}$,

being connected in each case by a linear relation.

The method exposed in this paper then completely establishes the nature and periodicity of elliptic and Abelian functions without, as remarked before, entering on any discussion of the reality of the roots of $f(z)$, or the real and imaginary parts of an integral, such as $I_i\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, a_1 and a_2 being any two roots of $f(z)$.

The results given in this paper will, I think, when applied to that important class of curves whose coordinates can be expressed in terms of Abelian functions, be found interesting and will furnish us with many new properties not unworthy of notice.

An Extension of Boltzmann's Minimum Theorem. *By* S. H. Burbury, M.A., F.R.S. Received May 31st, 1895. Communicated June 13th, 1895, by G. H. Bryan, F.R.S.

1. Let $f(p_1 \dots q_n)\, dp_1 \dots dq_n$, or shortly $f \,.\, dp_1 \dots dq_n$, denote the chance that a molecule of a gas shall at any instant have its n coordinates $p_1 \dots p_n$, and corresponding momenta $q_1 \dots q_n$ between the limits p_1, $p_1 + dp_1$, &c.

Similarly, let $F \,.\, dP_1 \dots dQ_n$ be the corresponding chance for the values $P_1 \dots P_1 + dP_1$ of the coordinates and momenta.

2. If at a given instant the variables $p_1 \dots Q_n$ stand to one another in a certain relation, an encounter between the two molecules ensues, that is, within a very short time after the given instant the variables $p_1 \dots Q_n$ will, by the mutual action of the two molecules alone, assume new values *conservatis conservandis*, which may be denoted by accented letters $p_1' \dots Q_n'$.

If we ask what is the number per unit of volume of pairs for which at this instant the variables are so related to one another, the answer usually given is that it is proportional to $Ff \,.\, dp_1 \dots dQ_n$. In other words, it is usual to assume the chances f and F to be independent.

3. On this assumption of independence, and on this assumption only, it has been proved that, if

$$H = \iiiint \dots f(\log f - 1)\, dp_1 \dots dq_n,$$

$\dfrac{dH}{dt}$ is necessarily negative. If, therefore, when the system is isolated, F and f continue to be independent, $\dfrac{dH}{dt}$ continues to be negative. H tends to a minimum, which it reaches when the distribution of momenta is according to the Boltzmann-Maxwell law.

4. Now, systems may exist in which that independence of f and F for encountering molecules cannot be conceded. I have myself propounded the doctrine that the independence of f and F is only a

consequence of the (generally) assumed rarity of the medium, and that they cease to be independent as the medium becomes denser, on the ground that in the dense medium the proximity of two molecules, implied by their encounter, affords a presumption that they have recently been exposed to the same influences, and have acquired some velocities in common. However this may be, and I am not now assuming the truth of it, it is worth while to consider whether and how we can prove the theorem without assuming the independence of f and F. I propose in this paper to treat only the simplest case, regarding the molecules as equal elastic spheres.

5. Let c be the diameter of a sphere. Consider two spheres A and B. Let R be their relative velocity. About O, the centre of A, suppose a circular area described of radius c, perpendicular to R. Let r be the distance of the centre of B from the plane of that area. Then

$$\frac{dr}{dt} = R.$$

Let a be the distance from O, the centre of A, to the point P, in which the line through the centre of B parallel to R cuts that plane. Then, if $a < c$, a collision will occur between A and B, unless any third sphere previously collides with either of them. Further, a^2+r^2 cannot be less than c^2. And, if a^2+r^2 is infinitely nearly equal to c^2, the chance of any third sphere colliding with either A or B before they collide with each other vanishes, and a collision necessarily occurs between A and B. The only effect of that collision is to change the direction of the relative velocity R; and the nature of that change depends only on a, and on the angle β, which OP makes with a fixed diameter of the circular area.

Let us call a, β the *collision coordinates*. If the velocities of the two colliding spheres before collision be denoted by

$$\left.\begin{array}{l} u \ \ldots \ u+du \\ v \ \ldots \ v+dv \\ w \ \ldots \ w+dw \end{array}\right\} \text{ for one sphere,}$$

and

$$\left.\begin{array}{l} U \ \ldots \ U+dU \\ V \ \ldots \ V+dV \\ W \ \ldots \ W+dW \end{array}\right\} \text{ for the other,}$$

and if the collision coordinates be $\alpha \ldots \alpha + d\alpha$, and $\beta \ldots \beta + d\beta$, then as the result of collision u, v, w, U, V, W become u', v', w', U', V', W', and α, β become α', β'. Conversely, if before collision the variables be denoted by the accented letters, their values after collision will be denoted by the unaccented letters, and, as is known,

$$du\, dv\, dw\, dU\, dV\, dW = du'\, dv'\, dw'\, dU'\, dV'\, dW'.$$

6. Let the number per unit volume of spheres whose velocities are $u \ldots u + du$, $v \ldots v + dv$, $w \ldots w + dw$ be $f(u, v, w)\, du\, dv\, dw$. Call these the class u. Similarly, let the number whose velocities are $U \ldots U + dU$, &c., be $F(U, V, W)\, dU\, dV\, dW$, and call these the class U. Now, let us suppose for a moment that no collisions are allowed to happen, except (1) direct collisions between the spheres of class u and spheres of class U, without restriction as to the values of α and β; and (2) reverse collisions between spheres of class u' and spheres of class U', with such values only of α' and β' as that u', v', w', &c., shall after collision become $u \ldots u + du$, &c.

If that were so, the only way by which any sphere could leave the class u would be by one of the direct collisions, and the only way by which any sphere could enter the class u would be by one of the reverse collisions. Hence on this supposition the increase per unit of time of the number of spheres in the class u, *i.e.*,

$$\frac{d}{dt} f(u, v, w), \quad \text{or} \quad \frac{df}{dt},$$

would be (number of reverse collisions per unit of time) — (number of direct collisions per unit of time).

7. Now, in the ordinary case, when f, F are independent, the number of direct collisions is

$$\iint \pi c^2 R f F .\, d\alpha\, d\beta.$$

And the number of reverse collisions is

$$\iint \pi c^2 R f' F' .\, d\alpha\, d\beta,$$

and so

$$\frac{df}{dt} = \iint \pi c^2 R\, (f'F' - fF)\, d\alpha\, d\beta.$$

8. But I now propose to treat the case in which f and F are not

independent, or the velocities u, U are *correlated*. And, therefore, for fF we must substitute a more general form

$$\phi\,(u, v, w, U, V, W, \alpha, \beta),$$

where ϕ is some function.

For the reverse collisions we shall have a corresponding function

$$\phi'\,(u', v', w', U', V', W', \alpha, \beta) \quad \text{or} \quad \phi'.$$

Then, collisions being still restricted as stated in 6, we should have

$$\frac{df}{dt} = \iint \pi c^2 R\,(\phi' - \phi)\,d\alpha\,d\beta.$$

9. But now we can make U, V, W assume successively all values, still maintaining u, v, w unaltered; and then we obtain for the complete variation of f with the time

$$\frac{df}{dt} = \iiiint\!\!\int \pi c^2 R\,(\phi' - \phi)\,d\alpha\,d\beta\,dU\,dV\,dW.$$

Now, let $$H = \iiint f\,(\log f - 1)\,du\,dv\,dw;$$

and therefore

$$\frac{dH}{dt} = \iiint \frac{df}{dt} \log f\,du\,dv\,dw$$

$$= \iiint du\,dv\,dw \iiiint\!\!\int \pi c^2 R\,(\phi' - \phi)\,\log f\,d\alpha\,d\beta\,dU\,dV\,dW.$$

10. In this integration, extending over all values both of u, v, w and U, V, W, these classes interchange, so that our integral includes the two terms

$$\pi c^2 R\,(\phi' - \phi)\log f$$

and $$\pi c^2 R\,(\phi' - \phi)\log F;$$

and therefore includes the term

$$\pi c^2 R\,(\phi' - \phi)\,\log\,(fF).$$

For a similar reason it includes the term

$$\pi c^2 R\,(\phi - \phi')\log\,(f F');$$

and therefore includes the term

$$\pi c^3 R\,(\phi'-\phi)\log\frac{fF}{f'F'},$$

and consists wholly of terms of this form.

11. Thus expressed, $\frac{dH}{dt}$ is not necessarily always of the same sign, unless of the two equations

$$\phi=\phi',\quad fF=f'F'$$

one involves the other (which condition, however, will be found to hold in the cases we shall consider). But in the permanent state $\frac{dH}{dt}$ must be zero, which can be by making either

$$\phi=\phi'\quad\text{or}\quad fF=f'F'.$$

Also $\frac{df}{dt}$ must be zero, which can only be by making

$$\phi=\phi'.$$

If, therefore, in any problem we find that the two equations

$$\phi=\phi'\quad\text{and}\quad fF=f'F'$$

cannot co-exist, but one must be taken and the other left, we must take

$$\phi=\phi'.$$

In our case, however, we shall find that the solution of

$$\phi=\phi'$$

involves

$$fF=f'F'.$$

12. A solution of this equation

$$\phi=\phi'$$

is obtained by making ϕ a function of the energy only, namely, the ordinary solution

$$\phi=Ke^{-h(u^2+v^2+w^2+U^2+V^2+W^2)}.$$

But, as Mr. Bryan has pointed out, in his "Report on Thermodynamics," that is not the only solution. And it must be rejected here because it makes the velocities of colliding spheres independent, which is assumed not to be true.

13. The following is another solution, namely,

$$\phi = K\epsilon^{-hQ},$$

in which $Q = A\,(u^2+v^2+w^2+U^2+V^2+W^2) + B\,(uU+vV+wW)$

and K, A, B are constant.

For, using this form, we have, after collision,

$$Q' = A\,(u'^2+v'^2+w'^2+U'^2+V'^2+W'^2) + B\,(u'U'+v'V'+w'W').$$

Now, by the conservation of energy,

$$u'^2+v'^2+w'^2+U'^2+V'^2+W'^2 = u^2+v^2+w^2+U^2+V^2+W^2,$$

and, by conservation of R or R^2,

$$(u'-U')^2+(v'-V')^2+(w'-W')^2 = (u-U)^2+(v-V)^2+(w-W)^2;$$

and therefore also

$$u'U'+v'V'+w'W' = uU+vV+wW;$$

and therefore $$Q' = Q$$

and $$\phi' = \phi.$$

14. Assuming that our function ϕ contains the velocities of two spheres only, we find $f(u, v, w)$ or f by integrating ϕ according to U, V, W between limits $\pm\infty$, and find F, f', and F' in the same way. Whence it is easily seen that

$$\phi = \phi'$$

involves $$fF = f'F',$$

and so the function H found on our hypothesis has all the properties of that function as usually found on the hypothesis that $B = 0$. But *the actual value attained by* H *when minimum* will be a function of B.

15. Now let us consider a more general case, that the velocities, not of two only, but of n, spheres are correlated.

Let us suppose that a certain spherical space S contains n spheres, and that, their positions being unknown, the chance of their having velocities

$$u_1 \dots u_1+du_1 \dots w_n \dots w_n+dw_n$$

is of the form $K\epsilon^{-hQ}$, in which

$$Q = (au_1^2 + bu_1u_2 + au_2^2 + bu_1u_3 + bu_2u_3 + \&c.),$$

and K, a, b are constants.

In order that our system may be permanent, that is, unaffected by collisions, it is necessary and sufficient that when, by collision of any pair of spheres, their velocities u_1, v_1, w_1, u_2, v_2, w_2, become u_1', v_1', &c., all the terms in the index should remain with u_1', v_1', &c., substituted for u_1, v_1, &c., that is, if Q contain au_1^2, Q' must contain $au_1'^2$, and so on.

16. Let us then denote by λ, μ, ν the direction cosines of the line of centres at collision between the two spheres whose velocities are before collision $u_1 v_1 w_1$, $u_2 v_2 w_2$. Their velocities resolved in the line of centres are $\lambda u_1 + \mu v_1 + \nu w_1$ and $\lambda u_2 + \mu v_2 + \nu w_2$. And these are interchanged by the collision, so that after collision

$$u_1' = u_1 - \lambda(\lambda u_1 + \mu v_1 + \nu w_1) + \lambda(\lambda u_2 + \mu v_2 + \nu w_2),$$

and, to determine $u_1' \ldots w_2'$, we have the six linear equations

$$\begin{aligned}
u_1' &= (1-\lambda^2)\, u_1 - \lambda\mu v_1 - \lambda\nu w_1 + \lambda^2 u_2 + \lambda\mu v_2 + \lambda\nu w_2,\\
v_1' &= -\lambda\mu u_1 + (1-\mu^2)\, v_1 - \mu\nu w_1 + \lambda\mu u_2 + \mu^2 v_2 + \mu\nu w_2,\\
w_1' &= -\lambda\nu u_1 - \mu\nu v_1 + (1-\nu^2)\, w_1 + \lambda\nu u_2 + \mu\nu v_2 + \nu^2 w_2;\\
u_2' &= \lambda^2 u_1 + \lambda\mu v_1 + \lambda\nu w_1 + (1-\lambda^2)\, u_2 - \lambda\mu v_2 - \lambda\nu w_2,\\
v_2' &= \lambda\mu u_1 + \mu^2 v_1 + \mu\nu w_1 - \lambda\mu u_2 + (1-\mu^2)\, v_2 - \mu\nu w_2,\\
w_2' &= \lambda\nu u_1 + \mu\nu v_1 + \nu^2 w_1 - \lambda\nu u_2 - \mu\nu v_2 + (1-\nu^2)\, w_2.
\end{aligned}$$

To solve these equations for u_1, v_1, &c., we have only to interchange the accents between the right and left hand members.

17. If, now, in the expression

$$Q = au_1^2 + bu_1u_2 + au_2^2 + bu_1u_3 + bu_2u_3 + \&c.,$$

we substitute for u_1, u_2, and similarly for v_1, v_2, w_1, w_2, their values in terms of u_1', u_2', &c., we find

(1) The coefficient of $u_1'^2$ is a. That is because u_1^2 and u_2^2 have the same coefficient a in Q.

(2) The coefficient of $u_1'u_2'$ is b, the same as before for u_1u_2.

(3) The coefficient of $u_1'u_3$ is b, the same as for u_1u_3, &c.

The form of the index is therefore unaltered, and the assumed law of distribution is unaffected by collisions.

18. We can now find the chance that two of the n spheres shall have velocities $u_1 \ldots u_1+du_1$, &c., and $u_2 \ldots u_2+du_2$, &c., whatever the velocities of the others may be, by integrating ϕ according to $u_3, v_3 \ldots w_n$ between the limits $\pm\infty$. As no products of the form uv, uw, or vw are supposed to occur in Q, it is sufficient to operate on the u's only.

Form then the determinant of the function Q, that is,

$$D = \begin{vmatrix} 2a, & b, & b, & \ldots & b \\ b, & 2a, & b, & \ldots & b \\ \ldots & \ldots & \ldots & \ldots & \end{vmatrix},$$

in all n^2 constituents. Let D_{11}, D_{12}, &c., be its first minors, and D_{1221} the coaxial minor formed by omitting the first and second rows and columns. Then the result of the integration is

$$\epsilon^{-hA(u_1^2+\ldots+u_2^2+\ldots)+B(u_1u_2+\ldots)},$$

in which $$A = \frac{D_{11}}{D_{1221}}, \quad B = \frac{D_{12}}{D_{1221}}.$$

But, evaluating the determinant, we find

$$D = (2a-b)^n + nb\,(2a-b)^{n-1}.$$

This is easily seen for $n=2$, $n=3$, and can be extended by induction.

Therefore also

$$D_{11} = (2a-b)^{n-1} + (n-1)\,b\,(2a-b)^{n-2},$$

$$D_{1221} = (2a-b)^{n-2} + (n-2)\,b\,(2a-b)^{n-3}.$$

Also we find $$D_{12} = b\,(2a-b)^{n-2}.$$

Therefore $$A = (2a-b)\,\frac{2a+\overline{n-2}\,b}{2a+\overline{n-3}\,b},$$

$$B = \frac{b\,(2a-b)}{2a+\overline{n-3}\,b}.$$

We can now treat the function H as in 13.

19. Again, if we integrate once more, we can find the chance that a single sphere shall have velocities $u \ldots u+du$, &c., in the form

$$K\epsilon^{-h\,D/D_{11}\,(u^2+v^2+w^2)}\,du\,dv\,dw = f(u, v, w)\,du\,dv\,dw,$$

whence also for two colliding spheres

$$fF = f'F'.$$

And from that result we find that H exceeds the value which it has when $b = 0$ by $\frac{1}{2}\log\frac{D}{D_{11}}$, and the function H has for this system all its ordinary properties, becoming minimum in the assumed distribution, and having when minimum the last stated value.

20. We must consider further the coefficients a and b.

The integration in 18 extended over n spheres supposed to be contained in a spherical space S, so that, ρ being the number of spheres in unit of volume, $n = \rho S$. As S becomes very large, the chances for the two spheres, whose positions within S are unknown, having given velocities must approach independence, that is, A becomes constant and B tends to zero. Comparing this with the values found above for A and B, we see that for large values of S (or of n) $2a-b$ is independent of S, and b tends to vanish. That is one condition which a and b have to satisfy. Another can be found as follows.

21. On the equilibrium of a vertical column of gas whose molecules are equal elastic spheres of diameter c.

If in the Clausian equation

$$\tfrac{2}{3}pV = \Sigma\tfrac{1}{2}mv^2 + \tfrac{1}{2}\Sigma\Sigma Rr,$$

we evaluate the virial term $\frac{1}{2}\Sigma\Sigma Rr$, we find it equal in case of our elastic spheres to $\frac{2}{3}\pi c^3\rho \,.\, 2\rho T_r$. Here T_r is the energy of the motion of the spheres in the volume considered relative to their common centre of gravity. (See *Science Progress*, November, 1894.)

$$\text{Let} \quad \tfrac{2}{3}\pi c^3\rho = \kappa = \frac{4 \times \text{aggregate volume of spheres in volume } V}{V}$$

Then we know that p, or the pressure per unit of surface, is equal to $(1+\kappa)\,\frac{2}{3}\rho T_r$.

22. I find now that in a vertical column of gas whose molecules are equal elastic spheres of diameter c it is not T that is constant, as

proved by Maxwell for the case where c (and therefore κ) is negligible, but

$$T+\kappa T_r.$$

(See Appendix.)

Also $T+\kappa T_r = a\,(u_1^2+u_2^2+\ldots+w_n^2)+b\,(u_1u_2+u_1u_3+\ldots+v_1v_2+\&c.)$,

if we make $$2a = 1+\frac{n-1}{n}\kappa,$$

$$b = -\frac{\kappa}{n} = -\tfrac{2}{3}\,\frac{\pi c^3}{S},$$

because $$n = \rho S.$$

The coefficients a and b so found satisfy the condition of 20 above. Also the distribution of velocities according to this law is unaffected by collisions, as shown in 17. Therefore it gives a more general solution of the problem of the motion of elastic spheres than the ordinary one in which the velocities of each sphere are supposed to be independent. Also, with these values of a and b,

$$D = (1+\kappa)^{n-1} \quad \text{and} \quad D_{11} = (1+\kappa)^{n-2}.$$

The value of H found for this system is

$$H = \tfrac{3}{2}\log h+\tfrac{3}{2}\log\frac{D}{D_{11}}+\text{constant}$$

$$= \tfrac{3}{2}\log h+\tfrac{3}{2}\log(1+\kappa)+\text{constant}.$$

The smaller n is, the greater in numerical value is the ratio $\frac{b}{a}$, and therefore the more intense the *correlation*.

23. The above values of a and b are to be regarded as limiting values which a and b assume when S, the space considered, is very large compared with $\frac{2}{3}\pi c^3$. For smaller values of S a correction is required, as follows.

Correction.

The limiting values assumed for a and b were

$$\left.\begin{aligned} 2a &= 1+\frac{n-1}{n}\kappa \\ b &= -\frac{\kappa}{n} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The corrected values are

$$\left.\begin{aligned} 2a &= \frac{n+\kappa}{n}\left(1+\frac{n-1}{n}\kappa\right) \\ b &= -\frac{n+\kappa}{n}\frac{\kappa}{n} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which also satisfy the condition of 20.

I assume the chance for a (spherical) group of n contiguous spheres, whose positions within the sphere are unknown, having velocities infinitely near to $u_1, v_1, \ldots w_n$, &c., to be $C\epsilon^{-hQ}\,du_1 \ldots$ with

$$Q = a\Sigma\,(u^2+v^2+w^2)+b\Sigma\Sigma\,(uu'+vv'+ww').$$

Now, if this be true, the chance for any single sphere having velocity in x, $u_1 \ldots u_1+du_1$ is found by integrating $\epsilon^{-hQ}\,du_1 \ldots dw_n$ for all the variables except u_1 between $\pm\infty$, and comes out in the form

$$\epsilon^{-h(D/D_{11})u_1^2}\,du_1,$$

in which

$$D = \begin{vmatrix} 2a, & b, & b & \ldots \\ b, & 2a, & b & \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix},$$

and D_{11} is its first coaxial minor.

Now, to be consistent, this chance must be the same, whether we regard the single sphere as a member of a group of n, or as a member of a group of $2n$, &c., so long at least as ρ or $\frac{n}{S}$ is constant. Therefore $\frac{D}{D_{11}}$ must for all values of n be independent of n, except as it appears in $\frac{n}{S}$. But, with the values (1) of a and b,

$$\frac{D}{D_{11}} = (1+\kappa)\frac{n}{n+\kappa}.$$

Therefore, with the values (2) of a and b,

$$\frac{D}{D_{11}} = (1+\kappa) = 1+\frac{ns}{S},$$

if

$$s = \tfrac{2}{3}\pi\sigma^3,$$

and that is the solution.

24. With these values (2) of a and b, the index becomes

$$h\frac{n+\kappa}{n}(T+\kappa T_r),$$

or

$$h\frac{S+s}{S}(T+\kappa T_r),$$

which becomes in limit, when $\frac{S}{s}$ is large, $T+\kappa T_r$ whatever the density, or $\frac{n}{S}$, may be. Without affecting the above results, we may by a further small correction of $2a$ and b cause the determinant to vanish when the density $\frac{n}{S}$ exceeds a certain point, beyond which point therefore the formulæ may cease to be applicable.

Appendix.

To prove the above stated result for the vertical column.

1. If p be the pressure per unit of surface, x the height of a point in the column above a fixed plane, f the vertical force, m the mass of a sphere, ρ the density, we have

$$\frac{dp}{dx} = -mf\rho,$$

also

$$p = \tfrac{2}{3}(1+\kappa)\rho T_r.$$

Here

$$\kappa = \tfrac{2}{3}\pi c^3\rho,$$

and T_r is kinetic energy of relative motion; whence, if we make

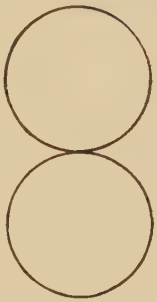
$$(1+\kappa)T_r = \text{constant} = \frac{3}{2h},$$

we find

$$\rho = \rho_0 \epsilon^{-hmfx},$$

which is the same equation as found for the ordinary case when $c=0$ and $\kappa=0$.

2. Again, consider N spheres crossing the plane $x=0$ with u for vertical component of velocity. Of these some, say $N-N'$, will reach the plane $x=dx$ without collision. N' will undergo collision before reaching dx. But for these N' there will be substituted, as the result of collisions, N' other spheres with the same vertical component u.



Now, if the impact were direct, *i.e.*, the line of centres at collision vertical, the substituted sphere would gain a vertical height c, *i.e.*, the diameter of a sphere, without losing in respect of that distance any kinetic energy to the force f. This is a consequence of the fundamental assumption of instantaneous impacts, for which I am not responsible. The conservation of energy is not affected, because whatever kinetic energy one sphere gains the other loses. If, therefore, all the N' collisions were direct, the average height of the N spheres at the end of the time $\frac{dx}{u}$ would be, not dx, but $dx + \frac{N'}{N}c$, while their loss of kinetic energy would be $Nmfdx$.

3. But all impacts will not be direct; we must consider then the result of indirect impacts. For this purpose consider two classes of collisions, in one of which the sphere A has vertical component u before collision, and in the other A' has vertical component u after collision. The effect of a pair of collisions, one from each class, is to substitute A' for A as the sphere with vertical component u. Now let l denote the vector line of centres at collision, and $\cos(ul)$ the cosine of the angle which l makes with the vertical. Then in the first of the pair of collisions the centre of A is below the point of contact by $\frac{1}{2}c\cos(ul)$. In the second, the centre of A' is above the point of contact by $\frac{1}{2}c\cos(ul)$. There is no reason why the point of contact should be higher or lower in one case than in the other. It will be on average at the same height. Therefore on average of all pairs of collisions substituting A' for A with vertical velocity u, A' is above A by

$$c\,\overline{\cos(ul)} = r, \text{ suppose.}$$

Let q be the relative velocity of the two colliding spheres. Then

$$\begin{aligned} r &= c\,\overline{\cos(ul)} \\ &= c\,\overline{\cos(uq)}\;\overline{\cos(ql)} \\ &= \tfrac{2}{3}c\,\overline{\cos(uq)}, \end{aligned}$$

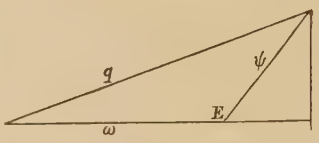
because $$\overline{\cos(ql)} = \frac{\int \cos^2\theta \sin\theta\, d\theta}{\int \cos\theta \sin\theta\, d\theta} = \tfrac{2}{3}.$$

Let ω be the absolute velocity of the sphere whose vertical component velocity is u, so that

$$\cos(u\omega) = \frac{u}{\omega}.$$

Then
$$r = \tfrac{2}{3}c\,\overline{\cos(uq)}$$
$$= \tfrac{2}{3}c\,\frac{u}{\omega}\,\overline{\cos(\omega q)}.$$

Let ψ be the velocity of the other colliding sphere, E the angle between ω and ψ. Then



$$r = \tfrac{2}{3}c\,\frac{u}{\omega}\,\frac{\omega-\psi\cos E}{q}.$$

We have to multiply this by the number of collisions which N spheres having velocity ω undergo with spheres of velocity $\psi \ldots \psi+d\psi$, making with ω angles $E \ldots E+dE$ in time dt, or $\frac{dx}{u}$, and then integrate for all values of ψ and E.

Let $\rho f(\psi)\,d\psi$ be the number of spheres in unit volume with velocity $\psi \ldots \psi+d\psi$. The result is

$$N\pi c^2\rho\int_0^\infty d\psi\, f(\psi)\int_0^\pi \tfrac{1}{2}\sin E\,dE\,q\,\tfrac{2}{3}c\,\frac{u}{\omega}\,\frac{\omega-\psi\cos E}{q}\,\frac{dx}{u}$$
$$= \tfrac{2}{3}\pi c^3\rho\,.\,N dx$$
$$= \kappa N dx.$$

Therefore at time dt the average height of the N spheres or their successors above the plane $x=0$ is $(1+\kappa)\,dx$.

But the energy which they lose in the ascent is $Nmfdx$. The loss takes place only during free path. It follows that the loss of energy due to the ascent dx is, allowing for substitutions, $\frac{mfdx}{1+\kappa}$ per sphere.

4. Now suppose that at $x=0$ the number per unit of volume of spheres having $\frac{1}{2}mu^2 \ldots d(u^2)$ for energy of vertical velocity to be

$$K\epsilon^{-h(1+\kappa)mu^2}\,d(u^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Then, by what has been proved in (2), the number which at height dx have $\frac{1}{2}mu^2 \ldots d(u^2)$ for energy of vertical velocity is

$$\epsilon^{-hmfdx}\,K\epsilon^{-h(1+\kappa)mu^2}\,d(u^2),$$

and the number which at height dx have

$$\tfrac{1}{2}mu^2 \dots d(u^2) - \frac{mfdx}{1+\kappa}$$

for energy of vertical velocity is

$$\epsilon^{-hmfdx} K \epsilon^{-h(1+\kappa)[mu^2-(mfdx)/(1+\kappa)]} d(u^2) = K\epsilon^{-h\overline{1+\kappa}mu^2} d(u^2) \dots\dots (2).$$

The two groups (1) and (2) are equally numerous, and therefore either can by ascending or descending, allowing for substitutions, exactly replace the other. Now this is the reasoning by which in the ordinary case, when $\kappa = 0$, we prove T to be constant. It now proves $(1+\kappa)\, T_r$ to be constant.

5. Further, any group of n spheres will generally have some energy of motion of their common centre of gravity, or, as Natanson calls it, apparent motion, of which we have as yet taken no account. Call this T_s. Then T_s is independent of x for the same reason that when $c = 0$ T is independent of x. Therefore $T_s + \overline{1+\kappa}\, T_r$ or $T + \kappa T_r$ is independent of x.

6. The pressure per unit of surface is increased in the ratio $1 : 1+\kappa$ as the molecules, from being material points, become spheres with finite diameter c.

But the pressure per unit of surface is the quantity of momentum which is carried through unit of surface in unit of time (Watson, *Kinetic Theory of Gases*). Now, so far as this momentum is carried through the surface by molecules *during their free path*, it is not altered in the least by κ acquiring finite value. The increase of the transfer of momentum consists wholly in the process above explained, namely, the instantaneous transfer of momentum through the distance c which occurs on collision.

Applications of Trigraphy. *By* J. W. Russell, M.A. Received May 8th, 1895. Communicated May 9th, 1895.

[*Note.*—The author obtained the properties mentioned in §§ 1–11 by the method of the following paper before he was aware that theorems, to a great extent equivalent, had been investigated by August, starting from trispective ("relatio duploprojectiva ac perspectiva") in his *Disquisitiones de superficiebus tertii ordinis*. In consequence, this preliminary matter has been put as briefly as possible.]

Definition of Trigraphic Ranges.

1. Take three lines l, l', l'' (called the bases) situated in any manner in space. Let x, x', x'' be the distances measured on l, l', l'' from any origins to X, X', X''. Then X, X', X'' are said to generate trigraphic ranges if x, x', x'' satisfy the relation

$$xx'x''+b_1x'x''+b_2x''x+b_3xx'+c_1x+c_2x'+c_3x''+d=0$$

(called the relation of the trigraphy), where b_1, b_2, b_3, c_1, c_2, c_3, d are seven constants.

If, instead of $x = OX$, we take $k = AX/XB$, where A and B are fixed origins, and similarly for X' and X'', then k, k', k'' satisfy a relation of the same form; for $x = bk/(1+k)$ if O is taken at A.

Again, if we take $t = AX/XB \div AC/CB$, where C is a third fixed point, and so on, the form of the relation is not altered. For $t = k/k_0$, where $k_0 = AC/CB$.

We immediately draw the following conclusions:—

If any two of the three points X, X', X'' are given, the third is *generally* known (but see § 3).

If one of them, say X, is fixed, then the other two generate homographic ranges.

If we replace each of the ranges by a homographic range, we obtain three trigraphic ranges. In fact, if we use the third form of the trigraphic relation, the relation is not altered, for t is a cross ratio. We may say that two such trigraphies are homographic. We see that a trigraphy projects into a homographic trigraphy.

The Vanishing Points.

2. If we take x, x', x'' as the variables, and if X', X'' are at infinity, then $X = I$, say, is given by $x+b_1 = 0$. So the other vanishing points I', I'' are given by $x'+b_2 = 0$ and $x''+b_3 = 0$.

Notice that I, I', I'' are not a triad of corresponding points.

The Vague Points.

3. Choosing the vanishing points as origins, the trigraphic relation is

$$xx'x''+c_1x+c_2x'+c_3x''+d = 0.$$

Now, if x' and x'' satisfy both the relations

$$x'x''+c_1 = 0 \quad \text{and} \quad c_2x'+c_3x''+d = 0,$$

x may have any value. The solutions of these equations are

$$\begin{cases} x' = (-d+s)/2c_2 = u', \text{ say, giving } X' = U', \text{ say,} \\ x'' = (-d-s)/2c_3 = v'', \text{ say, giving } X'' = V'', \text{ say,} \end{cases}$$

$$\begin{cases} x' = (-d-s)/2c_2 = v', \text{ say, giving } X' = V', \text{ say,} \\ x'' = (-d+s)/2c_3 = u'', \text{ say, giving } X'' = U'', \text{ say,} \end{cases}$$

where $$s^2 = d^2+4c_1c_2c_3.$$

Hence, if $X' = U'$, $X'' = V''$, or if $X' = V'$, $X'' = U''$, X may be anywhere.

Similarly, if X' is indefinite, we have

$$\begin{cases} x'' = (-d-s)/2c_3 = v'', & \text{giving } X'' = V'' \text{ again,} \\ x \;\; = (-d+s)/2c_1 = u, \text{ say,} & \text{giving } X = U, \text{ say,} \end{cases}$$

$$\begin{cases} x'' = (-d+s)/2c_3 = u'', & \text{giving } X'' = U'' \text{ again,} \\ x \;\; = (-d-s)/2c_1 = v, \text{ say,} & \text{giving } X = V, \text{ say.} \end{cases}$$

Finally, if X'' is indefinite, we have

$$\begin{cases} x = (-d+s)/2c_1 = u, \text{ giving } X = U \text{ again,} \\ x' = (-d-s)/2c_2 = v', \text{ giving } X' = V' \text{ again,} \end{cases}$$

$$\begin{cases} x = (-d-s)/2c_1 = v, \text{ giving } X = V \text{ again,} \\ x' = (-d+s)/2c_2 = u', \text{ giving } X' = U' \text{ again.} \end{cases}$$

Let us call the six points U, V, U', V', U'', V'', so determined, the vague points. Their properties can be briefly stated by saying that the following are triads, viz., $(U, V', ?)$, $(V, U', ?)$, $(U, ?, V'')$,

$(V, ?, U'')$, $(?, U', V'')$, $(?, V', U'')$. We can take any U with any V; but not a U with a U or a V with a V.

We see that the vague points are all real or all imaginary according as $d^2+4c_1c_2c_3$ is positive or negative.

Notice that U, U', U'' are not a triad, nor are V, V', V''.

In two homographic trigraphies the vague points correspond; for a definite point cannot correspond to an indefinite point.

4. Given the trigraphic relation, to construct the vague points.

Take any point A on l. On l' take any points $P', Q', \ldots$. Keeping X at A, let X'' be at $A_1'', A_2'', \ldots$, when X' is at $P', Q', \ldots$. Then $(P'Q'\ldots) = (A_1''A_2''\ldots)$. Similarly, replacing A by B, we get

$$(P'Q'\ldots) = (B_1''B_2''\ldots).$$

Hence $$(A_1''A_2''\ldots) = (B_1''B_2''\ldots).$$

Now (A, U', V''), (A, V', U''), (B, U', V''), (B, V', U'') are triads. Hence we have

$$(V''U''A_1''A_2''\ldots) = (V''U''B_1''B_2''\ldots).$$

Hence U'', V'' are known as the common points of two homographic ranges.

5. We can throw the trigraphic relation into the form

$$(UV, XA)(U'V', X'A')(U''V'', X''A'') = 1,$$

where A, A', A'' are particular positions of X, X', X''.

For, choosing the third form of the relation, viz.,

$$tt't''+b_1t't''+b_2t''t+b_3tt'+c_1t+c_2t'+c_3t''+d=0,$$

let X be at V and X' at U'; then X'' must be vague. Now

$$t = UX/XV \div UA/AV = \infty \quad \text{and} \quad t' = 0.$$

Hence t'' is given by $$b_2t''+c_1=0.$$

Hence $b_2=0$ and $c_1=0$. Similarly, U, V'' give $b_1=0$, $c_3=0$; and V', U'' give $b_3=0$, $c_2=0$. Also $X=A$, $X'=A'$, $X''=A''$ give $d=1$. Hence $tt't''=1$, as enunciated.

An equivalent statement is that $UX/XV \,.\, U'X'/X'V' \,.\, U''X''/X''V''$ is a constant (which we may call the constant of the trigraphy). If we take X' and X'' at infinity, then X is at I, and the above constant is UI/IV; so it is $U'I'/I'V'$, and also $U''I''/I''V''$.

As two simple examples of the above theorem, we see that a variable line cuts the side of a triangle ABC in points which generate

trigraphic ranges, A being V' and U'', B being V'' and U, and C being V and U'; also that the lines joining A, B, C to a variable point meet the opposite sides in points which also generate trigraphic ranges, the vague points being in the same positions as before.

Trispective.

6. If a variable plane through a fixed point O cuts three given lines l, l', l'' in points X, X', X'', and if the plane Ol cuts l', l'' in U', V'', and if Ol' cuts l'', l in U'', V, and if Ol'' cuts l, l' in U, V', then X, X', X'' generate trigraphic ranges of which U, V, U', V', U'', V'' are the vague points.

It will be sufficient to prove that

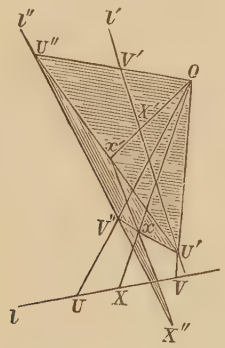
$$(UV, AX) \cdot (U'V', A'X') \cdot (U''V'', A''X'') = 1,$$

where A, A', A'' are particular positions of X, X', X''. Let OX, OA meet $V''U'$ in the points x, a, and let OX', OA' meet $U''U'$ in x', a'. Then

$$(UV, AX) = (V''U', ax),$$

and

$$(U'V', A'X') = (U'U'', a'x').$$



Hence we have to prove that

$$(V''U', ax) \cdot (U'U'', a', x') \cdot (U''V'', A''X'') = 1.$$

Now $xx'X''$ and $aa'A''$ are the lines of intersection of the plane $V''U'U''$ with the planes $OXX'X''$ and $OAA'A''$. Hence

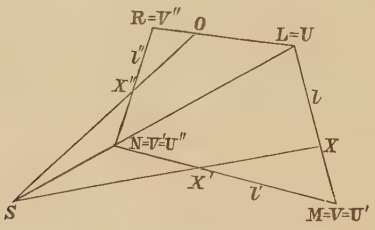
$$\frac{V''a}{aU'} \frac{U'a'}{a'U''} \frac{U''A''}{A''V''} = -1 = \frac{V''x}{xU'} \frac{U'x'}{x'U''} \frac{U''X''}{X''V''}.$$

Hence the required result is true.

Three ranges situated as above may be said to be in trispective, O being the centre of trispective.

7. Conversely, any three trigraphic ranges can be placed so as to be in trispective.

For construct a gauche quadrilateral by placing U at any point L, V and U' at M, V' and U'' at N, and V'' at R. Let the plane $AA'A''$



meet the line RL at O. Then the trigraphy determined on LM, MN, NR by planes through O is the given trigraphy; for the vague points and the triad (A, A', A'') are the same.

Notice that the above construction becomes imaginary if the vague points are imaginary. In fact, in this case, there can be no real position of O; for, if there were, the vague points would be real. Hence the disadvantage of starting with trispective in discussing trigraphy.

Given the vague points and one triad (A, A', A''), to construct the point X'' corresponding to X, X', we proceed as above to construct O; then the plane OXX' cuts RN in X''. Practically we should join O to the point S, in which XX' cuts LN.

Trigraphic Pencils.

8. Three axial pencils are said to be trigraphic if any three lines cut them in three trigraphic ranges.

Let the lines meet the planes $(a, b, c, p, \ldots)$, $(a', b', c', p', \ldots)$, $(a'', b'', c'', p'', \ldots)$ of the pencils in the points $(A, B, C, P, \ldots)$, and so on. Then, if (ab, pc) represents a cross ratio of the planes a, b, p, c, we have

$$(ab, pc) = (AB, PC).$$

Hence, if

$$tt't'' + \ldots + d = 0$$

is the trigraphic relation of the ranges, it is also the relation of the pencils if

$$t = (ab, pc),$$

and so on.

Let the equations of the plane p referred to the planes a and b be

$$\alpha = k\beta,$$

and so on. Then

$$t = \sin ap/\sin pb \div \sin ac/\sin cb = k/k_0, \text{ say,}$$

where $$k = \sin ap/\sin pb.$$

Substituting, we get a more convenient form of the trigraphic relation of the pencils generated by

$$\alpha = k\beta, \quad \alpha' = k'\beta', \quad \alpha'' = k''\beta'',$$

viz., $$kk'k'' + b_1k'k'' + \ldots + c_1k + \ldots + d = 0.$$

Conversely, any three lines cut three trigraphic pencils in three ranges which are trigraphic, and may be said to be homographic with the pencils.

9. The properties of trigraphic pencils now follow obviously from those of ranges. For instance :—

If we replace each pencil by a homographic pencil, we obtain trigraphic pencils.

Two vague planes can be drawn through each axis with properties analogous to those of the vague points.

But there are clearly no planes corresponding to the vanishing points.

If we take the vague planes as planes of reference, the trigraphic relation is $\alpha\alpha'\alpha'' = \lambda\beta\beta'\beta''$, where λ is a constant.

If a triad of trigraphic pencils is homographic with a triad of ranges, the vague planes are the homographs of the vague points.

The reciprocal of a triad of trigraphic pencils is a (homographic) triad of trigraphic ranges ; and the reciprocals of the vague planes are the vague points.

Three pencils are said to be in trispective if every three corresponding planes meet on a fixed plane. By reciprocating §§ 5 and 6, we see that pencils are trigraphic if they are in trispective, and that trigraphic pencils can be placed so as to be in trispective.

We may define trigraphic pencils of rays in a plane as those which are cut by three lines in trigraphic ranges ; and properties analogous to those of axial pencils hold.

Trigraphic Properties of a Quadric Surface.

10. If A, B, C are fixed points on a quadric, all the points on the quadric subtend at the axes AB, BC, CA pencils which are trigraphic.

The equation of the quadric referred to the tetrahedron $ABCD$ (where D is any fourth fixed point on the quadric) is

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta + \delta(f\alpha + g\beta + h\gamma) = 0.$$

Let the equations of the planes joining the point P on the quadric to BC, CA, AB be $\alpha = k\delta$, $\beta = k'\delta$, $\gamma = k''\delta$. Then

$$lk'k'' + mk''k + nkk' + fk + gk' + hk'' = 0.$$

Hence the pencils are trigraphic.

Also, since the relation is satisfied by $k = \infty$, $k' = \infty$, $k'' = \infty$, we see that ABC is a self-corresponding plane.

Conversely, the locus of the intersection of corresponding planes of three trigraphic pencils whose axes are AB, BC, CA is a quadric through A, B, C, if ABC is a self-corresponding plane of the trigraphy.

By reciprocation, we see that a variable tangent plane of a quadric generates trigraphic ranges on the three intersections of three fixed tangent planes; and, conversely, the planes joining corresponding points of three trigraphic ranges whose bases meet in a self-corresponding point touch a quadric which touches the planes formed by the pairs of bases.

The above method of constructing a quadric geometrically has an advantage over the homographic method in that it applies to quadrics which have not real rectilinear generators.

Trigraphic Property of a Cubic Surface.

11. The geometrical interpretation of the equation

$$\alpha\alpha'\alpha'' = \lambda\beta\beta'\beta''$$

is clearly that corresponding planes of three trigraphic axial pencils intersect on the surface

$$\alpha\alpha'\alpha'' = \lambda\beta\beta'\beta''$$

of the third order, the axes of the pencils being rectilinear generators of the surface. Conversely, all the points on a surface of the third order subtend at any three of the generators of the surface pencils which are trigraphic; for we can throw the equation of the surface into the above form.

Reciprocating, we get a property of a surface of the third class.

Trigraphic Interpretation of the Solution of a Cubic Equation.

12. Let the cubic equation be

$$ax^3+3bx^2+3cx+d=0.$$

Consider the symmetrical trigraphy on the same line given by

$$axx'x''+bx'x''+bx''x+bxx'+cx+cx'+cx''+d=0.$$

When $x=x'=x''$, we get the triple points of this trigraphy. Hence these triple points are the graphs of the roots of the equation. The trigraphic relation is also

$$UX/XV.\ UX'/X'V.\ UX''/X''V=UI/IV.$$

Hence the triple points are given by

$$UX/XV=\sqrt[3]{UI/IV}.$$

Hence the three positions of X are known, and the equations can be solved geometrically by a real construction if it has one real root only.

It is an interesting fact that the vague points are the graphs of the auxiliary quadratic, *i.e.*, of the Hessian. For the vague points are given by

$$ax'x''+bx''+bx'+c=0 \quad \text{and} \quad bx'x''+cx'+cx''+d=0.$$

Eliminating x', we get

$$(ac-b^2)\,x''^2+(ad-bc)\,x'+bd-c^2=0.$$

Constructions, given Seven Triads.

13. Given seven triads of corresponding points of three trigraphic ranges, to construct the point corresponding to two given points.

Let the seven triads be (1, 2, ... 7), (1′, 2′, ..., 7′), (1″, 2″, ... 7″); and let 8″ be the point corresponding to the given points 8, 8′. For brevity denote the line 1′1″ by l, 1″1 by l', and 11′ by l''. Then the pencils l (12 ... 78), l' (1′2′ ... 7′8′), l'' (1″2″ ... 7″8″) are trigraphic. Let the planes $l2$, $l'2'$, $l''2''$ meet in P_2, and so on. Then, since 11′1″ is a self-corresponding plane of the three pencils, the points 1, 1′, 1″, P_2, ... P_8 lie on the same quadric. To get 8″ therefore, we construct* the point P_8 in which the line of intersection of $l8$ and $l'8'$

* The problem—given nine points on a quadric, to construct the point in which any line through one of them cuts the quadric again—is solved by Hesse in volume XXIV. of *Crelle's Journal.*

meets again the quadric drawn through the nine points $1, 1', 1'', P_2, \ldots P_7$; then $8''$ lies on the plane $l''P_8$.

To construct the vague points, we notice that $11'U''$ and $1'1''V$ must be vague planes of the pencils. Hence their intersection must meet the quadric in an indefinite point. Hence they must be drawn through one of the rectilinear generators through $1'$. So for the other vague points.

To construct the vanishing points, we draw $11'X''$ parallel to l'' and $1'1''X$ parallel to l.

To construct the trigraphic relation—

Take two triads (A, A', A''), (B, B', B'') of corresponding points as double origins. The relation then is

$$k'k'' + b_2k''k + b_3kk' + c_1k + c_2k' + c_3k'' = 0.$$

Take X at A, and X' at B', and construct X'' at C'', say. Then $k = 0$, $k' = \infty$, $-c_2 = k'' = A''C''/C''B''$. Hence c_2 is known. So $k = 0$, $k'' = \infty$ gives c_3. Then $k' = 0$, $k'' = \infty$ gives b_2/c_3, and therefore b_2; and $k'' = 0$, $k' = \infty$ gives b_3/c_2, and therefore b_3. Then $k'' = 0$, $k = \infty$ gives b_3/c_1, and therefore c_1.

If the ranges are in the same plane, we first project the ranges on to lines which are not in the same plane, and finally project back again.

If pencils are given, we first consider sections of these pencils.

Plane Cubic Curves.

14. If we take three trigraphic pencils in the same plane, three corresponding rays will not generally meet in a point; but certain triads will be concurrent. The locus of the intersection of these triads is a curve of the third order which passes through the three vertices of the pencils. For let the trigraphic relation be

$$akk'k'' + b_1k'k'' + b_2k''k + b_3kk' + c_1k + c_2k' + c_3k'' + d = 0,$$

and let $\alpha = k\beta$, $\alpha' = k'\beta'$, $\alpha'' = k''\beta''$ be a concurrent triad. Then, substituting for k, k', k'', we get the locus of the intersection, viz.,

$$a\alpha\alpha'\alpha'' + b_1\alpha'\alpha''\beta + \ldots + c_1\alpha\beta'\beta'' + \ldots + d\beta\beta'\beta'' = 0,$$

which is as stated.

Conversely, all the points on a curve of the third order subtend at any three points on the curve three pencils which belong to two trigraphies, one of which merely expresses the fact that the three lines are concurrent.

Take the three points as vertices of reference. The cubic is

$$b_1\alpha\gamma^2+b_2\beta\alpha^2+b_3\gamma\beta^2+c_1\alpha\beta^2+c_2\beta\gamma^2+c_3\gamma\alpha^2=\alpha\beta\gamma.$$

Let the rays be $\beta=k\gamma$, $\gamma=k'\alpha$, $\alpha=k''\beta$. Since these are concurrent, we have

$$kk'k''=1,$$

the first trigraphy. Now writing the equation of the curve in the form

$$b_1\frac{\gamma}{\beta}+b_2\frac{\alpha}{\gamma}+b_3\frac{\beta}{\alpha}+c_1\frac{\beta}{\gamma}+c_2\frac{\gamma}{\alpha}+c_3\frac{\alpha}{\beta}=1,$$

and noticing that $\frac{\gamma}{\beta}=k'k''$, and so on, we get

$$b_1k'k''+b_2k''k+b_3kk'+c_1k+c_2k'+c_3k''=1,$$

the second trigraphy.

Notice that the subtended pencils belong also to every trigraphy given by the relation

$$b_1k'k''+\ldots+c_1k+\ldots-1+\mu\,(kk'k''-1)=0,$$

where μ is an arbitrary constant.

Taking the vague lines of any one of these trigraphies as lines of reference, the trigraphic relation becomes

$$kk'k''=\lambda;$$

and the equation of the cubic is therefore

$$uu'u''=\lambda vv'v''.$$

This gives a geometrical interpretation of the lines $u=0$, &c., in this form of the equation of a cubic.

The following are particular cases of the above theorems:—

If the vertices of the pencils lie on a line which is a self-corresponding ray of the pencils, the locus breaks up into this ray and a conic. For every point on this ray is on the locus.

All the points on a conic subtend at any three points, one of which is on the conic, pencils which belong to two trigraphies. Here the rest of the cubic is the line joining the other two points.

Construction of a Cubic Curve.

15. If nine points A, B, C, 1, 2, ..., 6 are given on a cubic, we can take A, B, C as vertices; then 1, 2, ..., 6 with the other points on the cubic subtend trigraphic pencils at A, B, C. But 1, 2, ..., 6 do

not determine the trigraphy. In fact we must give some definite value to μ. The most convenient value is $\mu = 0$. Then $k = \infty$, $k' = \infty$, $k'' = \infty$ satisfy the relation; hence AB, BC, CA are corresponding rays. A trigraphy to which AP, BP, CP belong is now completely defined, viz., by $A(B12 \ldots 6)$, $B(C12 \ldots 6)$, and $C(A12 \ldots 6)$; and, if AP, BP, CP belong to this trigraphy, P is on the cubic.

Given nine points on a cubic, to construct the tangent at any one of them.

The tangent at A is the ray corresponding to BA and CA in the above pencils.

Given nine points on a cubic, to construct the point in which the line joining two of them cuts the cubic again.

The chord AB cuts the cubic again in the point R, if CR corresponds to AB and BA.

Given nine points on a cubic, to construct the points in which any line through one of them cuts the cubic again.

Let the line be AD through A. Let AD, BY, CZ be three corresponding rays, Y and Z being on AD. Then, since AD is fixed, BY and CZ generate homographic pencils. The required points are the common points of the homographic ranges generated by Y and Z.

The last construction is a solution of the problem—Given nine points on a cubic, to construct the cubic.

For, by constructing the other points on every line through A, we find every point on the curve.

Notice that—To construct the points in which a given line cuts a cubic given by nine points—is the same problem as that of finding the triple points of the ranges determined on the line by the above trigraphic pencils.

Twisted Curves.

16. We know the projective property of a twisted cubic, viz., that all the points on it subtend, at any three chords, three homographic pencils.

There are two kinds of quartic curves, viz., the complete intersection of two quadric surfaces and the partial intersection of a quadric and of a cubic surface.

Consider the complete intersection of two quadrics. Take any three points A, B, C upon it. Then any point P on the quartic determines at AB, BC, CA pencils which belong to two trigraphies, ABC being a self-corresponding plane in each.

Consider the partial intersection of a quadric and a cubic. We know that the other portion of the complete intersection is two common rectilinear generators, say a and a'. Take any third generator a'' of the cubic, not in the same plane with a or a'. Then the quartic is such that the planes Pa, Pa' generate homographic pencils, and the planes Pa, Pa', Pa'' generate trigraphic pencils.

In the same way, the other twisted curves which are the intersections of a quadric and a cubic, or of a cubic and a cubic, may be discussed.

General Case.

17. If the n planes $\alpha_1 = k_1\beta_1$, $\alpha_2 = k_2\beta_2$, ... $\alpha_n = k_n\beta_n$ which pass through fixed axes, and are referred to fixed planes, are such that

$$(a, b_1, b_2, \ldots)(k_1+1)(k_2+1)\ldots(k_n+1) = 0,$$

the constants a, b_1, b_2, ... being put in after the multiplication $(k_1+1)(k_2+1)$ ... has been performed, then the planes are said to generate an n-graphy.

The general proposition is—The locus of the intersection of concurrent corresponding planes of an n-graphy is a surface of the n^{th} order which passes through the n axes. In fact, the equation of the locus is

$$(a, b_1, b_2 \ldots)(\alpha_1+\beta_1)(\alpha_2+\beta_2)\ldots(\alpha_n+\beta_n) = 0.$$

A similar definition and proposition apply to plane pencils, and give a curve of the n^{th} order.

The locus of P when Pa_1, Pa_2, ..., Pa_m belong to an m-graphy, and Pb_1, Pb_2, ..., Pb_n belong to an n-graphy, is a twisted curve of the mn^{th} order; being the intersection of surfaces of the m^{th} and n^{th} orders.

Notice that all the matter from § 14 onwards can be duplicated by reciprocation.

The Reciprocators of Two Conics discussed Geometrically. *By* J. W. Russell, M.A. Received May 8th, 1895. Communicated May 9th, 1895.

General Construction.

1. If two conics α and β have a common pole and polar, we can construct a conic Γ with respect to which they are reciprocal in the following way:—

Let U be the common pole; and let the common polar u cut α in A, A', and β in B, B'. Take X, X' the double points of the involution $(AB, A'B')$. Let any tangent q of α cut AA' in N. Let the line joining U to the fourth harmonic N' of N for XX' cut β in Q. Let QX cut q in M. Take R so that (QM, XR) is harmonic. Now take Γ as the conic touching UX at X, UX' at X', and passing through R.

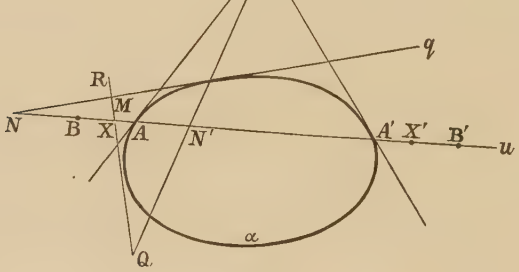


Fig. 1.

Then AA' is the reciprocal of U. Hence the reciprocal of B passes through U; and also through A, since (BA, XX') is harmonic. Hence UA is the reciprocal of B; so UB of A, UA' of B' and UB' of A'. Hence the reciprocal of α is a conic which touches UB at B, and UB' at B'. Now, since (NN', XX') is harmonic, the reciprocal of N is UQ; hence the reciprocal of Q passes through N; and also through M since (QM, XR) is harmonic. Hence q is the reciprocal of Q. Hence the reciprocal of α also passes through Q, and is therefore β.

2. It is convenient to call Γ a reciprocator of α and β.

Since we might have taken X, X' as the double points of the invo-

lution $(AB', A'B)$, and also have taken Q as the other point in which UN' meets β, the above construction gives four solutions of the problem if A, B, A', B' are distinct.

Notice that in all cases the reciprocators Γ constructed as above will have a self-conjugate triangle in common with α and β. For let V, W be the double points of the involution (AA', BB'). Now X, X' are the double points of the involution $(AB, A'B')$. Hence (AA', BB', XX') is an involution. Hence V, W, being the double points of this involution, are harmonic with X, X'. Hence V, W are conjugate for α, β, and Γ.

Particular Cases.

3. If the two conics intersect in four distinct points, we may take U at any vertex of the common self-conjugate triangle. Then A, A', B, B' will be distinct, and we get four solutions.

If the two conics touch a at A and have two distinct intersections D, E, there is one position of U, viz., the intersection of a with DE; and the common polar u of U is the line joining A to the intersection

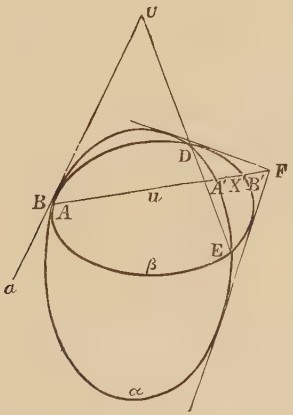


Fig. 2.

of the distinct common tangents. Here A, B coincide. If we take the involution $(AB, A'B')$, X coincides with A and B, and X' is such that $(AX', A'B')$ is harmonic. We cannot take the involution $(AB', A'B)$; for no two points can be found which are harmonic both with A, A' and with A, B'. Hence there will be two solutions.

If the two conics have double contact, we may take U at the common pole. Then A, B coincide, and so do A', B'. If we take the

involution $(AB, A'B')$, X is at A and X' at A'; and Γ has double contact with α and β. This gives two solutions. If we take the involu-

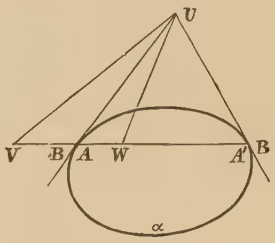


Fig. 3.

tion $(AB', A'B)$, X and X' are any two points which divide AA' harmonically; and we get a doubly infinite set of solutions.

If the two conics have four-point contact, we may take U at any point on the common tangent; for any such point has the same polar

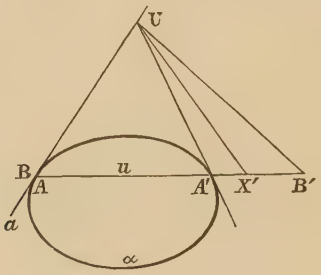


Fig. 4.

for both conics. Hence, as in the previous case, there is apparently a doubly infinite set of solutions (but see § 12).

Exceptional Construction.

4. If the conics have three-point contact and a distinct intersection, the above method fails; for then the conics have no common pole and polar. Let C be the point of contact, and K the distinct intersection. Let the common tangent k meet the tangent c at C in D, meet CK in L, and touch the conics α and β in A and B. Let the tangent b at K to α cut k in M. Take X, the fourth harmonic of C for K, L. Let MX cut KB in N, and take P such that (MN, XP) is harmonic.

Now take as Γ the conic which touches c at C, touches DX at X, and passes through P.

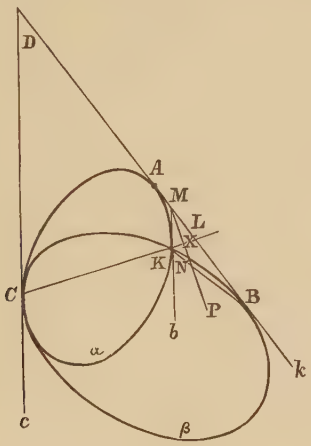


Fig. 5.

Then the reciprocal of D is CX. Hence the reciprocal of K is k for (CX, KL) is harmonic. But K is on α and k touches α; hence k touches and K is on α', the reciprocal of α for Γ. Now M is on k. Hence its reciprocal passes through K; and also through N, since (MN, XP) is harmonic. Hence KN is the reciprocal of M. Hence B is the reciprocal of MK which touches α. Hence B is on α'. Also since α and Γ touch at C, α' touches both at C. Hence α' and β coincide; for each touches c at C, touches k at B, and passes through K.

Notice that in this case Γ has three-point contact with α and β at C. For, if α and Γ have not three-point contact, they have a common pole and polar; hence α and β have a common pole and polar; which is not true.

The above Solution is Exhaustive.

5. We have now shown that in every case there is at least one solution of the problem. Let us next consider whether there are any solutions other than those already found.

Take first the exceptional case in which the conics have three-point contact. Here the conics α and β touch at C only. Hence Γ must touch both α and β at C, for coincident common points must reciprocate into coincident common tangents. Again, the unique intersection K must reciprocate into the unique common tangent k Hence X is known, and then P. Hence there is no other solution.

6. Take the case in which the conics α and β intersect in four distinct points. Then α and Γ must also intersect in four distinct points, for, if α and Γ touch, α and β touch. Hence α and Γ have one and only one common self-conjugate triangle UVW; and this reciprocates into itself and into a triangle which is self-conjugate for β and Γ. Hence UVW must be the common self-conjugate triangle of α and β. Hence α, β, Γ have the same common self-conjugate triangle. Hence the polar of U for Γ must be VW, *i.e.*, we must take the polar of U for α and β as the reciprocal of U. Hence the positions of X, X' are known. Also the reciprocal of q which passes through N and touches α must lie on the reciprocal of N (viz., UN'), and also on β. Hence Q is known, and then R. Hence there are four and only four reciprocators.

Take the case in which α and β touch at A, and intersect in two distinct points D, E. Then the line DE must reciprocate into F, the intersection of distinct common tangents. Also A must reciprocate into a. Hence U must reciprocate into AF.

Take the case in which α and β touch at A and A'. Then A must reciprocate into UA or UA', and A' into UA' or UA. In either case U reciprocates into AA'.

Take, lastly, the case in which the conics have four-point contact at A. Then A must reciprocate into a, the tangent at A. Hence the reciprocal of any point U on a must reciprocate into some line u through A, say $AA'B'$. Then the reciprocal of A' on α and u is the tangent to β from U, *i.e.*, is UB'. Hence u is the polar of U for β, and therefore for α also.

Hence in all cases we have obtained every reciprocator.

Self-reciprocal Conics.

7. Let us next inquire for what conics a given conic is its own reciprocal.

Let the point A on the conic α be the reciprocal of the tangent a of the conic α. Then the tangent a' at A is the reciprocal of the point of contact A' of a. Hence $(AA'BB'\ldots)$ of poles $= (aa'bb'\ldots)$ of polars $= (A'AB'B\ldots)$ of points of contact. Hence $(AA', BB', \ldots)$ is an involution. Hence AA', BB', ... meet in a fixed point, O say. Now let A be one, E, of the points of contact E, F of tangents from O. Then the reciprocal of E is the tangent at E. Hence the required reciprocator Γ touches OE at E. Similarly, Γ touches OF at F. Hence α and Γ have double contact.

Let AE cut $A'F$ in P, and let AF cut $A'E$ in P'. Then P and P' are on Γ. For, if a cut AF in L and EF in G,

$$(AL, FP') = A'(AL, FP') = (HG, FE),$$

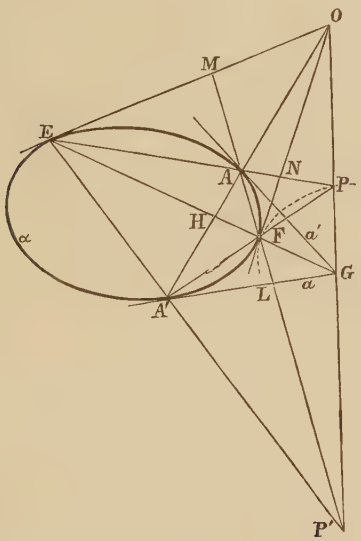


FIG. 6.

which is harmonic, for G is the pole of AA'. Now a is the reciprocal of A, and F is on Γ. Hence P' is on Γ, and similarly P.

We have still to verify that such a conic exists. Describe a conic Γ' touching OE at E, OF at F, and passing through P'. Then the reciprocal of a for this conic touches OE at E, and OF at F. Now, since (AL, FP') is harmonic, the reciprocal of A passes through L. Also AA' is the reciprocal of G, since (GH, FE) is harmonic. Hence the reciprocal of A passes through G. Hence a is the reciprocal of A. Hence the reciprocal of α touches OE at E, OF at F, and also touches a; and hence is α. Hence the reciprocal of a for Γ' is α.

Also the reciprocal of Γ' for a is Γ'. First notice that P is on the conic Γ' by the first part of the above proof. Also that PP' passes through O (and also through G) since PP', EF and AA' are the diagonals of the same quadrilateral. Now Γ' is generated from α by the intersection of EA' and AF, and α is generated from Γ' by the intersection of EP' and PF. Hence Γ' is self-reciprocal for α.

8. Such conics may be called a pair of self-reciprocal conics, each being its own reciprocal for the other. The fundamental property of self-reciprocal conics is that they are in harmonic homology, taking E as pole and OF as axis, or F as pole and OE as axis. For

$$(EN, AP) = O\,(EN, AP) = (EF, HG);$$

so for F. The simplest definition is that they have double contact, and have, at each point of contact, equal and opposite curvatures. For, since (MF, AP') is harmonic, $AF : FP' :: MA : MP' :: OF : OF$, when A and P' coincide with F. Hence $AF = FP'$ ultimately, and similarly $A'F = FP$. And the angles AFA' and $P'FP$ are equal. Hence the triangles AFA' and $P'FP$ are ultimately equal in all respects. Hence the curvatures of α and Γ at F are equal and opposite; and similarly at E. The above proof does not apply if F is at infinity, but in this case the proposition respecting curvature is nugatory.

Connexion between the Reciprocators.

9. Let us now study the connexion between the various conics Γ for which the same two conics α and β are reciprocal.

We notice that the reciprocators always occur in pairs which touch at two points X and X', say, and that another reciprocator cannot touch both of these conics at X, X'. We shall now prove that the two conics Γ_1, Γ_2 forming such a pair are self-reciprocal. For reciprocate for Γ_1. Then α reciprocates into β and β into α. Γ_1 reciprocates into itself. Also Γ_2 reciprocates into a reciprocator; for the proposition that α and β are reciprocal for Γ_2 reciprocates into the proposition that β and α are reciprocal for the reciprocal Γ_2' of Γ_2. Hence Γ_2' is a reciprocator having double contact with Γ_1 at X, X'. And the reciprocal for Γ_1 of no conic except Γ_1 can coincide with Γ_1. Hence Γ_2' coincides with Γ_2, *i.e.*, Γ_1 and Γ_2 are self-reciprocal.

10. If the two conics intersect in four distinct points, then each of the four conics Γ_1, Γ_2, Γ_3, Γ_4 is self-reciprocal for every other. For let X_1, X_1', X_2, X_2' be the double points of the above construction on VW; Y_1, Y_1', Y_2, Y_2' those on WU; and Z_1, Z_1', Z_2, Z_2' those on UV, where UVW is the common self-conjugate triangle of α and β. Let Γ_1, Γ_2 touch UX_1, UX_1' at X_1, X_1', and let Γ_3, Γ_4 touch UX_2, UX_2' at X_2, X_2'. Then Γ_1, Γ_2 are self-reciprocal, and so are Γ_3, Γ_4. Now two of the four touch VY_1, VY_1' at Y_1, Y_1'; but Γ_1, Γ_2 cannot touch again, nor can Γ_3, Γ_4. Hence Γ_1 must touch Γ_3, or Γ_2 touch Γ_4. Then Γ_2 touches Γ_4, or Γ_1 touches Γ_3 at Y_2, Y_2'. Hence Γ_1, Γ_3 are self-reciprocal,

and so are Γ_2, Γ_4. Similarly, Γ_1, Γ_4 touch on UV, and so do Γ_2, Γ_3. Hence Γ_1, Γ_4 are self-reciprocal, and so are Γ_2, Γ_3. Hence every one of the conics Γ_1, Γ_2, Γ_3, Γ_4 is self-reciprocal for every other.

If the two conics touch and intersect in two points, the two conics Γ_1, Γ_2 are self-reciprocal; for they have double contact.

11. If the two conics α and β have double contact at A and A', the two reciprocators Γ_1 and Γ_2 which have double contact with them at A and A' are, as before, self-reciprocal. Let Γ_3 and Γ_4 be the reciprocators touching at X, X', where X, X' are harmonic with A, A'; these also are, as before, self-reciprocal. Take V, W the double points of the involution (AA', XX'). Then V, W are conjugate for α, β, Γ_1, Γ_2, Γ_3, Γ_4. Hence UVW is a self-conjugate triangle for these conics. Also no other reciprocator has UVW as a self-conjugate triangle. For suppose the reciprocators γ_3', γ_4' touching at x, x' have UVW as a self-conjugate triangle. Then V, W are harmonic with x, x', which are harmonic with A, A'. Hence x, x' coincide with X, X'; for these are also harmonic with both A, A' and V, W. Now the four reciprocators obtained by using V or W as pole have also UVW as a self-conjugate triangle (see § 2, end). Hence these conics are the same as Γ_1, Γ_2, Γ_3, Γ_4. Hence, as in § 10, they are self-reciprocal in pairs. Hence we conclude that each of the variable conics Γ_3, Γ_4 has double contact with the fixed conics Γ_1, Γ_2 both on UV and on UW.

12. Finally, if the conics α and β have four-point contact, the reciprocators consist of one conic Γ' and a system of conics self-reciprocal for Γ'. For (see Fig. 4), the conic Γ' having four-point contact with both α and β at A, and passing through X' is a reciprocator. For the tangent at X' is UX', since the tangents at A' and X' to α and Γ' meet on AU; hence AA' is the reciprocal of U. Hence the reciprocal of UA' is B'. Hence the reciprocal of α for Γ' is a conic having four-point contact with β at A, and passing through B', *i.e.*, is β. Also Γ' is one of the reciprocators belonging to AA', for it is a reciprocator and touches UX' at X' and UA at A. And the other is self-reciprocal for Γ'.

13. We can extend the above construction to two quadrics. For instance, take the case in which the two quadrics have a common pole and polar, U and u. Let the quadrics be ϕ and ψ, and let their sections by the plane u be α and β. Take any reciprocator γ of the conics α and β. Let any tangent plane q of ϕ cut u in the line n.

Let the line joining U to the pole N' of n for γ cut ψ in the point Q. Let the line joining Q to any point X on γ meet q in M, and take the point R so that (QM, XR) is harmonic. Then a reciprocator of the quadrics ϕ and ψ is the quadric Γ which passes through R and touches the cone joining U to γ along γ.

For let ϕ' be the reciprocal of ϕ for Γ. Then, since ϕ touches the cone joining U to α along α, ϕ' touches the cone joining U to β along β; for U is the reciprocal of u. Now Q lies on UN'. Also N' is the reciprocal of Un. Hence the reciprocal of Q passes through the intersection of u and Un, *i.e.*, through n; and it passes through M, since (QM, XR) is harmonic; hence it is q. But q touches ϕ; hence Q is on ϕ'. Hence ϕ' is ψ.

Notice that, since Q may have two positions, we get twice as many reciprocators of the quadrics as there are of the conics α and β.

By A. E. Jolliffe. Received May 8th, 1895.
Communicated May 9th, 1895.

The following analytical investigation was undertaken with the object of confirming Mr. Russell's statements as to the number of the conics Γ in the different cases, concerning which some doubts had been expressed. For this reason the result in § 4 is obtained by rigorous analysis instead of being assumed as geometrically obvious. The result of § 2 and the existence of the infinite systems in the cases of double and four-point contact were first obtained as below by means of analysis. It is hoped that the additional matter contained in § 7 and onwards may be not without interest.

1. In the case of four distinct intersections, the equations of the two conics may be written

$$S \equiv x^2+y^2+z^2 = 0,$$

$$S' \equiv px^2+qy^2+rz^2 = 0.$$

Take the equation of Γ in the most general form possible, viz.,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0.$$

The reciprocal of S for Γ then has for its equation

$$(ax+hy+gz)^2+(hx+by+fz)^2+(gx+fy+cz)^2=0.$$

If this is S', we must then have

$$a^2+h^2+g^2=p, \quad h^2+b^2+f^2=q, \quad g^2+f^2+c^2=r,$$

$$gh+f(b+c)=0, \quad hf+g(c+a)=0, \quad fg+h(a+b)=0.$$

If f, g, h are none of them zero, we have

$$f^2=(a+c)(a+b), \quad g^2=(b+a)(b+c), \quad h^2=(c+b)(c+a),$$

with the condition

$$fgh+(b+c)(c+a)(a+b)=0.$$

The equation of the reciprocal of S then becomes

$$(a+b+c)^2(x^2+y^2+z^2)=0,$$

which is S again, and hence this solution must be rejected.

If f is zero, but not both g and h, suppose that g is zero. Then

$$a+b=0,$$

and the reciprocal of S is

$$(x^2+y^2)(a^2+h^2)+c^2z^2=0,$$

which has double contact with S, and therefore is not S'.

Hence we must have $f=0$, $g=0$, $h=0$, and Γ may be any one of the four conics

$$\sqrt{p}\,x^2 \pm \sqrt{q}\,x^2 \pm \sqrt{r}\,z^2=0.$$

Hence when two conics have four distinct intersections they are reciprocal for four conics and four only.

(It may be noticed that any one of these four conics is its own reciprocal for any of the other three.)

2. From the preceding analysis it is clear that the most general equation of a conic for which

$$S \equiv x^2+y^2+z^2=0$$

is its own reciprocal is

$$\Gamma \equiv (a+b+c)(x^2+y^2+z^2)-(x\sqrt{b+c}+y\sqrt{c+a}+z\sqrt{a+b})^2=0.$$

Hence Γ has double contact with S. The roots of the equation formed by equating to zero the discriminant of $\kappa S+\Gamma$ are $-(a+b+c)$, $-(a+b+c)$, $+(a+b+c)$, *i.e.*, the double root is equal to the remaining root but of opposite sign.

Now, if two conics touch at a finite point, their equations can be written in the form

$$U \equiv 2y+lx^2+2mxy+ny^2 = 0,$$

$$V \equiv 2y+l'x^2+2m'xy+n'y^2 = 0 \quad \text{(Cartesian coordinates)}.$$

Here, if we equate to zero the discriminant of $\kappa U+V$, the double root is -1, and the single root $-l'/l$. Hence, if these roots are equal but of opposite sign, $l'=-l$, and the conics U and V have equal but opposite curvatures at their point of contact.

Hence two conics are self-reciprocal if they have finite double contact and equal but opposite curvatures at each point of contact.

If the chord of contact is the line at infinity, the equations must evidently be of the form

$$S = \pm c,$$

where S is a homogeneous quadratic function of x and y, and c a constant.

3. If S and S' have double contact, their equations may be written

$$S \equiv x^2+y^2+z^2 = 0,$$

$$S' \equiv x^2+y^2+rz^2 = 0,$$

and from § 1 it is clear that they are reciprocals for the two conics

$$x^2+y^2 \pm \sqrt{r}\, z^2 = 0,$$

and any conic whose equation is of the form

$$a(x^2-y^2)+2hxy \pm \sqrt{r}\, z^2 = 0,$$

where $$a^2+h^2 = 1.$$

This system of conics can be obtained more simply thus: S and S' are by § 1, reciprocals for the four conics

$$x^2 \pm y^2 \pm \sqrt{r}\, z^2 = 0.$$

Now the equations of S and S' may be written in an infinity of ways in the form

$$X^2+Y^2+z^2 = 0,$$

$$X^2+Y^2+rz^2 = 0,$$

and are therefore reciprocals for the conics

$$X^2 \pm Y^2 \pm \sqrt{r}\, z^2 = 0.$$

The two conics $X^2 + Y^2 \pm \sqrt{r}\, z^2 = 0$

are the same as before, but the conics

$$X^2 - Y^2 \pm \sqrt{r}\, z^2 = 0$$

are not, and evidently belong to the system

$$a(x^2 - y^2) \pm \sqrt{r}\, z^2 + 2hxy = 0,$$

where $a^2 + h^2 = 1.$

It may be noticed that

$$x^2 + y^2 \pm \sqrt{r}\, z^2 = 0$$

are each self-reciprocal for the other and any conic of the doubly infinite system.

4. Next consider when two intersections of S and S' coincide. Their equations may always be written in the form

$$S \equiv 2yz + x^2 = 0, \quad S' = 2yz + px^2 + 2qxy + ry^2 = 0.$$

As before, assume

$$\Gamma \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

The reciprocal of S for Γ has for its equation

$$(ax + hy + gz)^2 + 2(hx + by + fz)(gx + fy + cz) = 0,$$

i.e., $x^2(a^2 + 2hg) + y^2(h^2 + 2bf) + z^2(g^2 + 2fc) + 2yz(gh + bc + f^2)$

$$+ 2zx(ag + gf + ch) + 2xy(ah + fh + bg) = 0;$$

therefore $g(a + f) + ch = 0, \quad g^2 + 2fc = 0.$

If neither g, f, c, nor h vanish, the reciprocal becomes

$$(bc + f^2 + gh)(2yz + x^2) + x^2(\overline{a+f}^2 - bc) - 2xy\frac{g}{c}(\overline{a+f}^2 - bc)$$

$$+ y^2\frac{g^2}{c^2}(\overline{a+f}^2 - bc) = 0,$$

which has double contact with S.

Suppose $g = 0$; then $f = 0$ and $c = 0$, and the reciprocal reduces to

$$(ax + hy)^2 = 0;$$

or $f=0$, $h=0$; or $c=0$. If $f=0$, $g=0$, $h=0$, the equation of the reciprocal is

$$a^2x^2+2bcyz=0,$$

which has double contact with S.

If $g=0$, $c=0$, then Γ touches S and S' at their common point, and we can find a, b, f, h.

5. Let O be the point of contact, P, Q the two other intersections, and OR the harmonic conjugate of the common tangent for OP, OQ. Let OR meet S in R, and take as triangle of reference OR, and the tangents to S at O and R. The equations of the conics may then be written

$$S\equiv 2yz+x^2=0,$$

$$S'\equiv 2yz+px^2+qy^2=0.$$

If $p=1$, we have the case of four-point contact.

Let $$\Gamma\equiv 2yz+ax^2+2hxy+by^2=0.$$

Then the reciprocal of S is

$$a^2x^2+2xya\,(h+1)+y^2\,(h^2+2b)+2yz=0;$$

therefore $$a^2=p,\quad h\,(a+1)=0,\quad h^2+2b=q,$$

and we get for Γ the two conics

$$2yz\pm\sqrt{p}\,x^2+\tfrac{1}{2}qy^2=0,$$

unless $p=1$, and then we get the conic

$$2yz+x^2+\tfrac{1}{2}q^2=0,$$

and the system $$2yz-x^2+2hxy+\tfrac{1}{2}\,(q-h^2)\,y^2=0,$$

or $$2yz+x^2+\tfrac{1}{2}qy^2-2\left(x+\frac{hy}{2}\right)^2=0.$$

Hence when two conics touch and have two other distinct intersections they are reciprocals for two conics, and two only (each of which is reciprocal for the other).

If they have four-point contact, they are reciprocals for one conic having four-point contact with each of them, and also an infinite system of conics touching them at their common point, and having

there an equal and opposite curvature. Each conic of this system is self-reciprocal for the single conic.

6. If the conics have three-point contact, we may take their equations as

$$S \equiv 2yz+(x+y)^2 = 0, \quad S' \equiv 2yz+(x-y)^2 = 0.$$

In this case there is no difficulty in proving that they are reciprocals for the single conic

$$\Gamma \equiv 2yz+x^2+\tfrac{1}{2}y^2 = 0.$$

7. In the case of four distinct intersections, taking the conics as

$$S \equiv x^2+y^2+z^2 = 0, \quad S' \equiv px^2+qy^2+rz^2 = 0,$$

their covariant is

$$F \equiv p\,(q+r)\,x^2+q\,(r+p)\,y^2+r\,(p+q)\,z^2 = 0;$$

we also have identically

$$F+\sqrt{p}\sqrt{q}\sqrt{r}\,(\sqrt{p}+\sqrt{q}+\sqrt{r})\,S+(\sqrt{q}\sqrt{r}+\sqrt{r}\sqrt{p}+\sqrt{p}\sqrt{q})\,S'$$
$$\equiv(\sqrt{q}+\sqrt{r})(\sqrt{r}+\sqrt{p})(\sqrt{p}+\sqrt{q})(\sqrt{p}\,x^2+\sqrt{q}\,y^2+\sqrt{r}\,z^2),$$

whence we may write the equation of Γ in the form

$$F+\Delta' S\left(\frac{1}{\sqrt{\kappa_1}\sqrt{\kappa_2}}+\frac{1}{\sqrt{\kappa_2}\sqrt{\kappa_3}}+\frac{1}{\sqrt{\kappa_3}\sqrt{\kappa_1}}\right)$$
$$+\Delta S'\,(\sqrt{\kappa_1}\sqrt{\kappa_2}+\sqrt{\kappa_2}\sqrt{\kappa_3}+\sqrt{\kappa_3}\sqrt{\kappa_1}) = 0,$$

where κ_1, κ_2, κ_3 are the roots of the equation

$$\Delta\kappa^3-\Theta\kappa^2+\Theta'\kappa-\Delta' = 0;$$

Δ, Θ, Θ', Δ' being the invariants of S and S'. The double signs of the surds give four distinct conics Γ. The equation of all four of them together may be obtained rationally thus.

The equation of any one of them is

$$F+aS+bS' = 0,$$

where $$a^2 = \Delta'(\Theta+2b), \quad b^2 = \Delta\,(\Theta'+2a).$$

Multiply the first of these equations by ab, b, a successively, and

eliminate a^2 and b^2 by means of the other two. We shall then obtain

$$\begin{aligned} abF + \Delta\Theta Sb + \Delta\Theta' S'a + 2\Delta\Delta'(\Theta S' + \Theta' S - 2F) &= 0, \\ abS + Fb + 2\Delta S'a + \Delta\Theta' S' &= 0, \\ abS' + 2\Delta' Sb + aF + \Delta'\Theta S &= 0, \\ \text{also} \qquad S'b + aS + F &= 0; \end{aligned}$$

therefore
$$\begin{vmatrix} F, & \Delta\Theta' S', & \Delta'\Theta S, & 2\Delta\Delta'(\Theta S' + \Theta' S - 2F) \\ S', & F, & 2\Delta' S, & \Delta'\Theta S \\ S, & 2\Delta S', & F, & \Delta\Theta' S' \\ 0, & S, & S', & F \end{vmatrix} = 0,$$

the required equation.

8. If S and S' have double contact, $p = q$, and two of the four equations

$$F + \Delta' S\left(\frac{1}{\sqrt{\kappa_1}\sqrt{\kappa_2}} + \ldots\right) + \Delta S'(\sqrt{\kappa_1}\sqrt{\kappa_2} + \ldots) = 0$$

give the two conics Γ having double contact with S and S'. The other two reduce to the identical relation connecting F, S, S'. It follows that the equation of these two conics Γ can be expressed linearly in terms of S and S', and there is no difficulty in verifying that their equation is

$$S^2\Delta'\sqrt{(\Theta^2 - 3\Delta\Theta')} = S'^2\Delta\sqrt{(\Theta'^2 - 3\Delta'\Theta)},$$

the signs of the surds being chosen so that

$$(\Theta\Theta' - 9\Delta\Delta') = 2\sqrt{(\Theta'^2 - 3\Delta'\Theta)}\sqrt{(\Theta^2 - 3\Delta\Theta')}.$$

9. When S and S' touch without having double contact, the preceding analysis is not applicable, but, if we take the equation of § 5 and examine what the four equations

$$F + \Delta' S\left(\frac{1}{\sqrt{\kappa_1}\sqrt{\kappa_2}} + \ldots\right) + \Delta S'(\sqrt{\kappa_1}\sqrt{\kappa_2} + \ldots) = 0$$

represent in this case, we shall find that two of them give the two conics Γ, and the other two, which become coincident, are the square of the equation of the common tangent.

The equation of the two conics Γ therefore is

$$\left(F + \frac{\Delta' S}{\kappa} + \kappa\Delta S'\right)^2 = 4\Delta\Delta'\left(S\sqrt{\kappa} + \frac{S'}{\sqrt{\kappa}}\right)^2,$$

where κ is the double root of

$$\kappa^3\Delta - \kappa^2\Theta + \kappa\Theta' - \Delta' = 0,$$

or $$\{F(\Theta\Theta' - 9\Delta\Delta') + 2\Delta' S(\Theta^2 - 3\Delta\Theta') + 2\Delta S'(\Theta'^2 - 3\Delta'\Theta)\}^2$$
$$= 8\Delta\Delta'(\Theta\Theta' - 9\Delta\Delta')\{S\surd(\Theta'^2 - 3\Delta'\Theta) + S'\surd(\Theta^2 - 3\Delta\Theta')\}^2,$$

the signs of the surds being determined as in § 8.

The square of the equation of the common tangent is

$$F - \frac{\Delta' S}{\kappa} - \kappa\Delta S' = 0,$$

or $$F(\Theta\Theta' - 9\Delta\Delta') - 2\Delta' S(\Theta^2 - 3\Delta\Theta') - 2\Delta S'(\Theta'^2 - 3\Delta'\Theta) = 0.$$

10. If S and S' have three-point contact, we can verify that one of these expressions for the equations of Γ still gives the equation of Γ, and the other three give the square of the equation of the common tangent. Hence the equation of Γ is

$$F + \frac{3\Delta' S}{\kappa} + 3\kappa\Delta S' = 0,$$

or $$F + \Theta' S + \Theta S' = 0,$$

and the square of the equation of the common tangent is

$$3F - \Theta S' - \Theta' S = 0.$$

11. If S and S' have four-point contact, we can easily verify that

$$3F - \Theta S' - \Theta' S = 0$$

is an identity, and the equation of the conic Γ having four-point contact with each of these is

$$F = 0 \quad \text{or} \quad \Theta S' + \Theta' S = 0.$$

On the Form of the Energy Integral in the Varying Motion of a Viscous Incompressible Fluid. *By* J. Brill, M.A. Received May 29th, 1895. Read June 13th, 1895. Received, in new form, September 11th, 1895.

1. In the varying motion of a viscous incompressible fluid, the energy integral can, in two special cases, be put into the same simple form as in the motion of the perfect fluid. These are the two-dimensional case and the case in which the motion is symmetrical about an axis. In the three-dimensional motion of the viscous fluid the energy integral is of a more complex form than in the corresponding case of motion of the perfect fluid.

2. We will first consider the case of two-dimensional motion. If we write

$$Q = \frac{p}{\rho} + \overline{V} + \tfrac{1}{2}q^2,$$

the equations of motion may be written in the form

$$\left.\begin{aligned} \frac{\partial Q}{\partial x} + \frac{\partial u}{\partial t} - 2v\zeta + 2\nu\frac{\partial \zeta}{\partial y} = 0 \\ \frac{\partial Q}{\partial y} + \frac{\partial v}{\partial t} + 2u\zeta - 2\nu\frac{\partial \zeta}{\partial x} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Eliminating Q from these equations, we obtain, for the equation controlling the vortex motion,

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left(u\zeta - \nu\frac{\partial \zeta}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\zeta - \nu\frac{\partial \zeta}{\partial y}\right) = 0 \quad \ldots\ldots\ldots\ldots (2).$$

Now, consider the differential equations

$$\frac{dx}{u\zeta - \nu\dfrac{\partial \zeta}{\partial x}} = \frac{dy}{v\zeta - \nu\dfrac{\partial \zeta}{\partial y}} = \frac{dt}{\zeta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

If $m = \text{const.}$ and $\beta = \text{const.}$ be two independent integrals of these equations, we have

$$\frac{u\zeta - \nu\dfrac{\partial \zeta}{\partial x}}{\dfrac{\partial (m, \beta)}{\partial (y, t)}} = \frac{v\zeta - \nu\dfrac{\partial \zeta}{\partial y}}{\dfrac{\partial (m, \beta)}{\partial (t, x)}} = \frac{\zeta}{\dfrac{\partial (m, \beta)}{\partial (x, y)}}.$$

Also, in virtue of equation (2), we see that m and β may be so chosen that we may write

$$\left.\begin{aligned} 2\left(u\zeta - \nu\frac{\partial\zeta}{\partial x}\right) &= \frac{\partial(m,\beta)}{\partial(y,t)} \\ 2\left(v\zeta - \nu\frac{\partial\zeta}{\partial y}\right) &= \frac{\partial(m,\beta)}{\partial(t,x)} \\ 2\zeta &= \frac{\partial(m,\beta)}{\partial(x,y)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

But we have

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\zeta = \frac{\partial(m,\beta)}{\partial(x,y)} = \frac{\partial}{\partial x}\left(m\frac{\partial\beta}{\partial y}\right) - \frac{\partial}{\partial y}\left(m\frac{\partial\beta}{\partial x}\right);$$

from which it follows that there exists a certain function α, such that

$$\left.\begin{aligned} u &= \frac{\partial\alpha}{\partial x} + m\frac{\partial\beta}{\partial x} \\ v &= \frac{\partial\alpha}{\partial y} + m\frac{\partial\beta}{\partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

If we substitute from equations (4) and (5) in the first of equations (1), it becomes

$$\frac{\partial Q}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial x} + m\frac{\partial\beta}{\partial x}\right) - \frac{\partial(m,\beta)}{\partial(t,x)} = 0,$$

which reduces to

$$\frac{\partial Q}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial\alpha}{\partial t} + m\frac{\partial\beta}{\partial t}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Similarly, the second of equations (1) may be reduced to the form

$$\frac{\partial Q}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial\alpha}{\partial t} + m\frac{\partial\beta}{\partial t}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

From equations (6) and (7) we immediately deduce the energy integral in the form

$$Q + \frac{\partial\alpha}{\partial t} + m\frac{\partial\beta}{\partial t} = f(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Now, if we write

$$u' = u - \nu \frac{\partial \log \zeta}{\partial x} = \frac{1}{\zeta}\left\{ u\zeta - \nu \frac{\partial \zeta}{\partial x} \right\},$$

$$v' = v - \nu \frac{\partial \log \zeta}{\partial y} = \frac{1}{\zeta}\left\{ v\zeta - \nu \frac{\partial \zeta}{\partial y} \right\},$$

then equations (3) assume the form

$$\frac{dx}{u'} = \frac{dy}{v'} = dt,$$

from which it follows that m and β satisfy the equations

$$\frac{\partial m}{\partial t} + u' \frac{\partial m}{\partial x} + v' \frac{\partial m}{\partial y} = 0,$$

$$\frac{\partial \beta}{\partial t} + u' \frac{\partial \beta}{\partial x} + v' \frac{\partial \beta}{\partial y} = 0.$$

3. In the case in which the motion is symmetrical about an axis, the equations of motion assume the form

$$\left.\begin{aligned} \frac{\partial Q}{\partial r} + \frac{\partial U}{\partial t} - 2V\omega + 2\nu \frac{\partial \omega}{\partial z} &= 0 \\ \frac{\partial Q}{\partial z} + \frac{\partial V}{\partial t} + 2U\omega - 2\nu \left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r} \right) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(9).$$

Eliminating Q, as before, we obtain

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial r}\left\{ U\omega - \nu \left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r} \right) \right\} + \frac{\partial}{\partial z}\left(V\omega - \nu \frac{\partial \omega}{\partial z} \right) = 0 \ldots\ldots(10).$$

Thus, if m and β be two independent integrals of the equations

$$\frac{dr}{U\omega - \nu \left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r} \right)} = \frac{dz}{V\omega - \nu \frac{\partial \omega}{\partial z}} = \frac{dt}{\omega} \quad \ldots\ldots\ldots\ldots(11),$$

we have

$$\frac{U\omega - \nu \left(\frac{\partial \omega}{\partial r} + \frac{\omega}{r} \right)}{\frac{\partial (m, \beta)}{\partial (z, t)}} = \frac{V\omega - \nu \frac{\partial \omega}{\partial z}}{\frac{\partial (m, \beta)}{\partial (t, r)}} = \frac{\omega}{\frac{\partial (m, \beta)}{\partial (r, z)}};$$

and, in virtue of equation (10), we see that m and β may be so chosen that we may write

$$\left.\begin{aligned} 2\left\{U\omega - \nu\left(\frac{\partial\omega}{\partial r} + \frac{\omega}{r}\right)\right\} &= \frac{\partial(m,\beta)}{\partial(z,t)} \\ 2\left(V\omega - \nu\frac{\partial\omega}{\partial z}\right) &= \frac{\partial(m,\beta)}{\partial(t,r)} \\ 2\omega &= \frac{\partial(m,\beta)}{\partial(r,z)} \end{aligned}\right\} \quad \text{.....(12).}$$

But $$\frac{\partial V}{\partial r} - \frac{\partial U}{\partial z} = 2\omega = \frac{\partial(m,\beta)}{\partial(r,z)} = \frac{\partial}{\partial r}\left(m\frac{\partial\beta}{\partial z}\right) - \frac{\partial}{\partial z}\left(m\frac{\partial\beta}{\partial r}\right).$$

Thus we see that there exists a function a, such that

$$\left.\begin{aligned} U &= \frac{\partial a}{\partial r} + m\frac{\partial\beta}{\partial r} \\ V &= \frac{\partial a}{\partial z} + m\frac{\partial\beta}{\partial z} \end{aligned}\right\} \quad \text{.....(13).}$$

Substituting from equations (12) and (13) in equations (9), we readily find that they reduce to the form

$$\frac{\partial Q}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\partial a}{\partial t} + m\frac{\partial\beta}{\partial t}\right) = 0,$$

$$\frac{\partial Q}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial a}{\partial t} + m\frac{\partial\beta}{\partial t}\right) = 0.$$

These equations at once give us the energy integral in the simple form

$$Q + \frac{\partial a}{\partial t} + m\frac{\partial\beta}{\partial t} = f(t) \quad \text{.....(14),}$$

which is exactly like equation (8).

Further, if we write

$$U' = U - \nu\frac{\partial}{\partial r}\log r\omega = \frac{1}{\omega}\left\{U\omega - \nu\left(\frac{\partial\omega}{\partial r} + \frac{\omega}{r}\right)\right\},$$

$$V' = V - \nu\frac{\partial}{\partial z}\log r\omega = \frac{1}{\omega}\left(V\omega - \nu\frac{\partial\omega}{\partial z}\right),$$

we see that equations (11) assume the form

$$\frac{dr}{U'} = \frac{dz}{V'} = dt.$$

Thus m and β satisfy the equations

$$\frac{\partial m}{\partial t} + U' \frac{\partial m}{\partial r} + V' \frac{\partial m}{\partial z} = 0,$$

$$\frac{\partial \beta}{\partial t} + U' \frac{\partial \beta}{\partial r} + V' \frac{\partial \beta}{\partial z} = 0.$$

4. We now come to the discussion of the three-dimensional case. We will write

$$u = \frac{\partial \alpha}{\partial x} + m \frac{\partial \beta}{\partial x}, \quad v = \frac{\partial \alpha}{\partial y} + m \frac{\partial \beta}{\partial y}, \quad w = \frac{\partial \alpha}{\partial z} + m \frac{\partial \beta}{\partial z},$$

from which it follows that

$$2\xi = \frac{\partial (m, \beta)}{\partial (y, z)}, \quad 2\eta = \frac{\partial (m, \beta)}{\partial (z, x)}, \quad 2\zeta = \frac{\partial (m, \beta)}{\partial (x, y)}.$$

The equations which control the vortex motion in this case are

$$\left.\begin{aligned}
&\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial y}\left\{ v\xi - u\eta + \nu\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)\right\} - \frac{\partial}{\partial z}\left\{ u\zeta - w\xi + \nu\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right)\right\} = 0 \\
&\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial z}\left\{ w\eta - v\zeta + \nu\left(\frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}\right)\right\} - \frac{\partial}{\partial x}\left\{ v\xi - u\eta + \nu\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)\right\} = 0 \\
&\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left\{ u\zeta - w\xi + \nu\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right)\right\} - \frac{\partial}{\partial y}\left\{ w\eta - v\zeta + \nu\left(\frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}\right)\right\} = 0
\end{aligned}\right\} \quad \text{.....(15).}$$

Guided by our former work, we will now make the assumptions

$$2\left\{ w\eta - v\zeta + \nu\left(\frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}\right)\right\} = \frac{\partial (m, \beta)}{\partial (x, t)} + a,$$

$$2\left\{ u\zeta - w\xi + \nu\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right)\right\} = \frac{\partial (m, \beta)}{\partial (y, t)} + b,$$

$$2\left\{ v\xi - u\eta + \nu\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)\right\} = \frac{\partial (m, \beta)}{\partial (z, t)} + c.$$

Substituting these values in equations (15), and taking account of the values for ξ, η, ζ given above, we obtain

$$\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z} = \frac{\partial a}{\partial z} - \frac{\partial c}{\partial x} = \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} = 0.$$

These equations indicate the existence of a function ϑ, such that

$$a = \frac{\partial \vartheta}{\partial x}, \quad b = \frac{\partial \vartheta}{\partial y}, \quad c = \frac{\partial \vartheta}{\partial z}.$$

Now the equations of motion of the fluid may be written in the forms

$$\frac{\partial Q}{\partial x} + \frac{\partial u}{\partial t} + 2(w\eta - v\zeta) + 2\nu\left(\frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}\right) = 0,$$

$$\frac{\partial Q}{\partial y} + \frac{\partial v}{\partial t} + 2(u\zeta - w\xi) + 2\nu\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right) = 0,$$

$$\frac{\partial Q}{\partial z} + \frac{\partial w}{\partial t} + 2(v\xi - u\eta) + 2\nu\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right) = 0.$$

By means of the results given above, these equations may be easily reduced to the forms

$$\frac{\partial Q}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial \alpha}{\partial t} + m\frac{\partial \beta}{\partial t}\right) + \frac{\partial \vartheta}{\partial x} = 0,$$

$$\frac{\partial Q}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial \alpha}{\partial t} + m\frac{\partial \beta}{\partial t}\right) + \frac{\partial \vartheta}{\partial y} = 0,$$

$$\frac{\partial Q}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial \alpha}{\partial t} + m\frac{\partial \beta}{\partial t}\right) + \frac{\partial \vartheta}{\partial z} = 0.$$

Hence we obtain for the form of the energy integral

$$Q + \vartheta + \frac{\partial \alpha}{\partial t} + m\frac{\partial \beta}{\partial t} = f(t).$$

The function ϑ satisfies the equation

$$\xi\frac{\partial \vartheta}{\partial x} + \eta\frac{\partial \vartheta}{\partial y} + \zeta\frac{\partial \vartheta}{\partial z} = 2\nu\left\{\xi\left(\frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}\right) + \eta\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right) + \zeta\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)\right\}$$

.....(16).

From equation (16) we see that the energy integral can only become reduced to the simple form that obtains for the motion of the perfect fluid if the vortex lines can continually be cut orthogonally by a family of surfaces. This is necessarily so in the two special cases we have considered. It, however, indicates a state of affairs that must be very rare in cases of three-dimensional motion.

If we write $u = u' + f, \quad v = v' + g, \quad w = w' + h,$

and determine f, g, h so as to satisfy the equations

$$\left.\begin{aligned} 2\left\{h\eta - g\zeta + \nu\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right)\right\} &= \frac{\partial\vartheta}{\partial x} \\ 2\left\{f\zeta - h\xi + \nu\left(\frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}\right)\right\} &= \frac{\partial\vartheta}{\partial y} \\ 2\left\{g\xi - f\eta + \nu\left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}\right)\right\} &= \frac{\partial\vartheta}{\partial z} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(17);$$

then we have

$$2(w'\eta - v'\zeta) = \frac{\partial(m, \beta)}{\partial(x, t)},$$

$$2(u'\zeta - w'\xi) = \frac{\partial(m, \beta)}{\partial(y, t)},$$

$$2(v'\xi - u'\eta) = \frac{\partial(m, \beta)}{\partial(z, t)}.$$

From these equations, we obtain

$$u'\frac{\partial(m, \beta)}{\partial(x, t)} + v'\frac{\partial(m, \beta)}{\partial(y, t)} + w'\frac{\partial(m, \beta)}{\partial(z, t)} = 0,$$

which may be replaced by the two equations

$$k\frac{\partial m}{\partial t} + u'\frac{\partial m}{\partial x} + v'\frac{\partial m}{\partial y} + w'\frac{\partial m}{\partial z} = 0,$$

$$k\frac{\partial\beta}{\partial t} + u'\frac{\partial\beta}{\partial x} + v'\frac{\partial\beta}{\partial y} + w'\frac{\partial\beta}{\partial z} = 0.$$

From these we easily deduce

$$k\frac{\partial(m, \beta)}{\partial(x, t)} = w'\frac{\partial(m, \beta)}{\partial(z, x)} - v'\frac{\partial(m, \beta)}{\partial(x, y)} = 2(w'\eta - v'\zeta).$$

Comparing this with the equation given above, we see that $k = 1$, and the above equations assume the form

$$\frac{\partial m}{\partial t} + u' \frac{\partial m}{\partial x} + v' \frac{\partial m}{\partial y} + w' \frac{\partial m}{\partial z} = 0,$$

$$\frac{\partial \beta}{\partial t} + u' \frac{\partial \beta}{\partial x} + v' \frac{\partial \beta}{\partial y} + w' \frac{\partial \beta}{\partial z} = 0.$$

Thus m and β are solutions of the equations

$$\frac{dx}{u'} = \frac{dy}{v'} = \frac{dz}{w'} = dt.$$

It is to be noted that equations (20) allow one degree of freedom in the choice of the quantities f, g, h, as should be the case.

On an Expansion of the Potential Function $1/R^{\kappa-1}$ in Legendre's Functions. *By* E. J. Routh. Received May 29th, 1895. Read June 13th, 1895.

1. When we require the potential of a body attracting according to the inverse square of the distance, we use Legendre's series

$$\frac{1}{R} = \Sigma P_n h^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $$R^2 = 1 - 2ph + h^2.$$

But, when the law of attraction is the inverse κ^{th} power of the distance, we require the expansion of $1/R^{\kappa-1}$. There are two ways of extending Legendre's series.

First, we may continue to make the expansion in powers of h^n, and put

$$\frac{1}{R^{\kappa-1}} = \Sigma P'_n h^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If $\kappa - 1$ is an odd integer equal to $2m+1$, we have

$$P'_n = \frac{1}{1 \cdot 3 \cdot 5 \ldots (2m-1)} \frac{d^m}{dp^m} P_{m+n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

If $\kappa-1$ is an even integer equal to $2m+2$,

$$P'_n = \frac{1}{2.4.6\ldots 2m}\,\frac{d^m}{dp^m}\,\frac{\sin(n+m+1)\,\theta}{\sin\theta} \quad \ldots\ldots\ldots\ldots(4),$$

where $$p = \cos\theta.$$

The properties of the function defined by (3) are discussed in the treatises of Todhunter, Ferrers, and Byerly. Before passing on to the subject of this paper, it may be noticed that they do not mention the property

$$(n+2)\,P'_{n+2} - p\,(2n+\kappa+1)\,P'_{n+1} + (n+\kappa-1)\,P'_n = 0,$$

which is satisfied by both the functions (3) and (4).

2. *Secondly*, we may retain Legendre's functions of p as the coefficients, but cease to expand in powers of h. We then have, when κ is even and >2,

$$\frac{1}{R^{\kappa-1}} = \Sigma\, P_n h^n \frac{\psi(h)}{(1-h^2)^{\kappa-3}}.$$

There is a similar expansion when κ is odd and >1, except that P_n is replaced by $\sin(n+1)\,\theta/\sin\theta$, and that the coefficients of the function $\psi(h)$ are different.

This form of the expansion has several advantages:—

(1) When for any reason it is necessary to use Legendre's functions in combination with the expansion, we have not the disadvantage of having two kinds of functions in the same problem.

(2) The function $\psi(h)$ is an integral rational function of h containing only even powers, the highest being $h^{\kappa-4}$. Thus the function does not increase in complexity as n increases, but has always the same number of terms.

(3) When the body considered is a thin spherical surface or a ring, h is the ratio of the radius to the distance of the attracted particle. Thus $\psi(h)$ is constant during an integration over the surface of any portion of the sphere or along the circumference of the ring.

Our first object in this paper is to obtain an expression for $\psi(h)$ as a function of h and κ.

3. To discover the general form of the function ψ, the first step is to find the scale of relation for different values of κ. We write

$$\frac{1}{R^{\kappa-1}} = \Sigma\, Q_n h^n \frac{\psi\,(h,\,\kappa-1)}{(1-h^2)^{\kappa-3}},$$

where Q_n is P_n or $\sin(n+1)\,\theta/\sin\theta$, according as κ is even or odd. Differentiating both sides with regard to h and noticing that

$$\frac{dR}{dh} = \frac{R^2+h^2-1}{2Rh},$$

we find, after some reduction, an expression for $\frac{1}{R^{\kappa+1}}$. Comparing the coefficients of Q_n, we find

$$\psi\,(h,\,\kappa+1) = \frac{1}{\kappa-1}\left\{2h\,(1-h^2)\frac{d\psi\,(h,\,\kappa-1)}{dh} + \left\{2n+\kappa-1-(2n-3\kappa+11)\,h^2\right\}\psi\,(h,\,\kappa-1)\right\}.$$

Assuming the known expansions for $1/R^3$ and $1/R^4$, we may deduce from this formula those for all integer inverse powers of R. For these two cases we know that

$$\psi\,(h,\,3) = 2n+1, \quad \psi\,(h,\,4) = (n+1)(1-h^2),$$

respectively. It immediately follows that for all integer values of κ the function ψ is an integral rational function of h^2.*

4. To deduce the general term of the expansion of $\frac{1}{R^{\kappa-1}}$, when κ is even, we put

$$\psi\,(h,\,\kappa) = \frac{A_0+A_2h^2+\ldots+A_{\kappa-3}h^{\kappa-3}}{1\,.\,3\,.\,5\,\ldots\,(\kappa-2)}.$$

Let A_0', A_2', &c., be the corresponding coefficients for $\psi\,(h,\,\kappa+2)$. Substituting in the scale of relation obtained above, we have

$$A_{2r}' = A_{2r}\,(2n+4r+\kappa) - A_{2r-2}\,(2n+4+4r-3\kappa).$$

Thus each coefficient may be calculated from those of the preceding function.

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* [It follows from the equation connecting the values of ψ for $\kappa-1$ and $\kappa+1$ that, if the first contains $1-h^2$ as a factor, the second, and therefore all that follow, must contain the same factor. Since this is the case with $\psi\,(h,\,4)$, the calculation of $1/R^{\kappa-1}$, *when* $\kappa-1$ *is even*, can be simplified by writing $\kappa-4$ instead of $\kappa-3$ as the index in the denominator on the right-hand side. Proceeding as before, we obtain the same equation to find $\psi\,(h,\,\kappa+1)$ in terms of $\psi\,(h,\,\kappa-1)$, except that we have 15 instead of 11 in the last term. Beginning, then, with $\psi\,(h,\,4) = n+1$, which is simpler than the corresponding form in the text, we find by easy stages each successive value of ψ. Another method is given in Art. 6.—*December 4th*, 1895.]

After calculating the forms of $\psi(h, \kappa)$ for several consecutive values of κ, the law of formation was discovered. We thus find

$$A_{2r} = (-1)^r LM(2n+1)N,$$

where

$L =$ coeff. of h^{2r} in $(1+h^2)^{\frac{1}{2}(\kappa-3)}$,

$M = (2n-\kappa+4)(2n-\kappa+6) \ldots (2n-\kappa+2r+2)$, r terms,

$N = (2n+2r+3)(2n+2r+5) \ldots (2n+\kappa-2)$, $\frac{1}{2}(\kappa-3)-r$ terms.

That this value of A_{2r} is correct for the smaller values of κ follows from the method by which it was obtained. That it is correct for all values of κ may then be shown by the inductive proof. Assuming its truth for κ, we find by substitution in the equation for A_{2r} given above that it is also true for $\kappa+2$. As the algebraical work of this substitution is tedious but elementary, it seems unnecessary to reproduce it here.

5. Substituting this value of A_{2r} in the expression for $\psi(h, \kappa)$, we immediately deduce the following compendious formula:—

$$\psi(h, \kappa) = \left(h^2 \frac{d}{dx} + \frac{d}{dy}\right)^{\frac{1}{2}(\kappa-3)} x^{-\frac{1}{2}(2n-\kappa+4)} y^{\frac{1}{2}(2n+\kappa-2)} \frac{2^{\frac{1}{2}(\kappa-3)}(2n+1)}{1.3.5 \ldots (\kappa-2)},$$

where x and y are to be put equal to unity after the differentiations have been performed. We recall to mind here that $\psi(h, \kappa)$ occurs in the general term of the expansion

$$\frac{1}{R^\kappa} = \Sigma P_n h^n \frac{\psi(h, \kappa)}{(1-h^2)^{\kappa-\frac{1}{2}}},$$

where κ is an odd integer greater than unity.

For example, suppose we require the expansion

$$\frac{1}{R^5} = \Sigma P_n h^n \frac{\psi(h, 5)}{(1-h^2)^3};$$

we find $$\psi(h, 5) = \left(h^2 \frac{d}{dx} + \frac{d}{dy}\right) x^{-\frac{1}{2}(2n-1)} y^{\frac{1}{2}(2n+3)} \frac{2(2n+1)}{1.3}$$

$$= \left\{-(2n-1)h^2 + 2n + 3\right\} \frac{2n+1}{3},$$

which may otherwise be shown to be correct.

It is evident that when h is less than unity the series is convergent.

6. The expansion when κ is even may be deduced from that when κ is odd by an easy rule. Putting

$$h + \frac{1}{h} = g,$$

we find, by differentiating the two series

$$\frac{1}{R} = \Sigma P_n h^n, \qquad \frac{1}{R^2} = \Sigma \frac{\sin(n+1)\theta}{\sin\theta} h^n,$$

$$\frac{1}{R^\kappa} = \Sigma \frac{(-2)^{\frac{1}{2}(\kappa-1)}}{1.3 \ldots (\kappa-2)} \frac{1}{h^{\frac{1}{2}\kappa}} \left\{ \left(\frac{d}{dg}\right)^{\frac{1}{2}(\kappa-1)} h^{n+\frac{1}{2}} \right\} P_n,$$

$$\frac{1}{R^{\kappa+1}} = \Sigma \frac{(-2)^{\frac{1}{2}(\kappa-1)}}{2\ 4 \ldots (\kappa-1)} \frac{1}{h^{\frac{1}{2}(\kappa+1)}} \left\{ \left(\frac{d}{dg}\right)^{\frac{1}{2}(\kappa-1)} h^{n+1} \right\} \frac{\sin(n+1)\theta}{\sin\theta}.$$

If, then, we write $n+\frac{1}{2}$ for n in the first of these coefficients, divide by $h^{\frac{1}{2}}$, and change $1 \ldots (\kappa-2)$ into $2 \ldots (\kappa-1)$, we obtain the second. Finally, we may write $\kappa-1$ for κ.

If, then, we write

$$\frac{1}{R^\kappa} = \Sigma \frac{\sin(n+1)\theta}{\sin\theta} h^n \frac{\psi(h, \kappa)}{(1-h^2)^{\kappa-3}},$$

when κ is an even integer greater than 2, we have

$$\psi(h, \kappa) = \left(h^2 \frac{d}{dx} + \frac{d}{dy}\right)^{\frac{1}{2}(\kappa-4)} x^{-\frac{1}{2}(2n-\kappa+6)} y^{\frac{1}{2}(2n+\kappa-2)} \frac{n+1}{1.2.3 \ldots \frac{1}{2}(\kappa-2)},$$

where x and y are to be equal to unity after the differentiations have been performed.

7. *Potential of a Spherical Shell.*—The scale of relation given in Art. 3 leads to a very useful theorem on the attraction of a thin heterogeneous spherical shell or ring. The following is an independent proof.

Let V_κ be the potential of a spherical surface at any internal or external point P when the law of force is the inverse κ^{th} power of the distance. If

$$R^2 = a^2 + r^2 - 2apr,$$

we have

$$V_\kappa = \frac{1}{\kappa-1} \int \frac{\rho\, d\sigma}{R^{\kappa-1}},$$

where ρ is the surface density and is not necessarily constant. Differentiating with regard to r and eliminating p, we have

$$(\kappa+1)\, V_{\kappa+2} = \frac{1}{a^2-r^2} \left\{ 2r \frac{dV_\kappa}{dr} + (\kappa-1)\, V_\kappa \right\}.$$

Thus, the potential of a heterogeneous shell for the law of the inverse square being well known, that for any even inverse power follows with ease.

By using the general expression in Art. 5, we may obtain an explicit form for the potential. If $\rho = Y_n$ be the surface density, we find that the potential at an external point is

$$V_\kappa = \frac{4\pi a^2}{\kappa - 1} \frac{Y_n'}{2n+1} \frac{1}{r^{\kappa-1}} \frac{h^n \psi(h, \kappa-1)}{(1-h^2)^{\kappa-3}}.$$

The potential at an internal point follows in the same way.

8. We may also expand $\frac{1}{R^{\kappa-1}}$ in Legendre's functions when κ is an odd integer, but not so conveniently as when κ is even. Let us first suppose that the force varies as the inverse cube.

Since $\sin(n+1)\theta / \sin\theta$ is an integral rational function of $\cos\theta$, it can be expanded in a finite series of Legendre's functions of p. It follows, from equations (2) and (4) of Art. 1, that we may put

$$\frac{1}{R^2} = \frac{1}{1-2ph+h^2} = \Sigma P_n \phi_n,$$

where ϕ_n is the function of h to be found. Following the usual rule, we have

$$\phi_n = \frac{2n+1}{2} \int_{-1}^{+1} \frac{P_n dp}{1-2ph+h^2}.$$

Let us put $$2f = h + \frac{1}{h},$$

and let F_n be a Legendre's function of f. Then

$$\phi_n = -\frac{2n+1}{4h} \int_{-1}^{+1} \frac{P_n dp}{p-f} = \frac{2n+1}{4h} \left\{ -\int_{-1}^{+1} \frac{P_n - F_n}{p-f} dp + F_n \log \frac{f+1}{f-1} \right\}.$$

Let $$I_n = \int_{-1}^{+1} \frac{P_n - F_n}{p-f} dp.$$

The series then becomes

$$\frac{-2h}{1-2ph+h^2} = \Sigma \frac{2n+1}{2} \left\{ I_n - F_n \log \frac{f+1}{f-1} \right\} P_n,$$

where $$2f = h + \frac{1}{h};$$

and f is therefore greater than unity, F_n is a Legendre's function of f, and I_n is a function of f only.

We know, from Arts. 78–80 of Todhunter's treatise, that

$$2Q_n = F_n \log \frac{f+1}{f-1} - I_n$$

is a Legendre's function of f of the second kind. We also know that

$$\tfrac{1}{2} I_n = \frac{2n-1}{n} \frac{F_{n-1}}{1} + \frac{2n-5}{n-1} \frac{F_{n-3}}{3} + \ldots + \frac{2n-4r+3}{n-r+1} \frac{F_{n-2r+1}}{2r-1} + \ldots,$$

the series stopping at the first negative suffix.

When the force varies as the inverse κ^{th} power, we have, by differentiation,

$$\left(\frac{-2h}{1-2ph+h^2}\right)^{\frac{1}{2}(\kappa-1)}$$

$$= \Sigma \frac{2n+1}{2} \frac{1}{1.2.3 \ldots \frac{1}{2}(\kappa-3)} \left(\frac{d}{df}\right)^{\frac{1}{2}(\kappa-3)} \left\{ I_n - F_n \log \frac{f+1}{f-1} \right\} P_n.$$

The differentiations can be effected by using the theorem

$$\frac{dF_r}{df} = \frac{r(r+1)}{2r+1} \frac{F_{r+1} - F_{r-1}}{f^2-1}.$$

For example, when the force varies as the inverse fifth power,

$$\left(\frac{-2h}{1-2ph+h^2}\right)^2 = \Sigma \frac{2n+1}{2} \left\{ \frac{I'_n}{f^2-1} - \frac{dF_n}{df} \log \frac{f+1}{f-1} \right\} P_n,$$

where $\tfrac{1}{2} I'_n = nF_n - n(n+1) \left\{ \frac{2n-3}{(n-1)n} \frac{F_{n-2}}{1.3} + \frac{2n-7}{(n-2)(n-1)} \frac{F_{n-4}}{3.5} + \ldots \right.$

$$\left. \ldots + \frac{2n-4r+1}{(n-r)(n-r+1)} \frac{F_{n-2r}}{(2r-1)(2r+1)} + \ldots \right\}.$$

9. We shall now show that the function I_n is "the other" solution of Legendre's equation of differences.*

To prove this, we recur to the definite integral

$$I_n = \int_{-1}^{+1} \frac{P_n - F_n}{p-f} dp,$$

where f is greater than unity. Substituting in the equation

$$(n+2) I_{n+2} - (2n+3) f I_{n+1} + (n+1) I_n = 0,$$

---

* [Regarding the two propositions of this article, the referees have kindly given me some references; on the first to Neumann's *Kugelf.*, page 70, and on the second to Heine, pages 134, 163, and 197.]

and remembering that P_n and F_n are Legendre's functions of p and f, we see that the equation is satisfied if

$$\int_{-1}^{+1} P_{n+1}\, dp = 0,$$

that is, if n is any positive integer including zero.

The general solution of the equation of differences

$$(n+2)\, u_{n+2} - (2n+3)\, p u_{n+1} + (n+1)\, u_n = 0$$

is therefore $$u_n = AP_n + \tfrac{1}{2} B I_n,$$

where $$\frac{I_n}{2} = \frac{2n-1}{n} P_{n-1} + \ldots + \frac{2n-4r+3}{n-r+1} \frac{P_{n-2r+1}}{2r-1} + \ldots .$$

We may also prove this by substituting the series in the equation of differences. Remembering that

$$P_n = \frac{(n+1)\, P_{n-1} + n P_{n-1}}{(2n+1)\, p},$$

we find that the coefficient of every P_n is zero.

We evidently have

$$I_0 = 0, \quad I_1 = 2, \quad I_2 = 3p, \quad I_3 = 5p^2 - \tfrac{4}{3}, \quad I_4 = \tfrac{35}{4} p^3 - \tfrac{55}{12} p, \quad \&c.$$

It may also be interesting to find the generating function of the complete integral. Let

$$V = u_0 + u_1 h + \ldots + u_n h^n + \ldots .$$

We then deduce from the equation of differences

$$(1 - 2ph + h^2) \frac{dV}{dh} + (h - p)\, V = u_1 - p u_0.$$

Regarding u_0 and u_1 as arbitrary, the equation of differences gives the values of u_2, u_3, &c. These arbitrary constants must be properly chosen if we wish to obtain the integrals separately. When

$$u_n = AP_n,$$

we have $$u_0 = A, \quad u_1 = Ap.$$

and the differential equation leads to

$$V = A/R.$$

When $$u_n = B \frac{I_n}{2},$$

we have $$u_0 = 0, \quad u_1 = B.$$

The solution of the differential equation is

$$V = \frac{B}{R} \log (h - p + R) + \frac{C}{R},$$

where, as before, $R^2 = 1 - 2ph + h^2$.

Since $u_0 = 0$, V must vanish when $h = 0$; hence

$$C = -B \log (1 - p).$$

The generating function is therefore

$$V = \frac{1}{R} \left\{ A + B \log \frac{h - p + R}{1 - p} \right\}.$$

This may also be written in the form

$$V = \frac{1}{R} \left\{ A + B \log \frac{R + 1 + h}{R + 1 - h} \right\}.$$

If we write for I_n its value given as a definite integral in Art. 8, we can easily sum the series $\Sigma I_n h^n$; after summation the integration can be effected. The result is

$$\Sigma \tfrac{1}{2} I_n h^n = \frac{1}{R} \log \frac{R + 1 + h}{R + 1 - h}.$$

10. Another proposition is suggested by the preceding theory. To connect the two methods of expanding $1/R^{\kappa+1}$ described in Arts. 1 and 2, it seems important to have the expansion of P'_n in Legendre's functions, where

$$P'_n = \frac{1}{1 \cdot 3 \cdot 5 \ldots (2m-1)} \frac{d^m}{dp^m} P_{m+n} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The expansion is given in Ferrers' and Todhunter's treatises for the cases in which $m = 1$ or 2; it is our object to find the general result.

Let $$P'_n = A_n P_n + A_{n-2} P_{n-2} + \ldots + A_r P_r + \ldots \quad \ldots\ldots\ldots\ldots (2),$$

where the series terminates at $A_0 P_0$ or $A_1 P_1$ according as n is even or odd. We know that P'_n satisfies the differential equation

$$(1 - p^2) \frac{d^2 P'_n}{dp^2} - (2m + 2) p \frac{dP'_n}{dp} + n (n + 2m + 1) P'_n = 0 \ldots\ldots (3);$$

a proof by Webb is given in the *Math. Mess.*, Vol. IX. We therefore have

$$\Sigma A_r \left[(1 - p^2) \frac{d^2 P_r}{dp^2} - (2m + 2) p \frac{dP_r}{dp} + n (n + 2m + 1) P_r \right] = 0.$$

Substituting from Legendre's equation,

$$\Sigma A_r \left[-2mp \frac{dP_r}{dp} + \{ n(n+2m+1) - r(r+1) \} P_r \right] = 0.$$

Integrating from $p = -1$ to p,

$$\Sigma A_r \left[-2m \left\{ pP_r - \int P_r dp \right\} + \{ \&c. \} \int P_r dp \right] = \Sigma (-1)^r 2m A_r.$$

Now
$$(2r+1) \int_{-1}^{p} P_r dp = P_{r+1} - P_{r-1},$$
$$(2r+1) P_r p = (r+1) P_{r+1} + r P_{r-1},$$

provided we agree that P_{-1} stands for -1.

Substituting, we find

$$\Sigma \frac{A_r}{2r+1} \left[\begin{matrix} (n-r)(n+r+2m+1) P_{r+1} \\ -(n+r+1)(n-r+2m) P_{r-1} \end{matrix} \right] = \Sigma (-1)^r 2m A_r.$$

Equating to zero the coefficient of P_{r+1}, we have

$$\frac{A_r}{A_{r+2}} = \frac{2r+1}{2r+5} \frac{n+r+3}{n-r} \frac{n-r-2+2m}{n+r+1+2m}.$$

The value of A_r may therefore be determined as soon as either the first or last term of the series is known. The final term contains A_0 or A_1 according as n is even or odd. To avoid the division into these two cases it is better to find the value of A_n. Comparing the coefficient of the highest power of p in P_n' and $A_n P_n$, we find

$$A_n = \frac{(2m+1)(2m+3) \ldots (2m+2n-1)}{1 \cdot 3 \ldots (2n-1)}.$$

Putting $r = n-2s$, we find, after some reductions,

$$\frac{A_{n-2s}}{2n-4s+1}$$
$$= \frac{m(m+1) \ldots (m+s-1)}{1 \cdot 2 \ldots s} \frac{(2m+1)(2m+3) \ldots (2m+2n-2s-1)}{1 \cdot 3 \ldots (2n-2s+1)}.$$

The general term of the expansion of P_n' in Legendre's functions has now been found.

11. In the same way we can find the expansion of

$$P_n'' = \frac{1}{2 \cdot 4 \ldots 2m} \frac{d^m}{dp^m} \frac{\sin(n+m+1)\theta}{\sin\theta},$$

in Legendre's functions. This satisfies a differential equation deduced from (3) by writing $2m+1$ for $2m$. Putting then

$$P''_n = B_n P_n + B_{n-2} P_{n-2} + \ldots + B_r P_r + \ldots,$$

we find exactly as before

$$\frac{B_r}{B_{r+2}} = \frac{2r+1}{2r+5} \cdot \frac{n+r+3}{n-r} \cdot \frac{n-r-1+2m}{n+r+2+2m}.$$

Again, since A_n or B_n can be found by comparing the coefficients of $p^n h^n$ in the equality

$$\frac{1}{(1-2ph+h^2)^{\frac{1}{2}(\kappa-1)}} = \Sigma (A_n P_n + \ldots)\, h^n,$$

according as $\kappa-1$ is odd or even, we see that B_n can be deduced from A_n by writing $2m+1$ for $2m$. We therefore have

$$\frac{B_{n-2s}}{2n-4s+1}$$

$$= 2^{n-2s} \frac{(2m+1)(2m+3)\ldots(2m+2s-1)}{1.2\ldots s} \cdot \frac{(m+1)(m+2)\ldots(m+n-s)}{1.3\ldots(2n-2s+1)}.$$

By writing $\kappa-1$ for $2m+1$ in the series found for P'_n (Art. 1), we obtain an expansion for

$$\frac{1}{R^{\kappa-1}} = \Sigma P'_n h^n$$

in Legendre's functions, which is true whether κ is an even or odd integer. Collecting the terms together, we can put this into the form

$$\frac{1}{R^{\kappa-1}} = \Sigma P_n F(h),$$

where the series $F(h)$ is the same function of h and κ, whether κ is even or odd. The series $F(h)$, however, contains an infinite number of terms, and we see by the results of Arts. 5 and 8 that the sum takes different forms according as κ is even or odd. It contains logarithms in the second case, and only algebraical expressions in the first.

On the most general Solution of given Degree of Laplace's Equation. *By* E. W. Hobson. Read May 9th, 1895. Received October 10th, 1895.

Let us assume that Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

is satisfied by $V = f_n(x, y, z, r)$,

where f_n denotes a function of degree n in x, y, z, r, and r denotes $\sqrt{x^2+y^2+z^2}$. We have

$$\frac{\partial V}{\partial x} = \frac{\partial f_n}{\partial x} + \frac{\partial f_n}{\partial r}\frac{x}{r},$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 f_n}{\partial x^2} + 2\frac{\partial^2 f_n}{\partial x \partial r}\frac{x}{r} + \frac{\partial^2 f_n}{\partial r^2}\frac{x^2}{r^2} + \frac{1}{r}\frac{\partial f_n}{\partial r} - \frac{x^2}{r^3}\frac{\partial f_n}{\partial r}.$$

Using the corresponding expressions for $\frac{\partial^2 V}{\partial y^2}$, $\frac{\partial^2 V}{\partial z^2}$, we find on addition

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) f_n + \frac{2}{r}\left(x\frac{\partial^2 f_n}{\partial x \partial r} + y\frac{\partial^2 f_n}{\partial y \partial r} + z\frac{\partial^2 f_n}{\partial z \partial r}\right) + \frac{2}{r}\frac{\partial f_n}{\partial r} + \frac{\partial^2 f_n}{\partial r^2} = 0.$$

Now, by Euler's theorem for homogeneous functions, we have

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + r\frac{\partial}{\partial z}\right)\frac{\partial f_n}{\partial r} = (n-1)\frac{\partial f_n}{\partial r};$$

hence the equation becomes

$$\frac{\partial^2 f_n}{\partial r^2} - \frac{2n}{r}\frac{\partial f_n}{\partial r} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f_n = 0,$$

or

$$\frac{\partial^2 f_n}{\partial r^2} - \frac{2n}{r}\frac{\partial f_n}{\partial r} - \nabla^2 f_n = 0,$$

where the operator ∇^2 affects f_n only so far as x, y, z occur explicitly, and not as they occur through r; we may regard this equation as an

ordinary differential equation with r as independent variable

$$\frac{d^2 f_n}{d\,(r\nabla)^2} - \frac{2n}{(r\nabla)}\,\frac{df_n}{d\,(r\nabla)} - f_n = 0,$$

where the operator ∇ is treated as a constant, since it does not affect r.

This differential equation is by means of a slight transformation reducible to Bessel's equation; in fact its complete primitive is

$$f_n = (\iota r\nabla)^{n+\frac{1}{2}} J_{-(n+\frac{1}{2})}(\iota r\nabla)\, A + (\iota r\nabla)^{n+\frac{1}{2}} J_{n+\frac{1}{2}}(\iota r\nabla)\, B,$$

where A, B are constants with respect to r, that is, they contain x, y, z explicitly, but not r; obviously A must be a function of degree n, and B a function of degree $-n-1$, in x, y, z; we may therefore write the value of f_n in the form

$$f_n(x, y, z, r) = (\iota r\nabla)^{n+\frac{1}{2}} J_{-(n+\frac{1}{2})}(\iota r\nabla)\, \phi_n(x, y, z) + (\iota r\nabla)^{n+\frac{1}{2}} J_{n+\frac{1}{2}}(\iota r\nabla)\, \psi_{-n-1}(x, y, z) \quad \ldots\ldots\ldots (1),$$

$$\text{or } f_n(x, y, z, r) = \left[1 - \frac{r^2\nabla^2}{2\,.\,2n-1} + \frac{r^4\nabla^4}{2\,.\,4\,.\,2n-1\,.\,2n-3} - \ldots\right] \Phi_n(x, y, z) + r^{2n+1}\left[1 + \frac{r^2\nabla^2}{2\,.\,2n+3} + \frac{r^4\nabla^4}{2\,.\,4\,.\,2n+3\,.\,2n+5} + \ldots\right] \frac{\Psi_n(x, y, z)}{r^{-n+1}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where ∇^2 denotes the operator $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$, which now operates on x, y, z as they occur explicitly or in r; ϕ_n, ψ_{-n-1}, Φ_n, Ψ_n are arbitrary functions of degrees indicated by the suffixes, subject to the condition that the series are convergent. The quantity n is unrestricted—it may have any real or imaginary value; we have therefore the theorem that (1) or (2) gives the most general solution of Laplace's equation of given (unrestricted) degree.

A particular case of the above expression has been already given in a paper* in which I showed that for a positive integral value of n the expression

$$\left(1 - \frac{r^2\nabla^2}{2\,.\,2n-1} + \frac{r^4\nabla^4}{2\,.\,4\,.\,2n-1\,.\,2n-3} - \ldots\right) \Phi_n(x, y, z)$$

is equal to $\quad (-1)^n \dfrac{2^n\,.\,n!}{(2n)!}\, r^{2n+1}\, \phi_n\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right) \dfrac{1}{r}$;

* "On a Theorem in Differentiation and its Application to Spherical Harmonics," *Lond. Math. Soc. Proc.*, Vol. XXIV., p. 55.

and therefore satisfies Laplace's equation; in this particular case, however, Φ_n was restricted to be a rational algebraical function, which restriction does not apply to the general theorem obtained above.

If in the first part of (2), we put

$$\Phi_n(x, y, z) = z^{n-m}(x \pm \iota y)^m,$$

we find, as a solution of Laplace's equation,

$$r^n \begin{matrix}\cos \\ \sin\end{matrix} m\phi \left\{ \mu^{n-m} - \frac{(n-m)(n-m-1)}{2.2n-1} \mu^{n-m-2} + \dots \right\},$$

where $\mu = \cos\theta$. When n and m are real integers and $n \geq m$, this gives the ordinary system of zonal and tesseral harmonics; when, however, n, m do not satisfy these conditions, the series is convergent only when $\mu > 1$. It may be observed that x, y, z are not restricted to be real, but may have complex values; consequently values of μ greater than unity are admissible, and in fact the series in this case represents a solution of Legendre's associated function for the case $\mu > 1$. If we put

$$\frac{\Psi_n(x, y, z)}{r^{2n+1}} = \frac{(x \pm \iota y)^m}{z^{n+m+1}},$$

we have, as a solution of Laplace's equation,

$$r^n \begin{matrix}\cos \\ \sin\end{matrix} m\phi \left\{ \frac{1}{\mu^{n+m+1}} + \frac{(n+m+1)(n+m+2)}{2.2n+3} \frac{1}{\mu^{n+m+3}} + \dots \right\}.$$

The series is convergent for values of μ greater than unity; it represents the function $Q_n^m(\mu)$ of the second kind, which occurs in potential problems with spheroidal boundaries.

Mr. W. D. Niven* has obtained expressions for the external and internal ellipsoidal harmonics of given type as series of spherical harmonics; these series are obtained by performing operations on certain spherical harmonics; the fact that both types of operators are Bessel's functions is explained by the form (1).

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* *Phil. Trans.*, Vol. CLXXXII., "On Ellipsoidal Harmonics."

Point-Groups in relation to Curves. *By* F. S. Macaulay, M.A.

Read June 13th, 1895. Received, in revised form, November 8th, 1895.

I.

1. Introduction.—The following paper deals with the properties of point-groups in relation to algebraic curves drawn through them; without considering any of their applications to the transformation or generation of curves. Its object is to treat the subject geometrically; and the work principally consists in developing and extending Sylvester's theory of residuation.* Geometrical proofs of some known theorems are given; but the greater part of the paper is taken up with proving new results; more especially, those on characterization, in Section III.

It is essential, at the outset, to adopt a proper convention with respect to the intersection of two curves at a common multiple point. Such a convention is attained, so far as it is needed in the paper, by proving that the intersection of two multiple points with p and q branches respectively, but no common tangents, may be regarded as the complete intersection of two infinitely small curves of orders p and q (Art. 8). It follows that the intersection of two such multiple points has practically the same properties as the complete intersection of two finite curves of orders p, q. We call this intersection a cluster of pq points, and divide it at will into smaller clusters.

The following are some of the more important consequences depending on the consideration of clusters. A point-group, regarded in general as any partial intersection of two algebraic curves with common multiple points, contains both ordinary points and clusters, the latter consisting of any parts of the several clusters common to the two curves. The condition that a curve should have an ordinary multiple point with p branches at a given point A, is shown (Art. 8) to be equivalent to the condition that the curve should pass through a general cluster of $\frac{1}{2}p(p+1)$ points, placed arbitrarily about A. Hence the condition that a curve should pass through given ordinary points, and have given multiple points, is equivalent to the condition that the curve should simply pass through a given point-group, con-

* Salmon, *Higher Plane Curves*, 2nd edition, Arts. 157–160.

taining ordinary points and clusters. Also the theorem that a curve through the complete intersection of two given curves C_l, C_m must be of the form $C_l S_{n-l} + C_m S_{n-m}$, is true, whether C_l, C_m have common clusters or not; from which it follows that the theorem of residuation (Art. 6) is applicable to all cases.

In considering the properties of a given point-group in relation to curves, we require to know how far the point-group affects the degrees of freedom of curves drawn through it; or, what is practically equivalent, the number of independent conditions supplied by the point-group for curves of any assigned order. For a group of N points this number of conditions cannot, under any circumstances, exceed N; and for curves of sufficiently high order it is equal to N; but for curves through the N points, of sufficiently low order, the number of conditions may be less than N. We call r_n the n-ic excess of a group of N points which supplies only $N - r_n$ independent conditions for n-ics. It will be found to be the rule, rather than the exception, that for point-groups such as we have to consider, viz. those obtained by the intersections of curves, the values of r_n are not zero for all values of n. Thus in the case of the complete intersection of two curves of orders l, m, the value of r_n is $\frac{1}{2}(l+m-n-1)(l+m-n-2)$, when $n \geq l \geq m < l+m$;* and is zero when $n \geq l+m-2$ (Art. 4, iv). For four points on a straight line, we have $r_1 = 2$, $r_2 = 1$, and $r_n = 0$ when $n \geq 3$. We are thus led to regard any given point-group as possessing a definite characterization, expressed by the number of its points N, and the values of r_n. The values of r_n are not connected by any law which holds in general; but for any given point-group, r_n diminishes as n increases, and cannot exceed certain limits.

The latter part of the paper consists of a general investigation of the characterization of point-groups; and it is shown from Theorem V (Art. 20), how to construct a non-composite point-group having any given characterization, by means of the intersections of curves.† Two other general problems are considered: viz., the determination of the absolute number of independent connexions of the points of a group whose construction is known; and the determination of the number of points that can be chosen arbitrarily on a curve of any given order which form part of such a point-group on the curve.

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* This is not to be taken to mean $l \geq m$; but only that $n \geq l$, $n \geq m$, and $n < l+m$. The same applies to all inequalities in the paper.

† In order, however, that the construction may be in all cases fully determinate, a knowledge is required of the relative position of multiple points on curves of given order, when their number is so great that they could not have an arbitrary position.

In writing the paper I have received invaluable help from Miss Scott, D.Sc., Professor of Mathematics at Bryn Mawr College, Pa., U.S.A. This has enabled me to make a number of alterations and corrections; but it is quite possible that the work is still not entirely free from numerical errors.

The theory of point-groups on curves is mostly contained in German and Italian mathematical publications.* Proofs of the principal known theorems are therefore given (Arts. 3–6, 9, 17–19), which serve also as examples of the methods followed in the paper.

2. Explanation of Terms.—(*a*) We shall denote a given curve of order m by C_m, and one whose coefficients are partially or wholly at disposal by S_m.

The *degree* of a point-group is the number of points it contains; and is equal to the sum of the degrees of its clusters added to the number of its ordinary points.

A point-group is denoted by a single letter Q, R, N, ..., either without a suffix, in which case the letter denoting the point-group denotes its degree also; or with a suffix, in which case the suffix denotes the degree of the point-group.

(*b*) Two point-groups Q, R on a given curve C_m are said to be *residual* to one another if they together make up the complete intersection of C_m with any other curve, proper or composite.

The point-group $Q+R$ is said to have a zero residual, since a curve can be drawn through it which does not cut C_m in any more points.

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* The following memoirs contain fundamental portions of the subject, and illustrate the different methods that have been employed.

Brill-Noether, "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie" (*Mathematische Annalen*, vii, p. 269). The greater part of this important memoir is reproduced with slight variation in the *Vorlesungen über Geometrie* of Clebsch-Lindemann; and in the translation of the same work by Benoist, *Leçons sur la Géométrie*, Tome ii, pp. 135–146, and Tome iii, p. 31 ff. Several papers are contributed separately by the same authors to other volumes of the *Math. Ann.*, some of which are referred to below. The work is mostly analytical.

Bacharach, "Ueber den Cayley'schen Schnittpunktsatz" (*Math. Ann.*, xxvi, p. 275); consisting chiefly of an examination of exceptions to Cayley's theorem, by the theory of residuation.

Cayley, "On the Intersection of Curves" (*Math. Ann.*, xxx, p. 85); a repl the preceding paper.

Castelnuovo, "Ricerche generali sopra i sistemi lineari di curve piane" (*Mem. della R. Accademia delle Scienze di Torino*, March, 1891, xlii, p. 3).

Bertini, "La geometria della serie lineari sopra una curva piana" (*Annali di Matematica*, April, 1894, xxii, p. 1); giving a valuable summary of known theorems on point-groups.

(*c*) The whole system of point-groups of the same degree, on a given curve C_m, which have a common residual, is called a complete *coresidual* or *equivalent* system. (Cf. Art. 6.)

The *equation of residuation* $R \equiv R'$, or $R - R' \equiv 0$, expresses the fact that R, R' are coresidual, *i.e.*, have a common residual Q, whether R, R' are of the same degree or not; and $Q + R \equiv 0$, that Q, R are residual.

The curve C_m on which Q, R, R', ... lie is called the *base-curve*.

(*d*) A *cluster* is any arrangement of points crowded infinitely near together; or a group of ordinary points on an infinitely small scale.

It has been already mentioned that we regard a point-group in general as containing both ordinary points and clusters; and it should be noticed that any ordinary point may be regarded as a cluster whose degree is unity. We shall only consider the simple case of those point-groups in which all the clusters and ordinary points are finitely separated; since the combination of any two gives rise to properties which require additional investigation.*

If on a given base-curve the ordinary points of a point-group containing clusters be denoted by Q, the whole point-group will be denoted by ΣQ.

(*e*) A curve *adjoined* to a given curve C_m is one which passes through every double point of C_m, and which has at each multiple point on C_m with three or more branches a multiple point with one branch less. Thus a curve C_n adjoined to C_m has an $(i-1)$-ple point at each i-ple point on C_m. We shall suppose, however, for the reason mentioned above, that C_m, C_n have no common tangent at any multiple point.

The definition of an adjoined curve is sometimes generalized as follows:—A curve adjoined to C_m is one which has *at least* $i-1$ branches at each i-ple point on C_m. (Art. 9, ii.)

(*f*) The term *general*, or *non-specialized*, is applied to curves, point-groups, &c., which only satisfy specified conditions. Thus we may

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* Thus we exclude, for example, the consideration of the intersection of two curves which have contact at any common multiple point; which corresponds to the case of one or more of the ordinary points being infinitely near to a cluster. See Bertini (*Math. Ann.*, xxxiv, p. 447); Noether (*Math. Ann.*, xl, p. 140); and Baker (*Math. Ann.*, xlii, p. 601).

speak of N general points on a given curve; or of a general n-ic through a given point-group.

A *point-group of special form** is one whose degree N exceeds the number of independent conditions it supplies for any one or more curves which can be drawn through it. The simplest example is any group of $\frac{1}{2}(m+1)(m+2)$ or more general points on an m-ic; and the most typical, the complete intersection of two curves of higher order than the second. A point-group which is not of special form may still be specialized; but we shall nevertheless refer to all such as general point-groups.

The same distinction is made between a general cluster and a cluster of special form, the latter being a point-group of special form on an infinitely small scale.

(g)† Two point-groups N, N' in a plane, which make up the complete intersection of any two curves, are called *rest-groups;* and any rest-group of N', *e.g.*, any coresidual of N, is called a second derived rest-group of N.‡

(h) The n-ic *excess*,§ r_n, of a point-group is the excess of its degree N over the number of independent conditions it supplies for n-ics.

An n-ic through $N-r_n$ of the N points passes necessarily through the remainder r_n, if the $N-r_n$ points supply $N-r_n$ independent conditions for n-ics (Art. 7).

The *characterization* of a point-group is expressed by its degree N and the several values of r_n.

(k) The n-ic *defect*,§ q_n, of a point-group N is the number of

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* The common term, *special point-group*, means a group of ordinary points on a given curve C_m, which has an excess for adjoined $(m-3)$-ics, or, more generally, for any given linear system of curves. Point-groups of special form differ from these—(i) in including clusters, and (ii) in having excess for general curves, instead of for curves belonging to a given linear system. Thus (ii) involves a restriction, which necessitates the employment of a distinguishing term.

† Definitions g, k, l, m refer more especially to Section III.

‡ Roughly speaking, we take the letters N, N' to denote rest-groups in a plane, whether they contain clusters or not; and Q, R, or ΣQ, ΣR, to denote residuals, *i.e.* rest-groups, on a given base-curve.

§ The n-ic excess r_n corresponds to the term *sovrabbondanza* (Castelnuovo), the former referring to the point-group N, and the latter to the general system of n-ics drawn through N. Also $N-r_n$ is the *postulation*, and q_n the *postulandum* (Cayley), of an n-ic drawn through N. The values of q_{m-3}, r_{m-3} for a whole point-group ΣR on a curve C_m, containing a general cluster of degree $\frac{1}{2}i(i-1)$ at each i-ple point on C_m, are the same as the values of q, r for the ordinary point-group R, in the Brill-Noether notation.

general points through which n-ics can be drawn, in addition to passing through the N points. (Cf. Art. 15, ii.) Thus the n-ic defect of N is the same as the degree of freedom of an n-ic drawn through N, or the dimensions of the general system of n-ics through N.

(l) A *redundant* point-group is one of special form which contains one or more general points having no connexion with the rest.

A *complete** point-group is one which includes all points in the plane which in reality belong to it. Any n-ic drawn through an incomplete point-group, for which r_n is not zero, must pass through those other points in the plane which complete the point-group; and may possibly necessarily pass through a second complete point-group.

A *simple* point-group is one which is neither redundant, nor incomplete, nor composed of two or more complete point-groups.

(m) If N be a given point-group on a given curve C_m, then the number of general points that can be chosen arbitrarily on C_m which form part of any group of N points coresidual to the given one is called the *multiplicity* or *manifoldness* of the coresidual system. This multiplicity is a definite number, since a coresidual system of degree N is completely determined by any single point-group of the system (Art. 6); and we may therefore call it the multiplicity of the given point-group on C_m, or of any point-group of the system.

The *absolute n-ic multiplicity*, x_n, of any point-group N which satisfies given conditions, is the number of general points that can be chosen arbitrarily on a given n-ic which form part of such a group of N points on the n-ic. It is implied that the given conditions are of such a kind that x_n has a definite (but not a given) value.

3. Theorem I.—*The number of independent conditions supplied for l-ics by the complete intersection of two curves C_l, C_m is*

$$lm-\tfrac{1}{2}(m-1)(m-2);$$

provided C_l, C_m have no common factor, and l is not less than $m-2$.

In dealing with point-groups on a given base-curve C_m this property is of fundamental importance. Expressed more fully, the theorem states that through any $lm-\frac{1}{2}(m-1)(m-2)-1$ of the lm

* The terms complete and incomplete, applied here to a single point-group, do not in any way correspond to the same terms when applied to a system of point-groups on a given curve. (See Def. *c*.)

points common to C_l, C_m an l-ic can be drawn which does not pass through all the rest; and that *an l-ic through any number whatever of the lm points does or does not necessarily pass through the rest according as they supply $lm-\frac{1}{2}(m-1)(m-2)$ independent conditions for l-ics, or less.* That the theorem is true when $l=m-1$ or $m-2$, follows at once from the fact that, in each of these cases, $lm-\frac{1}{2}(m-1)(m-2)$ is equal to $\frac{1}{2}l(l+3)$.

We shall suppose both here, and in Theorem II, that C_l, C_m have no common multiple points; leaving it to be shown in Art. 8 that the reasoning is also valid in the contrary case.

(i.) The number of independent conditions that all the points on C_m supply for l-ics is the number of general points on C_m which would require any l-ic drawn through them to be of the form $C_m S_{l-m}$; and this number is

$$\tfrac{1}{2}l(l+3)-\tfrac{1}{2}(l-m)(l-m+3) = lm-\tfrac{1}{2}(m-1)(m-2)+1;$$

since the curve $C_m S_{l-m}$ is still capable of passing through $\frac{1}{2}(l-m)(l-m+3)$ general points in the plane.

(ii.) Again, if an l-ic S_l through the lm points common to C_l, C_m be made to pass through one more point on C_m, it must pass through all points on C_m. For S_l meets C_m in more than lm points, and must therefore have a common factor with C_m. Let

$$S_l \equiv C_p S_{l-p}, \quad \text{and} \quad C_m \equiv C_p C_{m-p}.$$

Then, since S_l passes through all points common to C_l and C_m, S_{l-p} passes through all points common to C_l and C_{m-p}; hence

$$S_{l-p} \equiv C_{m-p} S_{l-m}, \quad \text{and therefore} \quad S_l \equiv C_m S_{l-m};$$

which had to be proved. Hence the number of independent conditions supplied by the lm points for l-ics is one less than that found in (i), viz. $lm-\frac{1}{2}(m-1)(m-2)$. In other words, *the l-ic excess of the lm points is $\frac{1}{2}(m-1)(m-2)$, and the degree of freedom of an l-ic through the lm points is $\frac{1}{2}(l-m+1)(l-m+2)$.**

The theorem can be easily extended to those cases in which C_l, C_m have common factors, provided each has one factor at least which is not a factor of the other.

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* Cf. Zeuthen, "Sur la détermination d'une courbe algébrique par des points donnés" (*Math. Ann.*, xxxi, p. 235), for a fuller development of the method followed here.

4. THEOREM II.*—*Any n-ic which passes through the complete intersection of two given curves C_l, C_m must be of the form*

$$S_n \equiv C_l S_{n-l} + C_m S_{n-m} = 0;$$

n being not less than l or m.

We shall suppose that C_l, C_m have no common factor. The truth of the theorem is, however, independent of any conditions imposed on the curves C_l, C_m; or any restriction placed on the value of n, assuming that S_{n-l}, S_{n-m} are zero when $n-l$, $n-m$ are negative.

(i.) To show that, when $n \geqslant l+m-2$, the lm points common to C_l, C_m supply lm independent conditions for n-ics.

If they do not, then either the lm points are such that an $(l+m-2)$-ic through all but *any* one must necessarily pass through the last, or some part N of the lm points possesses this property. We have then only to prove that this cannot be true of any N of the lm points.

Suppose $l \geqslant m$. Then among the lm points it is clear that $\frac{1}{2}m(m+1)$ can be chosen through all but any one or more of which a curve S_{m-1} can be drawn without passing through any of the rest. The other $lm - \frac{1}{2}m(m+1) \leqslant (l-1)m - \frac{1}{2}(m-1)(m-2)$ points lie on a curve S_{l-1} which does not contain C_m as a factor (Theorem I), and therefore does not pass through all the lm points. Now, if S_{l-1} passes necessarily through some of the $\frac{1}{2}m(m+1)$ points, we can still choose S_{m-1} so as to pass through all but one of the rest. The composite curve $S_{l-1}S_{m-1}$ then passes through $lm-1$ of the lm points without passing through the last. The same proof holds for any N of the lm points, except when the N points all lie on an $(m-1)$-ic or an $(l-1)$-ic; and each of these last cases is a simpler one than that already considered, to which similar reasoning applies.

Hence the degree of freedom of the general n-ic through the complete intersection of C_l, C_m is $\frac{1}{2}n(n+3) - lm$, when $n \geqslant l+m-2$.

(ii.) To find the degree of freedom D of the curve

$$S_n \equiv C_l S_{n-l} + C_m S_{n-m} = 0.$$

When $n \geqslant l+m$, choose the origin at a point not lying on C_m;

* The proof of this theorem was originally given in a memoir by Noether, "Ueber einen Satz aus der Theorie der algebraischen Functionen" (*Math. Ann.*, VI, p. 351). For other references, see Note to Art. 2 (*d*).

and consider the n-ic*

$$S'_n \equiv C_l S_{n-l}^{n-l-m+1} - C_m S'_{n-m} = 0,$$

where $S_{n-l}^{n-l-m+1}$ denotes any algebraic expression of order $n-l$, whose lowest terms are of order $n-l-m+1$. Let D' be the degree of freedom of S'_n; then S'_n can be made to pass through D' general points chosen arbitrarily in the plane. Let the coordinates of these and one more arbitrary point be substituted in $S'_n = 0$; then the resulting $D'+1$ independent equations for the coefficients of S'_n will necessitate that S'_n vanishes identically; *i.e.*, they will require that

$$C_l S_{n-l}^{n-l-m+1} \equiv C_m S'_{n-m}.$$

Now C_l has no factor in common with C_m, and $S_{n-l}^{n-l-m+1}$ cannot be divisible by C_m, since the origin is not on C_m. Hence the above identity requires that $S_{n-l}^{n-l-m+1}$ and S'_{n-m} should both vanish identically, *i.e.*, that all their coefficients, whose number is

$$\tfrac{1}{2}(n-l+1)(n-l+2) - \tfrac{1}{2}(n-l-m+1)(n-l-m+2) + \tfrac{1}{2}(n-m+1)(n-m+2) = \tfrac{1}{2}n(n+3) - lm + 1,$$

should be zero. Hence the $D'+1$ equations are equivalent to $\frac{1}{2}n(n+3) - lm + 1$ independent equations; and therefore

$$D' = \tfrac{1}{2}n(n+3) - lm.$$

But D is not less than D', since S_n is not less general than S'_n; and D is not greater than $\frac{1}{2}n(n+3) - lm$ by (i), since S_n is not more general than the general n-ic through the lm points. Hence also

$$D = \tfrac{1}{2}n(n+3) - lm.$$

When $n < l+m$, we can prove in the same way that S_n cannot vanish identically unless all the coefficients of S_{n-l}, S_{n-m} are zero; so that

$$D+1 = \tfrac{1}{2}(n-l+1)(n-l+2) + \tfrac{1}{2}(n-m+1)(n-m+2).$$

Hence finally we have

$$D = \tfrac{1}{2}n(n+3) - lm + \tfrac{1}{2}(l+m-n-1)(l+m-n-2), \quad \text{when } n < l+m;$$

and

$$D = \tfrac{1}{2}n(n+3) - lm, \quad \text{when } n \geq l+m-2.$$

* It can be easily shown that S_n can be changed to the form S'_n; but it is not required for the proof of the theorem.

(iii.) If $n \geq l+m-2$, it follows from (i) and (ii) that the general n-ic through the lm points common to C_l, C_m has the same degree of freedom as S_n, and can therefore be written in the form S_n.

If $n \geq l \geq m < l+m-2$, and C_n is an n-ic through the lm points, and $C_{l+m-n-1}$ another curve which does not pass through any of the lm points, then $C_n C_{l+m-n-1}$ is an $(l+m-1)$-ic through the lm points; and we therefore have

$$C_n C_{l+m-n-1} \equiv C_l S_{m-1} + C_m S_{l-1}.$$

Hence all the points common to $C_{l+m-n-1}$ and C_l lie on S_{l-1}; none of them lying on C_m, by hypothesis. Hence $C_{l+m-n-1}$ is a factor of S_{l-1}, and similarly of S_{m-1}, and, by dividing it out, we have

$$C_n \equiv C_l S_{n-l} + C_m S_{n-m},$$

which proves the theorem.

(iv.) *The n-ic excess of the lm points common to C_l, C_m is*

$$\tfrac{1}{2}(l+m-n-1)(l+m-n-2),$$

*when $n \geq l \geq m < l+m$; and is zero, when $n \geq l+m-2$.**

For any n-ic through the lm points common to C_l, C_m has the same degree of freedom D as S_n; and the number of independent conditions supplied by the lm points for n-ics is $\frac{1}{2}n(n+3)-D$; *i.e.*, is $lm-\frac{1}{2}(l+m-n-1)(l+m-n-2)$ when $n \geq l \geq m < l+m$, and lm when $n \geq l+m-2$; whence the theorem follows.

Cayley's theorem; viz., that an n-ic ($n \geq l \geq m < l+m$) through $N = lm-\frac{1}{2}(l+m-n-1)(l+m-n-2)$ points common to C_l, C_m must pass through the remainder; is not accurate without the addition of the proviso that the N points supply N independent conditions for n-ics. It is perhaps still more important to notice that, Cayley's theorem does not prove that an n-ic can be drawn through any $N-1$ points common to C_l, C_m, without passing through all the rest.

The simplest criterion as to whether the general n-ic, through any given group of points common to C_l, C_m, does or does not pass through the rest, is given at the end of Art. 20.

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* The n-ic excess of the lm points can be written in the form

$$\left[\tfrac{1}{2}(l+m-n-1)(l+m-n-2)\right] - \left[\tfrac{1}{2}(l-n-1)(l-n-2)\right] - \left[\tfrac{1}{2}(m-n-1)(m-n-2)\right];$$

the square brackets indicating that the terms enclosed by them are only to be retained when their factors are positive. This formula is correct for all values of n, reducing to $lm-\frac{1}{2}(n+1)(n+2)$ when $n < l < m$. (See Note, p. 507.)

5. THEOREM III.—*If a group of $m(n+n')$ points on a base-curve C_m have a zero residual (Art. 2, b); and mn of the $m(n+n')$ points have a zero residual; the remaining mn' points have a zero residual.*

Let $C_{n+n'}$, C_n be curves through the point-groups $m(n+n')$, mn respectively, which do not cut C_m in any more points. Then $C_{n+n'}$ passes through all the points common to C_m, C_n. Hence, if $n+n' \geq m$, we have

$$C_{n+n'} \equiv C_m S_{n+n'-m} + C_n S_{n'}.$$

Hence the curves $C_{n+n'}$, $C_n S_{n'}$ cut C_m in one and the same point-group, viz. $m(n+n')$; but C_n cuts C_m in the mn points, and in no others; hence $S_{n'}$ cuts C_m in the mn' points, and in no others; *i.e.*, the point-group mn' has a zero residual. Again, if $n+n' < m$, we must have $C_{n+n'} \equiv C_n S_{n'}$; and the same result follows.

This theorem may be stated as follows:—*Of the three equations* $N \equiv 0$, $N' \equiv 0$, $N+N' \equiv 0$, *any one is a consequence of the other two.**

For the theorem proves that, if $N \equiv 0$ and $N+N' \equiv 0$, then $N' \equiv 0$; and it is evident that, if $N \equiv 0$ and $N' \equiv 0$, then $N+N' \equiv 0$.

Hence the algebraic laws of addition and subtraction hold for equations of residuation.

For, if $M \equiv N$, $M' \equiv N'$; and Q be residual to M, N; and R to M', N'; then $Q+R$ is residual to $M+M'$ and to $N+N'$; hence we have $M+M' \equiv N+N'$. It follows also that $M+N' \equiv M'+N$, which is equivalent to $M-M' \equiv N-N'$ (Art. 2, c).

Points which occur both in the positive and negative terms on the same side of an equation cancel. If, for example, $L+M-L \equiv N$, then $L+M \equiv L+N$; from which it evidently follows that $M \equiv N$.

As examples in addition we may notice the following:—

(i.) Of the three equations $M \equiv N$, $M' \equiv N'$, $M+M' \equiv N+N'$, any one is a consequence of the other two; *i.e.*, two point-groups are coresidual if any parts of them are coresidual, and the remainders coresidual; and, if parts of two coresidual point-groups are coresidual, the remainders are coresidual.

(ii.) If $L \equiv M+N$ and $N+N' \equiv 0$, then $L+N' \equiv M$; *i.e.*, if two point-groups are coresidual, the point-groups obtained by taking away any part of one, and adding any residual of it to the other, are also coresidual.

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* Cf. BERTINI, *loc. cit.*, p. 497, for this way of presenting the subject.

6. The Theorem of Residuation.—*If two point-groups on a given base-curve have a common residual, then any residual of one is a residual of both.*

Let R, R' be the two point-groups, and Q their common residual; and let Q' be any residual of R. Then we are given the three equations $Q+R \equiv 0$, $Q+R' \equiv 0$, $Q'+R \equiv 0$; hence, by adding the last two and subtracting the first, we have $Q'+R' \equiv 0$; which proves the theorem. This theorem is therefore an immediate consequence, and may be regarded as another form, of Theorem III; and it is convenient to regard both theorems as included in the theorem of residuation.

Curves through a point-group Q on C_m determine so many coresidual point-groups R, R', ... on C_m, whose properties are to a certain extent independent of the particular point-group Q; since to any curve through R, which cuts C_m again in Q', corresponds a curve through R', which cuts C_m again in the same point-group Q'. In other words, two coresidual point-groups R, R' on a base-curve C_m are *equivalent* in respect to the point-groups Q, Q', ... determined on C_m by curves drawn through R, R'; and any two point-groups Q, Q' thus determined are also coresidual or equivalent. Also the condition that a point-group R' should have a given coresidual R is the same as that it should have a given residual, viz. any fixed residual of R; and a coresidual system is fully determined by any single point-group of the system.

From the theorem of residuation we have the following:—

If any number of point-groups be taken in succession on a base-curve C_m, each of which is residual to the preceding, then the p^{th} and q^{th} point-groups are residual or coresidual according as the difference between p and q is odd or even.

7. Point-Groups of Special Form.—An essentially fundamental property of curves, usually assumed without proof, is that *an n-ic can be drawn through any $\frac{1}{2}n\,(n+3)$ given points in a plane, no matter how they may be placed.* In other words, if the coordinates of any $\frac{1}{2}n\,(n+3)$ points be substituted in the general equation of an n-ic whose *constants* (as we shall call the coefficients for the time being) are all at disposal, the resulting $\frac{1}{2}n\,(n+3)$ equations for the ratios of the $\frac{1}{2}\,(n+1)(n+2)$ constants have always one finite solution at least. It is clear that we may suppose that the equations for the constants have all their coefficients finite, whether any of the given points are at infinity or not; and that it cannot follow from the equations

that all the $\frac{1}{2}(n+1)(n+2)$ constants are necessarily zero, since $\frac{1}{2}(n+1)(n+2)$ independent equations cannot be deduced from $\frac{1}{2}n(n+3)$ linear equations only. Again, it cannot follow from the equations that a single one of the constants is necessarily infinite; for, if the ratio of two constants is infinite, the antecedent of the ratio may still be assumed finite by taking the consequent zero. Hence the only hypothesis on which the equations cannot have a single finite solution is that they are inconsistent among themselves. But this is impossible, since any elimination of the constants from the equations would not lead to any inconsistency, but to an identity, viz. $0 = 0$.

It is, however, quite possible that the given points might be so placed that the elimination spoken of could be effected. When this is the case, the equations are not independent, and the given points do not all supply independent conditions for n-ics, while the point-group determined by them is of special form (Art. 2, f). The number of independent conditions supplied by the point-group for n-ics is exactly equal to the number of independent equations to which the system we have been considering is equivalent. It is important also to notice that the greatest number of points that can supply independent conditions for n-ics is $\frac{1}{2}(n+1)(n+2)$*; the coordinates of such points, when substituted in the equation $S_n = 0$, requiring all the coefficients of S_n to vanish.

If the coordinates of $\frac{1}{2}p(p+1)$ given points, which do not lie on a $(p-1)$-ic, be substituted in the general equation of the n^{th} order, viz.

$$S_n \equiv S_{p-1} + u_p + u_{p+1} + \ldots + u_n = 0,$$

u_q denoting a homogeneous function in x, y of the q^{th} order, the resulting $\frac{1}{2}p(p+1)$ equations will determine the coefficients of S_{p-1} in terms of the remaining coefficients of S_n; for, in eliminating all but one of the coefficients of S_{p-1}, the last will not disappear; since the given points do not lie on a $(p-1)$-ic. Hence we have the following theorem.

If an n-ic be drawn through $\frac{1}{2}p(p+1)$ given points, which do not lie on a $(p-1)$-ic, the coefficients of the terms in S_n of the $(p-1)^{th}$ and lower orders will be thereby completely determined in terms of the rest, which last can be chosen arbitrarily.

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* Hence we may say that the n-ic excess of any point-group N which does not lie on an n-ic is $N - \frac{1}{2}(n+1)(n+2)$, or more simply that *the n-ic defect is -1.* (Art. 15, ii.)

But if the $\frac{1}{2}p(p+1)$ *given points lie on a* $(p-1)$*-ic, or more generally, if they supply only* $\frac{1}{2}p(p+1)-\rho$ $(\rho \geqslant 1)$ *independent conditions for* $(p-1)$*-ics, then the coefficients of the terms in* S_n *of the* $(p-1)$ *and lower orders can be eliminated, giving rise to* ρ *equations among the coefficients of the terms in* S_n *of the* p^{th} *and higher orders; so that these last cannot be chosen arbitrarily.*

The ρ *equations will be equivalent to* $\rho-\rho'$ *independent equations only, if the* $\frac{1}{2}p(p+1)$ *points supply only* $\frac{1}{2}p(p+1)-\rho'$ *independent conditions for n-ics.*

This theorem shows that a point-group of special form has special properties in relation to any curve drawn through it; since any such point-group, in contrast to a general one of the same degree, is connected with the more complex shape of the curve dependent on the terms of higher order in its equation.

8. Multiple Points.—Ordinary multiple points, as well as multiple points of higher singularity, may be supposed to have complex shapes, which are not apparent because they are confined within infinitely small limits. Our object is to assign a shape to an ordinary multiple point with two or more branches, which is not inconsistent with theory, and which will provide a basis for reasoning about the intersection of two curves at a common multiple point with p and q branches respectively.

Suppose that we have a cluster of $\frac{1}{2}p(p+1)$ points at the origin, which do not lie on an infinitely small $(p-1)$-ic. Let the coordinates of any point in the cluster be denoted by κx_1, κy_1, where κ is an infinitely small constant, and x_1, y_1 are finite. Then, just as in the last article, an n-ic can be made to pass through the $\frac{1}{2}p(p+1)$ points of the cluster, and its equation will be of the form

$$S_n \equiv \kappa^p u_0+\kappa^{p-1}u_1+\ldots+\kappa u_{p-1}+u_p+\ldots+u_n=0,$$

where the coefficients of $u_p+u_{p+1}+\ldots+u_n$ may be chosen arbitrarily, and those of $\kappa^p u_0+\kappa^{p-1}u_1+\ldots+\kappa u_{p-1}$ are known in terms of them. Hence the coefficients of $u_p+u_{p+1}+\ldots+u_n$ can all be chosen finite, and those of $u_0, u_1, \ldots u_{p-1}$ are then also finite, and are known in terms of the coefficients of u_p; for when the coordinates κx_1, κy_1 of a point of the cluster are substituted in $S_n=0$, the terms arising from $u_{p+1}+\ldots+u_n$ will contain higher powers of κ than the preceding terms, and can be neglected, since the cluster does not lie on a $(p-1)$-ic. The curve S_n is then an ordinary n-ic with a p-ple point

at the origin, having all its finite coefficients, viz. those of $u_p + \ldots + u_n$, at disposal. There is thus no necessity that a curve drawn through a cluster should not be a curve of finite dimensions, as we might be naturally disposed to assume.

When, on the other hand, the given cluster is of special form; which is the case, for example, when the cluster lies on a $(p-1)$-ic; the coefficient of $u_p + \ldots + u_n$ will not be entirely arbitrary. In fact, if ρ_q be the q-ic excess of the cluster (Art. 2, *h*), and the coordinates of all the points of the cluster be substituted in $S_n = 0$, no terms being neglected; then by the theorem of the last article we can obtain ρ_{q-1} equations among the coefficients of $u_q + u_{q+1} + \ldots + u_n$; in which it is clear that κ will occur with all the coefficients of u_{q+1}, κ^2 with those of u_{q+2}, and so on. Hence it evidently follows that there are $\rho_{q-1} - \rho_q$ equations among the coefficients of u_q alone. If ρ_q is zero, the coefficients of $u_{q+1} + \ldots + u_n$ are not affected by the cluster.

It follows from the above that the only alteration required in the equation of a given curve C_n, in order that the new curve may pass through an arbitrary general cluster of degree $\frac{1}{2}p(p+1)$ at each and every p-ple point of C_n, is the addition of a series of infinitely small terms, possibly extending beyond the terms of highest order in C_n. We may then replace C_n by the new curve thus obtained; and those of its properties, which do not become indeterminate in the limit, will hold for the given curve C_n. The shape of this curve at a p-ple point (as may be easily seen by changing x, y to κx, κy and dividing out κ^p in the equation of S_n given above) is, to a first approximation, that of a p-ic on an infinitely small scale; whose asymptotes coincide in direction with the branches of C_n, but which may in all other respects be chosen at will. This then is the shape we assign to an ordinary p-ple point. By this means we resolve the intersection of two given curves at any common multiple point into a cluster of separate points; and, as a consequence, the theorems proved in Arts. 3–6 may be extended so as to include the case of curves with common multiple points.

Thus a given curve C_n with an ordinary p-ple point at A may be supposed ipso facto *to contain any given or arbitrarily chosen general cluster at A of any degree not higher than $\frac{1}{2}p(p+1)$.*

Also, if two given curves have a common multiple point at A with p and q branches respectively, but without common tangents, their common cluster pq at A may be regarded as forming the complete intersection of two infinitely small curves of orders p, q, whose asymptotes coincide in direction with the branches of the two curves at A.

The question arises as to how many points of the cluster pq a third curve with an r-ple point at A may be supposed *ipso facto* to pass through. From the theorem of the last article, it follows that the required number is the number of points of the cluster pq that supply independent conditions for $(r-1)$-ics; which is equal to

(i.) $\frac{1}{2}r(r+1)$, when $r \leqslant p \leqslant q$;

(ii.) $(r-1)p-\frac{1}{2}(p-1)(p-2)+1$
$= rp-\frac{1}{2}p(p-1)$, „ $r > p \leqslant q$;

(iii.) $pq-\frac{1}{2}(p+q-r)(p+q-r-1)$, „ $r>p>q<p+q$;

(iv.) pq, „ $r \geqslant p+q-1$.

Three curves with a common double point at A have a common cluster of three points at A, and the condition for a fourth common point at A is that the tangents at A to the three curves should be in involution. This follows from the property that conics with four common points cut the line at infinity in involution, so that the pairs of tangents to the three curves, which are parallel to the asymptotes of three such conics, are in involution. If one of the double points is a cusp, its tangent must be a double ray of the involution.

The results obtained above are all that we shall require to use in what follows; but it is necessary to repeat that they are only a first approximation. In order to decide whether a given curve does or does not pass through any cluster common to two given curves, we have in general to proceed to higher approximations. The following conditions are necessary and sufficient in order that a given curve C_n may be capable of being written in the form $C_l S_{n-l} + C_m S_{n-m}$; viz. that C_n vanishes at each ordinary point common to C_l, C_m; and that for each point A at which C_l has a p-ple point, and C_m a q-ple point, two curves S_{p-2}, S_{q-2} can be found such that the curve $C_n + C_l S_{q-2} + C_m S_{p-2}$ has a $(p+q-1)$-ple point at A. For it is evident that, if $C_n \equiv C_l S_{n-l} + C_m S_{n-m}$, all these conditions are satisfied; and conversely, if the condition as regards $C_n + C_l S_{q-2} + C_m S_{p-2}$ is satisfied, it follows, by (iv) above, that this curve, and therefore also the given curve C_n, contains the cluster pq common to C_l, C_m at A. When C_l, C_m have contact at A, similar but somewhat more complicated conditions can be obtained.*

* See references in Note, p. 498.

9. Noether's Theorem.—By a curve *non-adjoined* to C_m we shall understand any curve which has not exactly $(i-1)$ branches at each and every i-ple point on C_m. Such a curve may not pass at all through some of the multiple points, or may not pass through any of them. If the ordinary points in which any curve C cuts C_m be divided into any two point-groups Q, R, then Q, R are called adjoined or non-adjoined residuals according as the curve C is adjoined or non-adjoined to C_m. Similarly, if any number of curves be drawn through Q, *and have the same number of branches as* C *at each multiple point of* C_m, cutting C_m again in groups of ordinary points R, R', ..., then R, R', ... are called adjoined or non-adjoined coresiduals respectively. Thus adjoined and non-adjoined residuals are ordinary or true residuals deprived of the clusters which belong to them; and similarly for coresiduals. Noether's theorem consists of two parts, of which the first is as follows:—

(i.) *Adjoined coresiduals on* C_m *have the same system of adjoined residuals.**

In other words, if two point-groups R, R' on C_m have a common adjoined residual Q, then any adjoined residual of R is also an adjoined residual of R'. Let a general cluster $Q_{\frac{1}{2}i(i-1)}$ of degree $\frac{1}{2}i(i-1)$ be chosen arbitrarily at each i-ple point on C_m, and let ΣQ denote the point-group made up of Q and all the clusters $Q_{\frac{1}{2}i(i-1)}$. Then the adjoined curve through Q, R may be supposed to pass through the whole point-group ΣQ (Art. 8), and cuts C_m again in ΣR, which is made up of R and general clusters $R_{\frac{1}{2}i(i-1)}$. Thus R is part of a point-group ΣR which is a true residual of ΣQ. Similarly R' is part of a point-group $\Sigma R'$ residual to ΣQ, and ΣR, $\Sigma R'$ are true coresiduals. Again, by similar reasoning, if Q' is any adjoined residual of R, then Q' is part of a point-group $\Sigma Q'$ which is residual to ΣR, and therefore residual to $\Sigma R'$; hence Q', R' are adjoined residuals, which proves the theorem.

If we consider a point-group by itself, as we shall do later on, without reference to any particular curve C_m on which it lies, we must consider the whole point-group, including the ordinary points and clusters. If a point-group contains a general cluster of degree $\frac{1}{2}i(i-1)$, any curve through the point-group has simply an $(i-1)$-ple point at the cluster (Art. 8); and the points of the cluster may therefore be supposed to have any arbitrarily chosen general position, without affecting the character of the point-group.

* Brill-Noether, *loc. cit.*, p. 497.

(ii.) *Non-adjoined coresiduals on C_m are either specialized or general adjoined coresiduals.*

Let C, C' be two curves drawn through a point-group Q on C_m, each having k branches ($k \geqslant 0$) at any i-ple point A, and cutting C_m again in two non-adjoined coresiduals R, R'.

Then, if C, C' have at least $i-1$ branches at each and every i-ple point on C_m, the point-groups R, R' are general adjoined coresiduals.

For the curve C', having a k-ple point at A ($k \geqslant i-1$), passes through $ki-\frac{1}{2}i\,(i-1)$ points of the cluster ki common to C and C_m at A, this being the number of points of the cluster ki which supply independent conditions for $(k-1)$-ics. Thus C, C', C_m have a common cluster at A of degree $ki-\frac{1}{2}i\,(i-1)$, which belongs to a whole point-group ΣQ containing Q. Also, C, C' cut C_m again at A in two different general clusters of degree $\frac{1}{2}i\,(i-1)$, which belong respectively to point-groups ΣR, $\Sigma R'$ containing R, R'. Thus R, R' are parts of two coresidual point-groups ΣR, $\Sigma R'$ each of which contains a general cluster of degree $\frac{1}{2}i\,(i-1)$ at each i-ple point of C_m; *i.e.*, R, R' are general adjoined coresiduals. (Cf. Art. 2, *e*.)

Again, if the curves C, C' have less than $(i-1)$ branches at a single i-ple point of C_m, then R, R' are specialized adjoined coresiduals.

For, supposing k less than $i-1$, the curves C, C', C_m have a common cluster at A of degree $\frac{1}{2}k\,(k+1)$ which belongs to ΣQ; and C, C' cut C_m again at A in two clusters of degree $ki-\frac{1}{2}k\,(k+1)$, or $k\,(i-1)-\frac{1}{2}k\,(k-1)$, which belong to ΣR, $\Sigma R'$ respectively. But *any* adjoined curve through R contains the cluster $k\,(i-1)-\frac{1}{2}k\,(k-1)$ of ΣR (Art. 8, ii.); and therefore contains the whole point-group ΣR, and cuts C_m again in a point-group $\Sigma Q'$ which is residual to $\Sigma R'$. Hence R, R' are adjoined coresiduals.

The point-groups R, R' are however specialized, because curves could be drawn through ΣR, $\Sigma R'$ which have less than $(i-1)$ branches at the point A; which would not be the case if R, R' were general adjoined coresiduals.*

Thus, adjoined coresiduals include all other kinds of coresiduals, such as are usually considered. If however the curves C, C' through Q do not have the same number of branches at each multiple point

* The properties of non-adjoined coresiduals are fully worked out by Noether, "Ueber die Schnittpunktsysteme einer algebraischen Curve mit nicht-adjungirten Curven" (*Math. Ann.*, xv, p. 507).

of C_m, they will determine point-groups R, R' on C_m, which may be considered as non-adjoined coresiduals of the most general kind. The properties of such point-groups are more complicated than those of the non-adjoined coresiduals considered above; but may be investigated from the results of Arts. 8, 21.

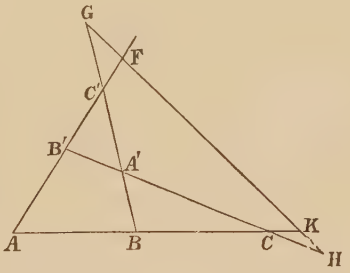
Theorem II (iv) gives the characterization of a point-group which forms the complete intersection of two curves of given order; but we have not yet determined the characterization of any point-groups formed by the partial intersections of curves. Before doing this, we give a few examples on the theorem of residuation; which will help to illustrate the general method of Section III.

II.

Examples on Residuation.

10. *If a complete* 5*-side can be inscribed to a quartic, then any number of complete* 5*-sides can be inscribed, all of which are circumscribed to the same conic.*

Adopting German nomenclature, we define a complete n-side as the figure formed by n straight lines, having $\frac{1}{2}n(n-1)$ corners; and a complete n-corner as the figure formed by n points, having $\frac{1}{2}n(n-1)$ sides.



Let the figure represent a complete 5-side inscribed to a quartic, the letters denoting its ten corners. Then, if any four straight lines be drawn through F, G, H, K, they will cut the quartic again in twelve more points, which lie on a cubic (Theorem III); and if the angles made by the four lines with FA, GB, HC, KA respectively be very small, the twelve points become six pairs close to A, B, C, A', B', C'. Hence a cubic can be drawn touching the quartic at these points. Similarly a conic can be drawn touching the cubic at A', B', C', and therefore also touching the quartic.

Let b', c' be two points on the quartic close to B', C', such that $b'c'$ is a tangent to the conic inscribed to the 5-side; and let the second tangents to the conic from b', c' meet in a'. Then a' lies on the quartic; for the triangles $A'B'C'$, $a'b'c'$ are circumscribed to a conic, and are therefore inscribed to another, viz. the conic which touches

the quartic at A', B', C'. Hence, corresponding to the line $b'c'$, we obtain another complete 5-side inscribed to the quartic and circumscribed to the conic, whence the theorem follows.

Hence if any triangle be circumscribed to the conic, two corners of which lie on the quartic, the third does also. In other words, the 13 corners of a triangle and 5-side circumscribed to the same conic supply only 12 independent conditions for quartics. Quartics through the 13 points are, however, specialized; for a complete 5-side cannot be inscribed to a general quartic. Similarly the 9 corners of a triangle and 4-side circumscribed to the same conic form the base of a pencil of cubics, which are not specialized. Several properties of cubics may be deduced from this, such as, for instance, the fundamental property that the lines joining any fixed point on the curve to pairs of corresponding points of the same kind* are in involution.

The general property, of which the examples given above are particular cases, is the following. *If an m-side and an n-side circumscribe a conic* ($m \leqslant n$), *then the* $(n-1)$*-ic excess of the point-group formed by their* $\frac{1}{2}m(m-1)+\frac{1}{2}n(n-1)$ *corners is* $\frac{1}{2}(m-1)(m-2)$. When $m<n-1$, the point-group forms the partial intersection of two $(n-1)$-ics; when $m=n-1$, it forms the base of a pencil of $(n-1)$-ics; and when $m=n$, it forms the complete intersection of an n-ic and $(n-1)$-ic.

11. *To find the general condition that a group of* 13 *points* Q_{13} *should supply only* 12 *independent conditions for quartics.*

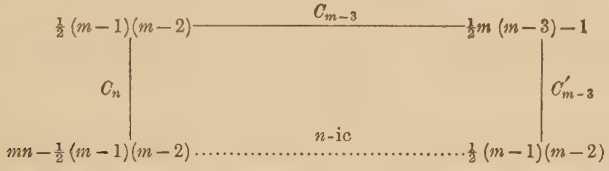
Take any fixed quartic C_4 through the point-group Q_{13} for base-curve. Then, since the 13 given points supply only 12 independent conditions for quartics, it follows that quartics can be drawn through Q_{13} cutting C_4 again in different groups of three points R_3, R_3', ..., which form a coresidual system. Now two point-groups R_3, R_3' on C_4 cannot be coresidual unless they lie on two straight lines which intersect in a point A on the curve. For, if a conic be drawn through R_3, it will cut C_4 again in a point-group Q_5, which is residual to R_3 and to R_3'. Hence more than one conic can be drawn through Q_5; and therefore four of the points Q_5 must lie on a straight line, and R_3, R_3' must lie on two straight lines through the fifth. Hence any two

* Two pairs of corresponding points of the same kind on a cubic are the ends of two diagonals of a complete 4-side inscribed to the curve. (*Higher Plane Curves*, Art. 151.)

quartics through Q_{13} cut again in 3 points on a straight line. Conversely, if any two quartics be drawn through 3 points on a straight line, they will cut again in 13 points which supply only 12 independent conditions for quartics.

12. *A group of* $mn-\frac{1}{2}(m-1)(m-2)$ *given points on a curve* C_m $(n \geq m-2)$ *has an infinite number of residual groups of* $\frac{1}{2}(m-1)(m-2)$ *points, i.e., does not supply* $mn-\frac{1}{2}(m-1)(m-2)$ *independent conditions for n-ics, provided one such residual lies on a curve of order* $m-3$.

Take C_m as base-curve; then the accompanying figure indicates the proof of the theorem.



Base-curve C_m.

Any two point-groups connected by a line in this figure are residual, the order of the curve on which any two residuals lie being marked on the connecting line. Also two point-groups separated by two lines are coresiduals; and two point-groups separated by three lines are residuals (Art. 6).

Starting with the given point-group $mn-\frac{1}{2}(m-1)(m-2)$, a curve C_n is supposed to be drawn through it which cuts the base-curve again in a group of $\frac{1}{2}(m-1)(m-2)$ points, lying on C_{m-3}, by hypothesis; the curve C_{m-3} cuts the base-curve again in a point-group $\frac{1}{2}m(m-3)-1$, through which a second $(m-3)$-ic C'_{m-3} can be drawn, determining a second group of $\frac{1}{2}(m-1)(m-2)$ points; this last point-group is residual to the given one, which proves the theorem.

By similar reasoning we can prove the following theorem.

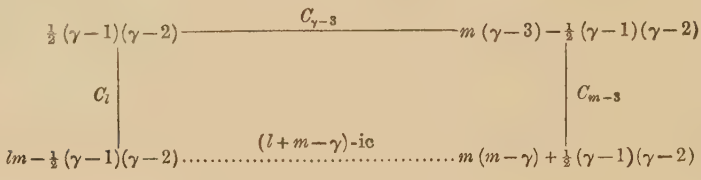
If $n \geq l \geq m < l+m$, $l+m-n=\gamma$; *and if* $\frac{1}{2}(\gamma-1)(\gamma-2)$ *of the points common to* C_l, C_m *lie on a* $(\gamma-3)$*-ic; then n-ics through the remaining* $lm-\frac{1}{2}(\gamma-1)(\gamma-2)$ *points common to* C_l, C_m *do not necessarily pass through the* $\frac{1}{2}(\gamma-1)(\gamma-2)$ *points.**

* Bacharach, *loc. cit.*, p. 497.

Take C_m as base-curve; then the $(\gamma-3)$-ic $C_{\gamma-3}$ through the point-group $\frac{1}{2}(\gamma-1)(\gamma-2)$ cuts the base-curve again in the point-group

$$m(\gamma-3)-\tfrac{1}{2}(\gamma-1)(\gamma-2) = (m-3)(\gamma-3)-\tfrac{1}{2}(\gamma-4)(\gamma-5),$$

through which a curve C_{m-3} can be drawn, without passing through all points on $C_{\gamma-3}$ (Theorem I), *i.e.*, without passing through the point-group $\frac{1}{2}(\gamma-1)(\gamma-2)$.



Base-curve C_m.

The curve C_{m-3} cuts the base-curve again in a point-group

$$m(m-\gamma)+\tfrac{1}{2}(\gamma-1)(\gamma-2),$$

residual to $lm-\frac{1}{2}(\gamma-1)(\gamma-2)$. Hence the $lm-\frac{1}{2}(\gamma-1)(\gamma-2)$ points lie on an $(l+m-\gamma)$-ic, that is an n-ic, which does not pass through the $\frac{1}{2}(\gamma-1)(\gamma-2)$ points; which had to be proved.

These two theorems are particular cases of more general ones, which are considered in the next section.

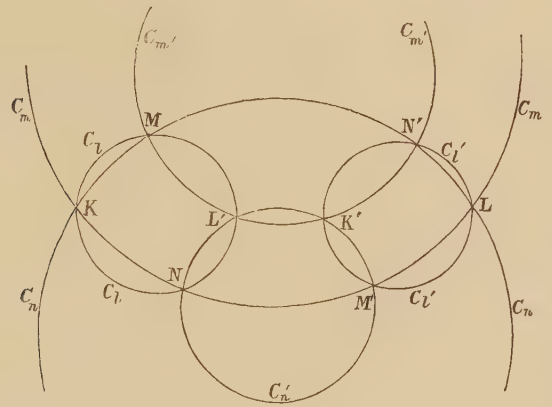
13. *If* $K = 1+\frac{1}{2}(k-1)(k-2)$ *general points be taken in a plane, and three proper curves* C_l, C_m, C_n *be drawn through them, cutting again in pairs in* L, M, *and* N *points; and on* C_l *any* $L_1 = \frac{1}{2}(l-k+1)(l-k+2)$ *more fixed general points be taken, and similarly any* M_1 *and* N_1 *points on* C_m *and* C_n; *and three other curves* C_ν, $C_{m'}$, $C_{n'}$, *of orders* $m+n-k$, $n+l-k$, $l+m-k$, *be drawn through the* $L+M_1+N_1$, $M+N_1+L_1$, *and* $N+L_1+M_1$ *points respectively; then, just as* C_l, C_m, C_n *have* K *points common in the plane, and their remaining points of intersection taken in pairs, viz.* L, M, N, *on* C_ν, $C_{m'}$, $C_{n'}$, *so also* C_ν, $C_{m'}$, $C_{n'}$ *have* K' *points common in the plane, and their remaining points of intersection taken in pairs, viz.* L', M', N' *on* C_l, C_m, C_n; *where* K' *is equal to*

$$mn+nl+lm-kl-km-kn+\tfrac{1}{2}(k-1)(k+4).*$$

---

* This is a generalization of a theorem given by Olivier, "Zur Theorie der Erzeugung geometrischen Curven" (*Crelle's Journal*, Bd. lxxi, p. 1). Cf. Study, "Ueber Schnittpunktfiguren ebener algebraischer Curven" (*Math. Ann.*, Bd. xxxvi, p. 216). The figure is a copy of one in Professor Study's paper.

The point-groups L_1, M_1, N_1 are not marked separately in the figure, but are included in the larger point-groups L', M', N'; their numerical values being connected by the equations written below.



$$\begin{aligned} L &= mn-K, & L' &= L_1+\tfrac{1}{2}(l-1)(l-2) &&= m'n'-K', \\ M &= nl-K, & M' &= M_1+\tfrac{1}{2}(m-1)(m-2) &&= n'l'-K', \\ N &= lm-K. & N' &= N_1+\tfrac{1}{2}(n-1)(n-2) &&= l'm'-K'. \end{aligned}$$

We shall first show that the conditions imposed on $C_{l'}$, $C_{m'}$, $C_{n'}$ are just sufficient to determine them. The curve $C_{l'}$ is required to pass through the $L+M_1+N_1$ points; but

$$\begin{aligned} L+M_1+N_1 &= mn-1-\tfrac{1}{2}(k-1)(k-2)+\tfrac{1}{2}(m-k+1)(m-k+2) \\ &\qquad +\tfrac{1}{2}(n-k+1)(n-k+2) \\ &= \tfrac{1}{2}(m+n-k+1)(m+n-k+2)-1 = \tfrac{1}{2}l'(l'+3). \end{aligned}$$

Hence a curve $C_{l'}$ can certainly be drawn through the $L+M_1+N_1$ points; and no other l'-ic can be drawn through the same points, provided they all supply independent conditions for l'-ics. Now the L points, which are common to C_m, C_n, supply L independent conditions; for if they did not, the remaining K points common to C_m, C_n would lie on an $(m+n-l'-3)$-ic, that is, a $(k-3)$-ic, by Art. 19 (iv).

Also the remaining M_1+N_1 points supply M_1+N_1 more independent conditions; for if they did not, any curve $S_{l'}$ through the L points and all but one of the M_1+N_1 points would necessarily pass through the last; and supposing this last to be one of the N_1 points, then $S_{l'}$

passes necessarily through a point chosen arbitrarily on C_n, and therefore must be of the form $C_n S_{m-k}$; and S_{m-k} passes through the

$$M_1 = \tfrac{1}{2}(m-k+1)(m-k+2)$$

points chosen arbitrarily on C_m, which is impossible.* The curves $C_{l'}$, $C_{m'}$, $C_{n'}$ are therefore completely determined by the given conditions.

Again, taking C_m as base-curve, $C_{l'}$ passes through the point-group $L+M_1$ on C_m, and cuts C_m again in $\frac{1}{2}(m-1)(m-2)$ points, since

$$L+M_1 = ml'-\tfrac{1}{2}(m-1)(m-2).$$

Now the $M_1+\frac{1}{2}(m-1)(m-2)$ points on C_m are residual to the L points, and therefore coresidual to the K points, and therefore residual to the N points. Hence the

$$N+M_1+\tfrac{1}{2}(m-1)(m-2) = mn'$$

points lie on an n'-ic; but $C_{n'}$ passes through the $N+M_1$ points (which supply $N+M_1$ independent conditions), and therefore passes through the $\frac{1}{2}(m-1)(m-2)$ points. Thus $C_{l'}$, $C_{m'}$ intersect in a point-group

$$M' = M_1+\tfrac{1}{2}(m-1)(m-2)$$

on C_m. Hence the three curves $C_{l'}$, $C_{m'}$, $C_{n'}$ intersect in pairs in three groups of L', M', N' points on C_l, C_m, C_n respectively, as shown in the figure.

The curves $C_{m'}$, $C_{n'}$ have the point-group L' common, and cut again in a point-group

$$K' = m'n'-L'.$$

Hence the complete intersection of the two curves $C_m C_{m'}$, $C_n C_{n'}$, is made up of the $K+L+M+N+K'+L'+M'+N'$ points; but of these $K+L'+M+N = l(m+m')$ lie on C_l; therefore the remaining $K'+L+M'+N' = l'(m+m')$ points lie on a curve of order l', viz. $C_{l'}$. This proves the theorem.

The value of K' is given by

$$\begin{aligned} K' &= m'n'-L' \\ &= (l+m-k)(l+n-k)-\tfrac{1}{2}(l-k+1)(l-k+2)-\tfrac{1}{2}(l-1)(l-2) \\ &= mn+nl+lm-kl-km-kn+\tfrac{1}{2}(k-1)(k+4). \end{aligned}$$

Similar properties hold for curves drawn through a point-group K of special form, in which case K might have any numerical value.

* It is assumed that k is a positive integer; if k were zero, the above reasoning would lead to the result that $C_{l'} \equiv C_m C_n$.

III.

Characterization of Rest-Groups.

14. For definitions see Art. 2, $(g\text{–}m)$. We shall suppose that the point-groups, whose characterization we are about to investigate, are made up of general clusters (including ordinary points), of given degree, finitely separated; that the degree of each cluster is a triangular number $\frac{1}{2}i\,(i+1)$; and that any curve drawn through a point-group has at a cluster $\frac{1}{2}i\,(i+1)$ either i or $i+1$ branches. Such a curve has therefore at an ordinary point of the point-group $(i=1)$, either one or two branches; and at any point in the plane not belonging to the point-group $(i=0)$, either no branch or one branch only. Also two curves C_l, C_m drawn through a point-group N will cut again in a rest-group N', of the same type as N. Thus, if N has a cluster $\frac{1}{2}i\,(i+1)$ at A, and C_l, C_m both have i branches at A, then N' has a cluster $\frac{1}{2}i\,(i-1)$ at A; if C_l has $i+1$ and C_m i branches, then N' has a cluster $\frac{1}{2}i\,(i+1)$; and if C_l, C_m both have $i+1$ branches, N' has a cluster $\frac{1}{2}\,(i+1)(i+2)$; all these clusters being general ones.

We shall define a K point-group on a given curve C_m as one which contains a general cluster of degree $\frac{1}{2}i\,(i-1)$ at each i-ple point of C_m, but no ordinary points; so that *a curve adjoined to C_m, and a curve through a* K *point-group, are equivalent terms.* If p is the deficiency of C_m, the degree of a K point-group is

$$\mathrm{K} = \Sigma\,\tfrac{1}{2}i\,(i-1) = \tfrac{1}{2}\,(m-1)(m-2)-p.$$

Also the number of ordinary points in which an adjoined $(m-3)$-ic cuts C_m is

$$m\,(m-3)-\Sigma i\,(i-1) = m\,(m-3)-(m-1)(m-2)+p = 2p-2\,;$$

and an adjoined $(m-3+r)$-ic cuts C_m in $2p-2+rm$ ordinary points.

An $(m-3)$-ic adjoined to C_m cannot be drawn when $p=0$, and does not cut C_m in any ordinary points when $p=1$. Hence, when considering adjoined $(m-3)$-ics, we shall assume that $p>1$. It should be noticed however, that when $p=0$ or 1, a K point-group supplies K independent conditions for $(m-3)$-ics. (Cf. Art. 19, i.)

A square bracket enclosing a triangular number, as in Art. 15 (iii), denotes that the number is to be retained when its factors, in the form in which they are written, are positive; and rejected when negative.

15. (i.) *A simple rest-group* (*Art.* 2, *l*) *can be immediately derived from an incomplete point-group, but not from a redundant one.*

For, if N is incomplete, a simple rest-group N' can be immediately derived from N by drawing two curves through it, of which one at least does not pass through the remaining points of the plane which complete N. If however N is redundant, any immediately derived rest-group N' must be incomplete; since the general points of N, which have no connexion with the rest, will, if added to N', at the least make N' more complete than before.

(ii.) Since the number of independent conditions supplied by a point-group N for n-ics is $N-r_n$, and the degree of freedom of an n-ic through N is q_n, we have the relation

$$N-r_n+q_n=\tfrac{1}{2}n(n+3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

(iii.) *If n-ics are drawn through any point-group N on a base-curve* C_m, *the multiplicity of the system of point-groups in which they cut* C_m *again is*

$$q_n-\left[\tfrac{1}{2}(n-m+1)(n-m+2)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let S_n be any n-ic through N, cutting C_m again in $mn-N$. Then the n-ic defect of N, viz. q_n, is the number of general points on C_m, together with the number of additional general points in the plane, through which S_n can be drawn; and the former number is the multiplicity of $mn-N$; and the latter is $\frac{1}{2}(n-m+1)(n-m+2)$ or 0, according as $n \geqslant m$ or $n<m$. Hence the multiplicity of $mn-N$ is

$$q_n-\left[\tfrac{1}{2}(n-m+1)(n-m+2)\right].$$

16. Example.—*The n-ic excess of a point-group* $N=lm+\alpha$ *on a base-curve* C_m, *coresidual to a group of* $\alpha<\frac{1}{2}(n-l+1)(n-l+2)>0$ *general points, none of which belong to N, is* $\frac{1}{2}(l+m-n-1)(l+m-n-2)$; *provided* $n>l<l+m$, *and also* $n>m-3$.

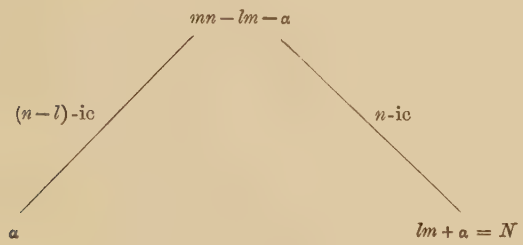
We take $\alpha>0$, because the theorem has been already proved for the case $\alpha=0$ (Theorem II, iv); and $\alpha<\frac{1}{2}(n-l+1)(n-l+2)$, because otherwise an $(n-l)$-ic could not be drawn through α, nor an n-ic through N, except one of the form $C_m S_{n-m}$. Also we take $n>l$, because $N>lm$; $n<l+m$, because the $(l+m-2)$-ic excess is zero; and $n>m-3$, because no point-group on C_m coresidual to a general point-group can lie on a curve of lower order than $m-2$.*

* It can be proved that no residual of α general points can lie on an $(m-3)$-ic, or curve of lower order, which does not pass through the point-group α.

If $l<m$, then it can be proved that $\alpha>\frac{1}{2}(m-l-1)(m-l-2)$; if however $l>m-3$, α might have any value from 1 to $\frac{1}{2}m(m-3)$.

Now an $(n-l)$-ic through the α general points cuts C_m again in a group of $mn-lm-\alpha$ points, whose multiplicity (since $n-l<m$) is

$$\tfrac{1}{2}(n-l)(n-l+3)-\alpha.$$



Base-curve C_m.

The point-groups $mn-lm-\alpha$ and N are residual. Hence an n-ic can be drawn through the point-group N; and, by Art. 15 (iii), the multiplicity of $mn-lm-\alpha$ (since $n>m-3$) is

$$q_n-\tfrac{1}{2}(n-m+1)(n-m+2).$$

Equating this to the value found above, we have

$$q_n=\tfrac{1}{2}(n-m+1)(n-m+2)+\tfrac{1}{2}(n-l)(n-l+3)-\alpha.$$

But, by Art. 15 (ii),

$$lm+\alpha-r_n+q_n=\tfrac{1}{2}n(n+3);$$

therefore
$$r_n=lm+\alpha+\tfrac{1}{2}(n-m+1)(n-m+2)$$
$$+\tfrac{1}{2}(n-l)(n-l+3)-\alpha-\tfrac{1}{2}n(n+3)$$
$$=\tfrac{1}{2}(l+m-n-1)(l+m-n-2).$$

Thus r_n is zero, when $n=l+m-2$; and therefore also when n has any higher value.

Suppose that α contains a K point-group (Art. 14); then so also does N. Hence, taking $\alpha=\mathrm{K}+\beta$ and $N=\mathrm{K}'+R$, the result proved may be expressed as follows:— If a group of $R=lm+\beta$ ordinary points on a base-curve C_m, of deficiency p, has a non-specialized adjoined coresidual of $\beta<\tfrac{1}{2}(n-l+1)(n-l+2)-\tfrac{1}{2}(m-1)(m-2)+p>0$ general points on C_m, then the excess of R for adjoined n-ics is equal to $\tfrac{1}{2}(l+m-n-1)(l+m-n-2)$; provided a K point-group has no excess for $(n-l)$-ics, and $n>l<l+m$, $n>m-3$.

17. Theorem IV.—*The necessary and sufficient condition that a point-group N, on a base-curve C_m, should lie on an $(m-3)$-ic, is*

$$\rho \geq N-\tfrac{1}{2}m\,(m-3)\,;$$

ρ being the multiplicity of N.

We regard the given point-group N as determining a series of other point-groups N, forming a complete coresidual system, of multiplicity ρ (Art. 2, m).

(i.) Suppose that the given point-group N lies on an $(m-3)$-ic; and that a fixed $(m-3)$-ic is drawn through it, cutting C_m again in N' $(N+N'=m\,.\,\overline{m-3})$. Then every point-group N is residual to N';

therefore $$\rho \geq \tfrac{1}{2}m\,(m-3)-N' \geq N-\tfrac{1}{2}m\,(m-3).$$

(ii.) Conversely, suppose that $\rho \geq N-\frac{1}{2}m\,(m-3)$. Then we have to prove that N lies on an $(m-3)$-ic. All the point-groups N may possibly have a certain number of fixed points in common; but, whether they have or not, it is clear that ρ points, say A, B, C, ... H, can be chosen out of any point-group N, which do not all belong to any other point-group of the system.

Take another point-group of the system containing B, C, ... H, but not A; and denote the remainders of the two point-groups, when B, C, ... H are taken away, by $R_{N-\rho+1}$, $R'_{N-\rho+1}$ respectively. Then

$$R_{N-\rho+1} \equiv R'_{N-\rho+1}. \qquad \text{(Art. 5)}$$

Draw any straight line through A, cutting C_m again in Q_{m-1}; and any $(m-3)$-ic through the remaining $N-\rho$ $(\leqslant \tfrac{1}{2}m\,.\,\overline{m-3})$ points of $R_{N-\rho+1}$, cutting C_m again in the point-group $Q_{m(m-3)-N+\rho}$. Then

$$Q_{m-1}+Q_{m(m-3)-N+\rho}+R_{N-\rho+1} \equiv 0\,;$$

therefore $$Q_{m-1}+Q_{m(m-3)-N+\rho}+R'_{N-\rho+1} \equiv 0.$$

These last $m(m-2)$ points therefore lie on an $(m-2)$-ic, which must break up into the straight line containing the $(m-1)$ points Q_{m-1} and an $(m-3)$-ic. But the straight line through Q_{m-1} cuts C_m again in A; hence the point A must be included in

$$Q_{m-1}+Q_{m(m-3)-N+\rho}+R'_{N-\rho+1}\,;$$

i.e., A is necessarily included in $Q_{m(m-3)-N+\rho}$. Hence any $(m-3)$-ic through the $N-\rho$ points of N passes necessarily through A, and

similarly through all the ρ points. Thus N lies on an $(m-3)$-ic, *and its* $(m-3)$*-ic excess is not less than* ρ.

To apply the theorem to the case of adjoined curves, we suppose that N consists of a K point-group (Art. 14), and R ordinary points on C_m. Then

$$N = \mathrm{K} + R = \tfrac{1}{2}(m-1)(m-2) - p + R;$$

therefore $$N - \tfrac{1}{2}m(m-3) = R - p + 1.$$

Hence the necessary and sufficient condition that N lies on an $(m-3)$-ic, or that R lies on an adjoined $(m-3)$-ic, is

$$\rho \geq R - p + 1;$$

where ρ is the multiplicity of N, and therefore also of R.

18. The Riemann-Roch Theorem.—*The multiplicity of any point-group on a base-curve* C_m *is equal to its* $(m-3)$*-ic excess.*

If the point-group N does not lie on an $(m-3)$-ic, then the number of independent conditions it supplies for $(m-3)$-ics is $\frac{1}{2}(m-1)(m-2)$, by Art. 7; and its $(m-3)$-ic excess is therefore $N - \frac{1}{2}(m-1)(m-2)$. Also its multiplicity on C_m is less than $N - \frac{1}{2}m(m-3)$, by Theorem IV; and is not less than $N - \frac{1}{2}(m-1)(m-2)$, since this number of points at least can be chosen arbitrarily for determining a point-group on C_m belonging to a given coresidual system of degree N (Theorem I). Hence the multiplicity of N is $N - \frac{1}{2}(m-1)(m-2)$; *i.e.*, is equal to its $(m-3)$-ic excess.

If the point-group N lies on an $(m-3)$-ic, let an $(m-3)$-ic be drawn through it, cutting C_m again in N' $(N + N' = m \cdot \overline{m-3})$; and let q_{m-3}, r^-, ρ and q'_{m-3}, r'_{m-3}, ρ' denote the $(m-3)$-ic defects, $(m-3)$-ic excesses, and multiplicities, of N and N' respectively.

Then, by Art. 15 (iii), $\rho = q'_{m-3}$, $\rho' = q_{m-3}$;

and, by Art. 17, $r_{m-3} \geq \rho \geq q'_{m-3}$, $r'_{m-3} \geq \rho' \geq q_{m-3}$;

and, by Art. 15 (ii), $r_{m-3} - q_{m-3} = N - \frac{1}{2}m(m-3)$

$$= N - \tfrac{1}{2}(N + N') = \tfrac{1}{2}(N - N') = q'_{m-3} - r'_{m-3}.$$

But neither $r_{m-3} - q'_{m-3}$ nor $r'_{m-3} - q_{m-3}$ is negative, from above;

therefore $r_{m-3} = q'_{m-3} = \rho$, and $r'_{m-3} = q_{m-3} = \rho'$.

Thus, if $N+N' = m(m-3) \equiv 0$, *the* $(m-3)$*-ic defect and excess of* N *are equal respectively to the* $(m-3)$*-ic excess and defect of* N'*; and the difference of the multiplicities of* N, N', *viz.* $\rho-\rho'$, *is equal to* $\frac{1}{2}(N-N')$.

19. The theorem of residuation and the Riemann-Roch theorem express the two fundamental properties of point-groups on curves. Some of the immediate consequences of the latter theorem are the following.

(i.) *A* K *point-group on* C_m *supplies* K *independent conditions for* $(m-3)$*-ics; and an* $(m-3)$*-ic through* K, *i.e., an adjoined* $(m-3)$*-ic, does not of necessity pass through any fixed ordinary point on* C_m.

Draw any $(m-3)$-ic through K, cutting C_m again in a point-group $K'+2p-2$, of which $2p-2$ are ordinary points on C_m, and K' is a point-group of the same kind as K. Then, if the $(m-3)$-ic excess of K is not zero, the $(m-3)$-ic defect of $K'+2p-2$ is not zero (Art. 18), and an $(m-3)$-ic can be drawn through $K'+2p-2$, and one or more other arbitrary points on C_m. We should then have an adjoined $(m-3)$-ic cutting C_m in more than $2p-2$ ordinary points, which is impossible. Hence the $(m-3)$-ic excess of a K point-group is zero.

Again, if an $(m-3)$-ic through K passes necessarily through a fixed ordinary point A on C_m, the $(m-3)$-ic excess of the point-group consisting of K and the point A is 1; and the $(m-3)$-ic defect of a residual point-group $K'+2p-3$ is also 1. Hence an $(m-3)$-ic through $K'+2p-3$, which passes necessarily through A, by hypothesis, can be made to pass through another arbitrary point on C_m, and cuts C_m in $2p-1$ ordinary points altogether, which is impossible.

Since the $(m-3)$-ic excess of K is zero, the multiplicity of the groups of $2p-2$ ordinary points cut from C_m by adjoined $(m-3)$-ics is

$$\tfrac{1}{2}m(m-3) - K = p-1.$$

(ii.) If the $(m-3)$-ic excess of any point-group $K+2$ on C_m is 1, the two ordinary points of $K+2$ belong to a system of adjoined coresidual point-pairs on C_m. For the multiplicity of $K+2$ is 1 (Art. 18); and any coresidual of $K+2$, of the same degree, is a point-group $K'+2'$, containing 2 ordinary points.

Also any adjoined $(m-3)$-ic through one point of a point-pair passes necessarily through the other, by (i); and therefore the $2p-2$ ordinary points, in which any adjoined $(m-3)$-ic cuts C_m, consist of $(p-1)$ point-pairs. Curves which contain such systems of point-pairs are called *hyperelliptic* curves, when $p>1$. A curve whose deficiency is 2 is

necessarily hyperelliptic. If N is any point-group for which $r_{m-3} = 1$; and an m-ic C_m can be described having an i-ple point ($i > 2$) at each cluster of N of degree $\frac{1}{2}i(i-1)$, and a double point at each of the ordinary points of N except two, and passing through the last two; then C_m is hyperelliptic.

(iii.) *If an n-ic C_n cuts C_m in two residual point-groups N, N', and r_n is not zero; then N' lies on an $(m-3)$-ic, which does not pass through N.*

For, if $n \geq m-2$, the multiplicity of N' on C_m is

$$\begin{aligned}\rho' &= q_n - \tfrac{1}{2}(n-m+1)(n-m+2) \qquad \text{(Art. 15, iii)}\\ &= \tfrac{1}{2}n(n+3) - (mn-N') + r_n - \tfrac{1}{2}(n-m+1)(n-m+2)\\ &= N' - \tfrac{1}{2}m(m-3) + (r_n - 1)\\ &\geq N' - \tfrac{1}{2}m(m-3).\end{aligned}$$

Hence N' lies on an $(m-3)$-ic (Theorem IV), which does not pass through N; since $n \geq m-2$.

Similarly, if $n \leq m-3$,

$$r'_{m-3} = \rho' = q_n = \tfrac{1}{2}n(n+3) - (mn-N') + r_n;$$

therefore
$$\begin{aligned}N' - r'_{m-3} &= mn - \tfrac{1}{2}n(n+3) - r_n\\ &= (m-3)n - \tfrac{1}{2}(n-1)(n-2) - (r_n - 1)\\ &\leq (m-3)n - \tfrac{1}{2}(n-1)(n-2).\end{aligned}$$

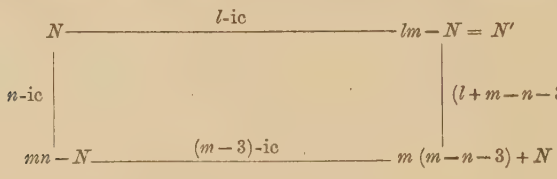
But this is the condition that an $(m-3)$-ic can be drawn through N', which is not of the form $C_n S_{m-3-n}$, or which does not pass through N.

If the point-group N is redundant, any $(m-3)$-ic through N' passes necessarily through the redundant points of N, but not through the remainder.

Conversely, if r_n is zero, the point-group N' does not lie on an $(m-3)$-ic, when $n \geq m-2$; and can only lie on an $(m-3)$-ic which passes through N, when $n \leq m-3$.

(iv.) *If two curves C_l, C_m through a point-group N cut again in a rest-group N', and if $q_n \geq 0$, $r_n \geq 1$, $r_{n+1} = 0$; then the curve of lowest order through N', which does not pass through N, is an $(l+m-n-3)$-ic.*

For an n-ic through N cuts C_m again in a point-group $mn-N$; which lies on an $(m-3)$-ic, by (iii); this cuts C_m again in a point-



Base-curve C_m.

group $N+m(m-n-3)$, which does not contain all the N points. Hence N' lies on an $(l+m-n-3)$-ic, which does not pass through N; and by similar reasoning, since r_{n+1} is zero, N' cannot lie on an $(l+m-n-4)$-ic which does not pass through N.

20. Theorem V.—*If any two curves C_l, C_m be drawn through a point-group N, cutting again in a rest-group N' $(N+N' = lm)$; then, provided $q_n \geq 0$, $r_n \geq 1$,*

$$r'_{l+m-n-3} = q_n + 1 - \left[\tfrac{1}{2}(n-l+1)(n-l+2)\right] - \left[\tfrac{1}{2}(n-m+1)(n-m+2)\right] \quad \ldots\ldots\ldots\ldots\ldots (3),$$

$$q'_{l+m-n-3} = r_n - 1 + \left[\tfrac{1}{2}(l-n-1)(l-n-2)\right] + \left[\tfrac{1}{2}(m-n-2)(m-n-2)\right] \quad \ldots\ldots\ldots\ldots\ldots (4).$$

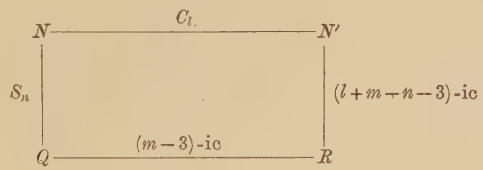
This theorem practically determines the complete characterization of N', if that of N is known; for $q_n \geq 0$, $r_n \geq 1$ are the necessary and sufficient conditions that an $(l+m-n-3)$-ic can be drawn through N', without passing through N (Art. 19, iv). In the case of these curves drawn through N' which, as a consequence, pass through N, the excess of N' is that of $N+N'$ diminished by N.* (See Note, p. 504.)

Also, assuming $l \leqslant m$, it should be noticed that the general n-ic S_n through N, although it may necessarily pass through N', has no factor in common with C_m, and therefore cuts C_m again in a finite rest-group Q.

Equation (4) is only another form of (3), and can be deduced from it by means of Art. 15 (ii). Hence it is only necessary to prove (3).

* By taking $q_n = -1$ when N does not lie on an n-ic (Note, p. 507), the equations (3), (4) can be proved to hold for all values of n.

(i.) Suppose that S_n does not pass through N', so that Q does not contain all the N' points. Then, since r_n is not zero, Q lies on an



Base-curve C_m.

$(m-3)$-ic (Art. 19, iii). Let any $(m-3)$-ic through Q cut C_m again in R. Then N', R are residual and lie on an $(l+m-n-3)$-ic.

By Art. 15 (iii), the multiplicity of R is

$$q'_{l+m-n-3} - \left[\tfrac{1}{2}(l-n-1)(l-n-2)\right];$$

and the multiplicity of Q is

$$q_n - \left[\tfrac{1}{2}(n-m+1)(n-m+2)\right].$$

But, by Art. 18, the difference of these is $\frac{1}{2}(R-Q)$; hence

$$\begin{aligned} q'_{l+m-n-3} - \left[\tfrac{1}{2}(l-n-1)(l-n-2)\right] - q_n + \left[\tfrac{1}{2}(n-m+1)(n-m+2)\right] \\ = \tfrac{1}{2}(R-Q) = R - \tfrac{1}{2}(Q+R) \\ = m(l+m-n-3) - N' - \tfrac{1}{2}m(m-3); \end{aligned}$$

therefore

$$\begin{aligned} r'_{l+m-n-3} &= N' + q'_{l+m-n-3} - \tfrac{1}{2}(l+m-n)(l+m-n-3) \qquad \text{(Art. 15, ii.)} \\ &= q_n + \left[\tfrac{1}{2}(l-n-1)(l-n-2)\right] - \left[\tfrac{1}{2}(n-m+1)(n-m+2)\right] \\ &\quad + m(l+m-n-3) - \tfrac{1}{2}m(m-3) - \tfrac{1}{2}(l+m-n)(l+m-n-3) \\ &= q_n + \left[\tfrac{1}{2}(l-n-1)(l-n-2)\right] - \left[\tfrac{1}{2}(n-m+1)(n-m+2)\right] \\ &\quad - \tfrac{1}{2}(l-n)(l-n-3) \\ &= q_n + 1 - \left[\tfrac{1}{2}(n-l+1)(n-l+2)\right] - \left[\tfrac{1}{2}(n-m+1)(n-m+2)\right]. \end{aligned}$$

(ii.) Suppose that S_n necessarily passes through N'. Then Q contains N', and any $(m-3)$-ic through R passes through a point-group Q, and therefore passes through all the N' points. Hence the $(l+m-n-3)$-ic excess of N' is zero (Art. 19, iii). Again, since S_n necessarily passes through N', n cannot be less than l; and, if n is less than m, S_n must contain C_l as a factor; therefore

$$q_n = \tfrac{1}{2}(n-l)(n-l+3) + \left[\tfrac{1}{2}(n-m+1)(n-m+2)\right].$$

Hence the relation (3) gives $r'_{l+m-n-3} = 0$, which is the correct value. Conversely, if $r'_{l+m-n-3} = 0$, an n-ic drawn through N must also pass through N'.

Hence an n-ic ($n \geqslant l \geqslant m < l+m-2$) through N points common to two given curves C_l, C_m does or does not necessarily pass through the remaining N' points according as the $(l+m-n-3)$-ic excess of N' is or is not zero.

21. Example. — Suppose that three proper curves C_l, C_m, C_n ($l \leqslant m$, $m \leqslant n$) have N points common in all, which are moreover N general points of intersection of an l-ic and m-ic. To find the excesses of the rest-group N' common to C_m, C_n.

From the conditions of the question we must have

$$N < lm - \left[\tfrac{1}{2}(l+m-n-1)(l+m-n-2)\right];$$

but this, and the condition $N \geqslant \frac{1}{2}l(l+3)$, are included in those found below. From the relation (3) of the last article, we have

$$r'_{m+n-p-3} = q_p + 1 - \left[\tfrac{1}{2}(p-m+1)(p-m+2)\right] - \tfrac{1}{2}\left[(p-n+1)(p-n+2)\right],$$

provided $q_p \geqslant 0$, and $r_p \geqslant 1$.

Now the condition $q_p \geqslant 0$ is equivalent to $p \geqslant l$. Also the condition $r_p \geqslant 1$ is equivalent to $N + q_p \geqslant \frac{1}{2}(p+1)(p+2)$, by Art. 15 (ii); and requires p to be less than n, since $r_n = 0$. Hence the last term in the value of $r'_{m+n-p-3}$ above, must be omitted.

Again, since the N points are general points of intersection of an l-ic and m-ic, and r_p is not zero; a p-ic through N must have C_l for a factor if $p < m$, and must pass through all points common to C_l, C_m, if $p \geqslant m$; hence we have

$$q_p = \tfrac{1}{2}(p-l)(p-l+3) + \left[\tfrac{1}{2}(p-m+1)(p-m+2\right].$$

Substituting this in the value of $r'_{m+n-p-3}$, we have

$$r'_{m+n-p-3} = \tfrac{1}{2}(p-l+1)(p-l+2),$$

provided $p \geqslant l < n$, and $N + q_p \geqslant \frac{1}{2}(p+1)(p+2)$.

Changing p to $m+n-p-3$, we may express the result in the form

$$r'_p = \tfrac{1}{2}(m+n-l-p-1)(m+n-l-p-2),$$

provided $N > l(m+n-p) - \frac{1}{2}l(l+3) - \left[\frac{1}{2}(n-p-1)(n-p-2)\right]$, and $p < m+n-l > m-3$.

This result is a similar one to that in Art. 16, but more general.

22. Theorem VI.—*If a point-group of given degree N has, as a consequence of satisfying certain given conditions, a definite number k of absolute connexions, and a definite absolute n-ic multiplicity x_n (Art. 2, m); then*

$$N - x_n + r_n = k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The number k is the number of independent geometrical connexions that exist between any N points which satisfy the given conditions; and $2N-k$ is the least number of parameters in terms of which the $2N$ coordinates of any such N points can be expressed.

The given conditions might be of such a kind that, although n-ics could be drawn through any point-group N, such n-ics would be necessarily specialized. In that case the equation (5) will still hold if x_n and r_n denote the absolute multiplicity and excess of N for the specialized n-ics. A simple example of this is the point-group formed by the vertices and orthocentre of any triangle ($N = 4$, $k = 2$); all conics through any such point-group being equilateral hyperbolas.

Since x_n has a definite value, it follows that a point-group N can be placed on a general n-ic; and this implies that the given conditions do not necessitate the point-groups having any clusters. This, however, does not prevent some of the x_n arbitrary points on a given n-ic being chosen at the multiple points (if there are any) so as to form clusters; although it will generally happen that there is a limit, less than x_n, to the number that can be chosen in this way. As bearing on this, it should be noticed that x_n is the number of *general* points that can be chosen arbitrarily on a given n-ic (Art. 2, m); so that, if the x_n points, or part of them, are chosen in a special position, the problem of finding the remainder may become a porismatic one, either having no solutions, or else having a greater multiplicity of solutions than it would have in the general case.

The relation (5) may be regarded as determining k if r_n and x_n are known for one particular value of n; and, when k is known, as determining x_n for all values of n for which r_n is known.

The theorem can be applied to a point-group with a given characterization, by finding the simplest construction for such a point-group, and deducing therefrom the value of k (Arts. 26–31).

(i.) Suppose that $r_n = 0$; then we have to prove that

$$N - x_n = k, \quad \text{or} \quad x_n = N - k.$$

It is clear that x_n is not less than $N-k$, since the k absolute connexions cannot make more than k points of any point-group N on a given curve to be dependent on the rest.

Again, x_n is not greater than $N-k$. For the coordinates of all the points of any point-group N are expressible in terms of $2N-k$ parameters, and if these coordinates so expressed be substituted in the general equation $S_n = 0$ of a curve of the n^{th} order, we obtain N equations involving the coefficients of S_n and the $2N-k$ parameters; and these equations are independent, since $r_n = 0$. Now, regarding S_n as a given curve, each point on it has one parameter; and each of the parameters of the N points on S_n are expressible in terms of the coefficients of S_n and the $2N-k$ parameters of the point-group N. If therefore more than $N-k$ general points of a group N on S_n could be chosen arbitrarily, we should, by equating their parameters to general arbitrary values $\alpha, \beta, \ldots$, have more than $N-k$ additional independent equations; making with the N equations more than $2N-k$ altogether. The $2N-k$ parameters of the point-group N could therefore be eliminated, giving rise to one or more equations between the coefficients of S_n and the arbitrary quantities $\alpha, \beta, \ldots$, which is impossible. Hence x_n must be equal to $N-k$.

(ii.) If $r_n > 0$; choose $N-r_n$ points of any point-group N which supply $N-r_n$ independent conditions for n-ics. Then if $q_n > 0$, *i.e.*, if $N-r_n < \frac{1}{2}n\,(n+3)$, the remaining r_n points of N are dependent on the $N-r_n$ points, since they lie on all n-ics through the $N-r_n$ points; and the $N-r_n$ points have therefore only $k-2r_n$ absolute connexions. Hence, applying the result of (i) to the $N-r_n$ points, we have

$$x_n = (N-r_n) - (k-2r_n), \quad \text{or} \quad N - x_n + r_n = k.$$

If $r_n \geqslant 0$, and $N-r_n = \frac{1}{2}n\,(n+3)$; only one n-ic C_n can be drawn through a given point-group N; and all the $\frac{1}{2}n\,(n+3)$ coefficients of C_n, and the x_n parameters of x_n general points on C_n, are expressible in terms of the $2N-k$ parameters of N; and vice versa. Hence

$$\tfrac{1}{2}n\,(n+3) + x_n = 2N-k, \quad \text{or} \quad N - x_n + r_n = k.*$$

* A difficult problem in enumerative geometry, is to find the number of point-groups N on a given n-ic, corresponding to x_n general points chosen arbitrarily on

23. The two following examples illustrate Theorem VI.

(i.) *To find the number of absolute connexions of the point-group formed by the complete intersection of any l-ic and m-ic.**

If $l<m$, we have

$$N = lm, \quad x_m = \tfrac{1}{2}l(l+3), \quad r_m = \tfrac{1}{2}(l-1)(l-2);$$

hence $$k = lm - \tfrac{1}{2}l(l+3) + \tfrac{1}{2}(l-1)(l-2) = lm - 3l + 1.$$

If $l = m$, we have

$$N = m^2, \quad x_m = \tfrac{1}{2}m(m+3) - 1, \quad r_m = \tfrac{1}{2}(m-1)(m-2);$$

hence $k = m^2 - \tfrac{1}{2}m(m+3) + 1 + \tfrac{1}{2}(m-1)(m-2) = (m-1)(m-2)$.

(ii.) *To find the number x_n of general points that can be chosen on a given curve C_n which belong to a point-group on C_n formed by the complete intersection of an l-ic and m-ic.*

We assume that an n-ic ($n \geq l \geq m$) through all the points common to a general l-ic and m-ic is not a specialized n-ic; from which it follows that a point-group forming the complete intersection of an l-ic and m-ic can be placed on a given n-ic.

If $l<m$; then

$$N = lm, \quad r_n = \left[\tfrac{1}{2}(l+m-n-1)(l+m-n-2)\right], \quad k = lm - 3l + 1;$$

therefore

$$lm - x_n + \left[\tfrac{1}{2}(l+m-n-1)(l+m-n-2)\right] = lm - 3l + 1,$$

or $$x_n = 3l - 1 + \left[\tfrac{1}{2}(l+m-n-1)(l+m-n-2)\right].$$

If $l = m$; then $$k = m^2 - 3m + 2;$$

and $$x_n = 3m - 2 + \left[\tfrac{1}{2}(2m-n-1)(2m-n-2)\right].\dagger$$

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the n-ic. The solution will, of course, depend on the nature of the given conditions. Castelnuovo (*Rendiconti della Reale Accademia dei Lincei*, v, p. 130) has given a solution for the number of ordinary point-groups, of lowest degree, on a given curve C_m, having a given multiplicity r; when p, the deficiency of C_m, is divisible by $r+1$. Brill (*Math. Ann.*, xxxvi, p. 321), and Zeuthen (*Math. Ann.*, xl, p. 119), have also given solutions for the number of ordinary point-groups, of given degree, on C_m, cut out by adjoined n-ics; when the excess of each point-group for adjoined n-ics is 1.

* Cf. Jacobi (*Crelle's Journal*, xv, p. 285).

† Cf. Cremona (*Teoria Geometrica delle Curve Piane*, Bologna, p. 46), and Clebsch-Lindemann (*Leçons sur la Géométrie*, T. iii, p. 129), for the case $l = m$.

24. Theorem VII.—*If k, k' are the numbers of absolute connexions of two point-groups N, N' $(N+N'=lm)$, which satisfy such given conditions that any l-ic and m-ic through any point-group N or N' cut again in a point-group N' or N respectively; then*

$$k-r_l-r_m = k'-r'_l-r'_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

The enunciation imposes a strict limitation to the given conditions, in addition to the limitations explicitly stated in Theorem VI; but the theorem certainly applies when either N or N' is a rest-group, derived by any number of steps, from a general point-group with no absolute connexions.

Suppose that we take any fixed base-curve C_m, and find the least number of parameters in terms of which the positions of all the points of two residuals N, N' on C_m, satisfying the given conditions, can be expressed. The number of general points that can be chosen arbitrarily which belong to a point-group N on C_m, is x_m; but when N is determined, we can still choose ρ' points of N' arbitrarily, ρ' being the multiplicity of N' on C_m. Hence $x_m+\rho'$ is the least number of parameters required. Similarly $x'_m+\rho$ is also the least number of parameters; hence

$$x_m+\rho' = x'_m+\rho,$$

or
$$x_m-x'_m = \rho-\rho'.$$

But
$$\rho = q'_l-\left[\tfrac{1}{2}(l-m+1)(l-m+2)\right],$$

and
$$\rho' = q_l-\left[\tfrac{1}{2}(l-m+1)(l-m+2)\right]; \qquad \text{(Art. 15, iii)}$$

therefore
$$x_m-x'_m = q'_l-q_l$$
$$= (N-r_l)-(N'-r'_l), \qquad \text{(Art. 15, ii)}$$

or
$$N-x_m-r_l = N'-x'_m-r'.$$

But
$$N-x_m = k-r_m,$$

and
$$N'-x'_m = k'-r'_m; \qquad \text{(Theorem VI)}$$

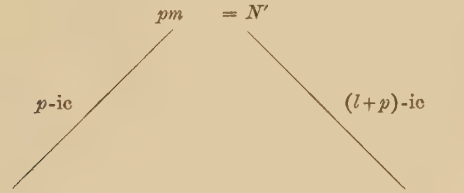
therefore
$$k-r_l-r_m = k'-r'_l-r'_m.$$

Some, or all, of the quantities r_l, r_m, r'_l, r'_m, may be zero.

25. Example.—*To find the number of absolute connexions of a group* $N = lm+a$ *on* C_m, *which has a coresidual of* $a < \frac{1}{2}(m-1)(m-2)$ *general points.*

This point-group is considered in Art. 16, where the value of r_n is found. Suppose $p\ (\leqslant m-3)$, the order of the lowest curve through a, cutting C_m again in $pm-a = N'$. Then the N' points are general points on a p-ic; and, by Theorem VII,

$$k-r_m-r_{l+p} = k'-r'_m-r'_{l+p};$$



Base-curve C_m.

but $r'_{l+p} = 0, \quad k'-r'_m = N'-x'_m$

$$= (pm-a)-\tfrac{1}{2}p\,(p+3),$$

$$r_{l+p} = \tfrac{1}{2}\,(m-p-1)(m-p-2)\,; \qquad \text{(Art. 16)}$$

hence $k = r_m+\frac{1}{2}\,(m-p-1)(m-p-2)+pm-a-\frac{1}{2}p\,(p+3)$

$$= r_m-a+\tfrac{1}{2}\,(m-1)(m-2).$$

And, according as only one m-ic C_m, or more than one m-ic can be drawn through N; the conditions for the latter case being that $m > l$, and $a < \frac{1}{2}\,(m-l+1)(m-l+2)$; we have

$$r_m = lm+a-\tfrac{1}{2}m\,(m+3), \quad \text{or} \quad \tfrac{1}{2}\,(l-1)(l-2)\,; \qquad \text{(Art. 16)}$$

and $k = lm-3m+1, \quad \text{or} \quad \frac{1}{2}\,(l+m-1)(l+m-2)-N+1.$*

Hence, since k is known, and also the value of r_n for all values of n, the value of x_n is also known (Theorem VI).

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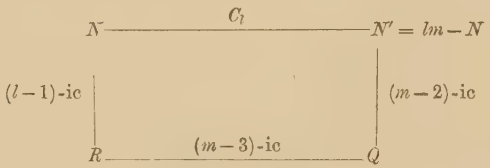
* In this case, $k > lm-3m+1 < lm-3l+1$; for $a < \frac{1}{2}\,(m-l+1)(m-l+2)$, from above; and $a > \frac{1}{2}\,(m-l-1)(m-l-2)$, since $l < m$ (Note, Art. 16).

26. Construction of Point-Groups whose Characterization is given.—By means of Theorem V, we can prove that from a non-composite point-group N, whose characterization is given, we can derive a series of rest-groups in succession, each one of which has a less complex characterization than the preceding, until we arrive ultimately at a general point-group.

Suppose that m is the order of the lowest curve C_m that passes through N. By means of the relation $N - r_n + q_n = \frac{1}{2}n(n+3)$, we can write down the values of q_m, q_{m+1}, ..., those of r_m, r_{m+1}, ... being given. If q_m is not zero, a second m-ic can be drawn through N, cutting C_m again in a rest-group $m^2 - N$; but, if q_m is zero, then no other m-ic besides C can be drawn through N. In this case we take l to be the order of the lowest curve which passes through N, and does not contain C_m as a factor; and we shall consider this case, as being the more general, since it reduces to the former case when $l = m$. The value of l is found from the first term of the series $q_m, q_{m+1}, q_{m+2}, \ldots$ which is greater than the corresponding term of the series 0, 2, 5, 9, ... Let n be the order of the highest curve for which the excess of N does not vanish; so that $r_{n+1} = 0$, $r_{n+2} = 0$, &c. Then the order of the lowest curve which passes



Base-curve C_m.

through N' is $l + m - n - 3$ (Art. 19, iv). Again, the point-group N' cannot have an excess for any curve of order higher than $m-3$; for, if it had an $(m-2)$-ic excess, it will be seen by referring to the figure that Q lies on an $(m-3)$-ic (Art. 19, iii); and that R, N lie on an $(l-1)$-ic; which is contrary to the hypothesis that an l-ic is the lowest curve through N which does not contain C_m as a factor. It may, however, happen that the $(m-3)$-ic, $(m-4)$-ic, &c., excesses of N' vanish; such, in fact, will be the case when l-ics, $(l+1)$-ics, &c., which pass through N, necessarily pass through N'.

Hence we have only to consider the excesses and defects of N' for curves of all orders from $l + m - n - 3$ up to $m-3$. Now in (4), Theorem V, the factors of the triangular numbers are negative for all these cases; and we therefore have

$$q'_{l+m-n-3} = r_n - 1, \quad q'_{l+m-n-2} = r_{n-1} - 1, \quad \ldots \quad q'_{m-3} = r_l - 1;$$

also $$r'_{m-3} = q_l - \tfrac{1}{2}(l-m+1)(l-m+2),$$

by (3), Theorem V. From these results we can arrange the following table of reduction:—

N	m	$m+1$	l	$l+1$	n
Defect	$q_m = 0$	$q_{m+1} = 2$	q_l	q_{l+1}	q_n
Excess	r_m	r_{m+1}	r_l	r_{l+1}	r_n
$N' = lm - N$			$l+m-n-3$	$l+m-n-2$	$m-3$
Defect			$r_n - 1$	$r_{n-1} - 1$	$r_l - 1$
Excess					$q_l - \tfrac{1}{2}(l-m+1)(l-m+2)$

In this table, there are three rows corresponding to each point-group; the first containing the degree N, and the orders m, $m+1$, ... of all the curves for which N has any excess; and the second and third, the excesses and defects of N corresponding to the several curves. The positions of the columns which are not inserted are indicated by the thick vertical lines. The order $l+m-n-3$ of the lowest curve through N' is placed in the same column as the order l of the lowest curve through N which determines a rest-group of N, viz. N'; and the order $m-3$ of the highest possible curve for which N' can have an excess thus falls in the last column. If $q_m > 0$, so that $l = m$, the defects and excesses of N' will occupy all the columns.

The defects of N' will then be the excesses of N written in the reverse order and diminished by unity. Only the last of the excesses of N' has been inserted, the remainder being more easily obtained from (1), Art. 15, than from (3), Theorem V, as explained in the examples given below.

To give general examples of this method of reduction, in which the n-ic excess of N is expressed as an algebraical function of n, would be too complicated; because, even in simple cases, the functions change in form, when n passes through certain values. We shall therefore only consider particular examples; choosing the excesses or defects of the particular point-group N arbitrarily, within certain limits. Some results, for a general case, are stated in Art. 31.

27. Suppose it is required to find a construction for a point-group N, characterized by the numbers $N = 67$, $r_{12} = 1$, $r_{11} = 3$, $r_{10} = 7$, $r_9 = 13$; from which we obtain $q_9 = 0$, $q_{10} = 5$, $q_{11} = 13$, $q_{12} = 24$. The reduction of this point-group is given in the following table :—

	N	m	l		n	
(1)	67	9	10	11	12	
	Defect	0	(4) 5	(7) 13	(10) 24	
	Excess	13	(6) 7	(4) 3	(2) 1	
(2)	$N' = lm - N$ = 23		4	5	6	
	Defect		0	(1) 2	(3) 6	
	Excess		9	(4) 5	(3) 2	
(2′)	24		4	5	6	7
	Defect		0	(1) 2	(3) 6	(5) 12
	Excess		10	(4) 6	(3) 3	(2) 1

The letters N, m, l, n are placed immediately above the numbers in the first row to which they correspond. Between successive defects in the second row the differences *diminished by unity* are interpolated; and when any one of these is subtracted from the order of the curve in the column to the right, it gives the difference of the corresponding excesses. This follows from the relations

$$N - r_n + q_n = \tfrac{1}{2}n(n+3), \quad N - r_{n-1} + q_{n-1} = \tfrac{1}{2}(n-1)(n+2);$$

which give by subtraction,

$$r_{n-1} - r_n = n - (q_n - q_{n-1} - 1).$$

The numbers in the three rows of stage (2) of the table are then written down as follows :—The orders of the curves are obtained by writing $m - 3 = 6$ in the last column, followed by 5, 4, ... until the

column l is reached; the defects and their interpolated numbers are found by diminishing the numbers in the row of the excesses in stage (1) by unity, and reversing; and the excesses are written down beginning with the last, the differences being found as already explained. In the same way, the numbers in any stage may be written down from those in the preceding stage.

In the given example, the process of reduction may be regarded as finished at stage (2); for N' forms a recognisable point-group, viz. that formed by 23 points of intersection of a 4-ic and 6-ic. Hence, since N is the rest-group of N' determined by an l-ic and m-ic, it follows that the point-group $N = 67$ may be constructed by drawing a 9-ic and 10-ic through 23 points of intersection of a 4-ic and 6-ic. This point-group has the given characterization.

It is important to notice that N' is incomplete, and that the process of reduction cannot be safely used for incomplete or redundant point-groups.* As the derived point-groups are often possibly incomplete, it follows that, by making corrections in the table for them, a variety of ways of reducing a characterized point-group may be obtained; and a corresponding variety of ways of constructing the characterized point-group. This is considered more fully in Art. 29.

Stage (2′) in the table represents the complete point-group $N'+1 = 24$, of which N' forms a part; and only differs from stage (2) in respect to the excesses, and the addition of a 7-ic to the series of curves. For one step further in the reduction, either (2) or (2′) might be used.

28. Another example of reduction is that of a point-group 369, whose characterization is given in the first three rows of the table on the next page.

In stage (4), the third derived rest-group 18 is recognisable as 18 points of intersection of a 4-ic and 5-ic, which is an incomplete group, as in the preceding example. Hence we have the following construction: —Draw two 10-ics through 18 points common to a 4-ic and 5-ic; through the rest-group 82 draw two 17-ics; and through the rest-group 207, two 24-ics; these cut again in a rest-group 369, which has the given characterization.

When, as in stage (3), one or more excesses become zero, the curves to which they correspond are simply left out of consideration in forming the next stage (Art. 19, iv).

* The reduction, if no correction were made, would lead eventually to a composite rest-group: to which would correspond composite curves, with a common factor.

(1) 369	24	25	26	27	28
Defect	3	(7) 11	(12) 24	(18) 43	(23) 67
Excess	48	(18) 30	(14) 16	(9) 7	(5) 2
(2) 207	17	18	19	20	21
Defect	1	(4) 6	(8) 15	(13) 29	(17) 47
Excess	38	(14) 24	(11) 13	(7) 6	(4) 2
(3) 82	10	11	12	13	14
Defect	1	(3) 5	(6) 12	(10) 23	(13) 37
Excess	18	(8) 10	(6) 4	(3) 1	(1) 0
(4) 18	4	5	6	7	
Defect	0	(2) 3	(5) 9	(7) 17	
Excess	4	(3) 1	(1) 0	(0) 0	

In this example, no defect is stated for a 23-ic; and a point-group 369, constructed as above, does not lie on a 23-ic. If, however, another group of 369 points has the same characterization, except that the 23-ic defect is zero, it will reduce by the same process; the fourth derived rest-group consisting of 3 points on a straight line.

29. The following example illustrates how an incomplete point-group may be recognised. As explained below, stage (2) gives an incomplete rest-group; which is completed in (2′), by adding 3 to the degree, and all the excesses, without changing the defects. From (2′) we obtain (3) and (4); the last representing 19 points of intersection of a 4-ic and 5-ic. We therefore have the following construction for the point-group 569. Draw two 13-ics through 19 points common to a 4-ic and 5-ic; through the rest-group 150 draw two 22-ics; and through 331 points of the rest-group 334 draw two 30-ics; then the rest-group 569 has the given characterization.

(1) 569	30	31	32	33	34	35	
Defect	4	(9) 14	(13) 28	(17) 46	(23) 70	(28) 99	
Excess	78	(22) 56	(19) 37	(16) 21	(11) 10	(7) 3	
(2) 331	22	23	24	25	26	27	
Defect	2	(6) 9	(10) 20	(15) 36	(18) 55	(21) 77	
Excess	58	(17) 41	(14) 27	(10) 17	(8) 9	(6) 3	
(2′) 334	22	23	24	25	26	27	28
Defect	2	(6) 9	(10) 20	(15) 36	(18) 55	(21) 77	(24) 102
Excess	61	(17) 44	(14) 30	(10) 20	(8) 12	(6) 6	(4) 2
(3) 150	13	14	15	16	17	18	19
Defect	1	(3) 5	(5) 11	(7) 19	(9) 29	(13) 43	(16) 60
Excess	47	(11) 36	(10) 26	(9) 17	(8) 9	(5) 4	(3) 1
(4) 19	4	5	6	7	8	9	10
Defect	0	(2) 3	(4) 8	(7) 16	(8) 25	(9) 35	(10) 46
Excess	5	(3) 2	(2) 0	0	0	0	0

If, commencing at the last column, the first two or three excesses $r_n, r_{n-1}, r_{n-2}, \ldots$ of a point-group N are the triangular numbers 1, 3, 6, ..., or multiples of them a, $3a$, $6a$, ... ; then the a^{th} rest-group derived from N will be found to simplify very much; the excesses of the a^{th} derived rest-group vanishing, in the last columns, when a is even; and the defects of the a^{th} derived rest-group forming the series 0, 2, 5, ..., in the first columns, when a is odd. If then the excesses $r_n, r_{n-1}, r_{n-2}, \ldots$ of N are such that when the same number is added to them, they become consecutive (but not necessarily the first) terms

of the series a, $3a$, $6a$, ..., we should assume N to be incomplete; subject to a condition as to the maximum value that r_n can have.

Hence N would be *possibly* incomplete if a positive number ρ could be found such that $r_n+\rho$, $r_{n-1}+\rho$ are respectively the same multiple of two consecutive triangular numbers, and r_n does not exceed a certain limit; and such a number ρ could be found as often as not. We should not, however, generally consider that this alone would be a sufficient indication that N is incomplete; although the effect of adding on ρ to N, r_n, r_{n-1}, ... might be tried, if the reduction of N appears to lead to an impossible result. We may take as a test for an incomplete point-group that a positive number ρ can be found, such that the three following equations shall hold at least ($p \geqslant 0$) :—

$$r_n+\rho = \frac{a}{2}(p+1)(p+2), \quad r_{n-1}+\rho = \frac{a}{2}(p+2)(p+3),$$

$$r_{n-2}+\rho = \frac{a}{2}(p+3)(p+4).$$

This only requires

$$r_{n-1}-r_n = a(p+2), \quad r_{n-2}-r_{n-1} = a(p+3).$$

Hence, if the differences $r_{n-1}-r_n$, $r_{n-2}-r_{n-1}$, which are given in the table, are such that, when their H.C.F. a is divided out, they become consecutive integers $p+2$, $p+3$ ($p \geqslant 0$), we should assume N to be incomplete. We should, in that case, substitute a complete point-group for N, by increasing the degree and all the excesses by

$$\rho = \frac{a}{2}(p+1)(p+2)-r_n;$$

at the same time adding p curves of orders $n+1$, $n+2$, ... $n+p$ with excesses, $\frac{a}{2}p(p+1)$, ... $3a$, a, to the table. The condition that must be satisfied, if the point-group N is fully characterized, is that the $(n+1)$-ic excess of the completed point-group, viz. $\frac{a}{2}p(p+1)$, must be equal to or less than ρ. This requires $r_n \leqslant a(p+1)$; or that r_n should not exceed the first term of the A.P. whose second and third terms are $r_{n-1}-r_n$ and $r_{n-2}-r_{n-1}$.

The rest-group 331, in the example above, has been replaced by the complete point-group 334; with the result that we obtain, two stages later, a much simplified rest-group.

The original point-group N should be tested not only for incompleteness, but also for redundancy; for, if N were redundant, the process of reduction would eventually lead to composite rest-groups. If N is a simple point-group, then among the derived rest-groups it is only necessary to test for incompleteness; since an incomplete rest-group will always precede a redundant one (Art. 15, i).

To test whether or not the original point-group N is redundant, we examine the differences of the defects from the beginning. If $q_{m+1}-q_m$, $q_{m+2}-q_{m+1}$ have the values $a(p+2)$, $a(p+3)$, where $p \geq 0$, then N should be assumed redundant; and the number

$$\rho = \frac{a}{2}(p+1)(p+2) - q_m - 1$$

should be added to all the defects, and subtracted from the degree N; while p curves of orders $m-p$, $m-p+1$, ... $m-1$, with defects $a-1$, $3a-1$, ... $\frac{a}{2}p(p+1)-1$, should be added at the beginning of the table. If N is fully characterized, it is necessary that the condition $q_m+1 \leqslant a(p+1)$ should be satisfied.

Unless the first given defect and last given excess of a point-group N are comparatively small, it should be assumed that the characterization of N is only partially given; and curves should be added to the table, with such assumed values for the defects and excesses as will make N reduce as rapidly as possible. (Cf. Art. 31.)

30. If the point-group N contains a general cluster $\frac{1}{2}i(i+1)$ at a point A, or any number of such clusters; we may choose the two lowest curves C_l, C_m through N to have i-ple points at A, and to have no common multiple points except at the clusters of N. Then C_l, C_m will intersect again at A in a general cluster $\frac{1}{2}i(i-1)$, which will belong to N'. Thus, if the reduction can be continued far enough, we shall arrive eventually at a rest-group which has only a single point at A. The process of reduction may however finish before the cluster disappears, the result being that the general point-group at which we finally arrive contains one or more general clusters. In this case the reduction may possibly lead to a point-group which has no excess for curves of any order, but which has necessarily a specialized form, which is not determined by the reduction. (See Note, p. 496.) In reversing the process, in order to construct N, the two curves drawn through a general cluster $\frac{1}{2}i(i-1)$ at

A will each have an i-ple point at A, and intersect again in a general cluster $\frac{1}{2}i(i+1)$ at A.

Thus it follows, as a consequence of Theorem V, that any non-composite point-group of special form is a rest-group, whose construction can in general be found, if its characterization is given. At the same time, Theorem V determines the complete characterization of a point-group, if its construction is given.

31. The Number of Absolute Connexions of a Point-Group.—When the construction of a point-group is known, we can employ Theorems VI and VII to determine the number of its absolute connexions, and its absolute multiplicity for a curve of any order.

Suppose it required to find the smallest possible number of absolute connexions of a point-group N, of which nothing more is known than that its n-ic defect and n-ic excess are q and r respectively.

Suppose that the n-ic excess r of N is put into the form

$$r = \frac{a_1}{2}(p_1+1)(p_1+2) + \frac{a_2}{2}(p_2+1)(p_2+2) + \ldots + \frac{a_\mu}{2}(p_\mu+1)(p_\mu+2),$$

where $p_1, p_2, \ldots p_\mu$ *are all different*, and in descending order; and that we assume for the complete characterization of N that given by the equations (a having both positive and negative integral values)

$$r_{n+a} = \frac{a_1}{2}(p_1-a+1)(p_1-a+2) + \ldots + \frac{a_\mu}{2}(p_\mu-a+1)(p_\mu-a+2), \quad \ldots\ldots\ldots\ldots\ldots(a)$$

the triangular numbers disappearing from the end as soon as they become zero. Thus the highest curve for which N has an excess is of order $n+p_1$, and $r_{n+p_1} = a_1$.

The number k of absolute connexions of a point-group, with this characterization, may be determined by finding its construction, and by repeated applications of Theorem VII. The work is too long to be given here, and we merely state the result, viz.,

$$k = (q+1)\Sigma a - \tfrac{1}{2}(\Sigma a)^2 + \tfrac{1}{2}\Sigma a^2 - \tfrac{1}{2}(\Sigma ap)^2 + \tfrac{1}{2}\Sigma ap^2 + n\Sigma ap$$
$$- \tfrac{3}{2}\{a_1.\overline{a_1-1}.\overline{p_1-p_2} + \overline{a_1+a_2}.\overline{a_1+a_2-1}.\overline{p_2-p_3} + \ldots$$
$$+ \overline{a_1+\ldots+a_\mu}.\overline{a_1+\ldots+a_\mu-1}.p_\mu\}, \ldots(\mathrm{A})$$

which may also be written in the form

$$k = a_1(q_{n+p_1}+1) + a_2(q_{n+p_2}+1) + \ldots + a_\mu(q_{n+p_\mu}+1); \ldots\ldots(\mathrm{A}')$$

with the condition that the minimum value of

$$\frac{q}{c} + \tfrac{1}{2}(c+3)(1+\Sigma a) + \Sigma ap,$$

for positive integral values of c, must not exceed $n+2$.

Again, if $q+1$ be put into the form

$$q+1 = \frac{a_1}{2}(p_1+1)(p_1+2) + \ldots + \frac{a_\mu}{2}(p_\mu+1)(p_\mu+2),$$

with the same conditions regarding the p's as before; and we assume the characterization of N to be that given by the equations

$$q_{n-\alpha}+1 = \frac{a_1}{2}(p_1-\alpha+1)(p_1-\alpha+2) + \ldots + \frac{a_\mu}{2}(p_\mu-\alpha+1)(p_\mu-\alpha+2), \quad \ldots\ldots\ldots\ldots(b)$$

the triangular numbers being omitted if their factors are negative; then the number of connexions is

$$k = r\Sigma a - \tfrac{1}{2}(\Sigma a)^2 + \tfrac{1}{2}\Sigma a^2 - \tfrac{1}{2}(\Sigma ap)^2 - \tfrac{1}{2}\Sigma ap^2 + n\Sigma ap$$

$$-\tfrac{3}{2}\{a_1.\overline{a_1-1}.\overline{p_1-p_2} + \overline{a_1+a_2}.\overline{a_1+a_2-1}.\overline{p_2-p_3} + \ldots$$

$$\ldots + \overline{a_1+a_2+\ldots+a_\mu}.\overline{a_1+\ldots+a_\mu-1}.p_\mu\}, \quad \ldots(B)$$

which may also be written in the form

$$k = a_1 r_{n-p_1} + a_2 r_{n-p_2} + \ldots + a_\mu r_{n-p_\mu}; \quad \ldots\ldots\ldots\ldots(B')$$

with the condition that the minimum value of

$$\frac{r}{c} + \tfrac{1}{2}(c+3)(\Sigma a - 1) + \Sigma ap,$$

for positive integral values of c, must not exceed n.

The smallest possible number of connexions is not greater than either of the minimum values of k given by (A), (B), subject to the conditions stated.

The condition given for (A) and (A′) is obtained by assuming that $n-c+1$ is the order of the lowest curve through the point-group N, and that q_{n-c} would turn out negative according to the characterization given in (*a*).

Thus $N-r_{n-c} > \frac{1}{2}(n-c)(n-c+3)$, $N-r_n+q_n = \frac{1}{2}n(n+3)$;

and, by subtracting,

$$c(n+\tfrac{3}{2})-\tfrac{1}{2}c^2 > q_n+r_{n-c}-r_n,$$

or $c(n+\frac{3}{2}) > q+\frac{1}{2}\Sigma a(p+c+1)(p+c+2)-\frac{1}{2}\Sigma a(p+1)(p+2)+\frac{1}{2}c^3$,

or $$n+3 > \frac{q}{c}+\Sigma a(p+\tfrac{3}{2})+\tfrac{1}{2}c\Sigma a+\tfrac{1}{2}(c+3),$$

or $$n+2 \geqslant \frac{q}{c}+\tfrac{1}{2}(c+3)(\Sigma a+1)+\Sigma ap.$$

This is a sufficient condition that a point-group N, with the characterization given in (a), can be placed on an n-ic.

Similarly, the condition given for (B) and (B′) is obtained by assuming that $n+c-1$ is the order of the highest curve for which the point-group N has an excess, and that r_{n+c} would turn out zero or negative according to the characterization given in (b).

Point-groups with the characterization given in (a) or (b) are simple point-groups, all of whose successive derived rest-groups, of lowest degree, are also simple. If the point-group has an excess for n-ics only, and not for curves of any other order; then $p_1 = 0$, $a_2 = a_3 = \ldots = 0$, and $c = 1$. In this case both (A) and (B) give $k = r(q+1)$, with the conditions $n \geqslant q+2r$, $n \geqslant r+2q$ respectively. Hence it would appear that the known formula $r(q+1)$ does not always give the correct value of k; and that it certainly does not, if n is less than the smaller of the two numbers $q+2r$, $r+2q$. This can be easily proved independently.

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APPENDIX.

The Society's death-roll for the Session 1894–5 is a heavy one. We have lost one of our most distinguished and oldest members, Professor Cayley. For the following sketch, which has been drawn up at the request of the Council, we are indebted to Mr. Samuel Roberts, who sat for so many years at the Council-table with Professor Cayley:—

The death of Professor Cayley has naturally been the subject of world-wide regret. At home and abroad numerous biographical notices and appreciative comments on his life and work constitute a

remarkable tribute to his genius, and testify to a feeling of social and intellectual loss extending far beyond scientific circles.

For many reasons, the members of the London Mathematical Society will be peculiarly sensible of the great loss they have sustained. It is fitting, therefore, to say something here of Cayley's work as member and officer of our Society. For such general accounts and estimates of his mathematical discoveries as were possible within very limited space, and so soon after his removal from amongst us, I must refer to the notices I have alluded to. In them will be found information concerning the honours conferred upon him by the chief British and foreign scientific societies, and much also to illustrate his attractive personality, contributed by friends, who enjoyed the privilege of his intimacy.*

Throughout his long connexion with our Society, Professor Cayley rendered signal services, and particularly so during the somewhat critical years when its status was being determined under the disadvantage of extremely limited resources. It must be remembered that the idea of a national society for the cultivation of pure and applied mathematics exclusively was somewhat of a novelty, and its successful realization depended a great deal on the cooperation of men of established eminence in the science.

Professor Cayley was elected a member on the 19th June, 1865, and was therefore one of the earliest members, the first meeting having been held on the 15th January preceding. He brought with him the prestige of a great name at a very opportune time. His contributions to our *Proceedings* commenced almost immediately. They number more than eighty, varying very much in length, character, and form, but well adapted to the needs of an association of men of different kinds and degrees of special or general mathematical acquirement. The list is too long to be given here in full. It comprises numerous short communications, in the first instance oral, but of which some account or summary is printed. These, for the most part, require only elementary proficiency for their

* For a very interesting and full account of Cayley in his personal and scientific relations, see Professor A. R. Forsyth's obituary notice of him in the *Proceedings of the Royal Society*, Vol. LVIII. (and in the eighth volume of the *Collected Works*).

Earlier notices have been published in 1895, by J. W. L. Glaisher (*Cambridge Review*, 7th February), Canon Venables (*Guardian*, 6th February), Ch. Hermite (*C. R.*, 4th February), Ch. A. Scott (*Bulletin of the American Society*, March), F. Brioschi (*Rendiconti d. R. Accad. dei Lincei*, 8th March), M. Noether (*Math. Annalen*, Band XLVI.), who treats of Cayley as a mathematician and at some length. Reference may also be made to the article by Dr. Salmon (*Nature*, 20th September, 1883).

satisfactory appreciation. Others, again, of greater length and more developed, lead to the very frontiers of realized knowledge. They naturally require from the student advanced attainments, and cannot fairly be said to be easy reading. Of course, Cayley's greater memoirs must be looked for elsewhere. Many, I may say most, of these were published before the inauguration of the Society. Nevertheless his contributions to our *Proceedings* present good examples of his methods and force in relation to the most noteworthy subjects which his genius favoured.

It may be permitted to call to mind some of the more important memoirs. As might be expected, a predominant number relate to curves and surfaces. We find a series of three memoirs on "Nodal Quartic Surfaces," *i.e.*, quartic surfaces with from one to sixteen nodes. These researches were, as the author states, suggested by Kummer's memoir (*Berl. Abh.*, 1866), and contain a reproduction of many of its results, with, however, further and important developments. The form of the papers is strikingly characteristic. While their general method is that of "modern algebra," the geometrical insight of the author and his power of realizing external spatial conceptions, hidden under algebraical formulæ, are throughout in evidence. There are passages, too, that show how his mind worked towards positive conclusions. At times he anticipates demonstration and tells us what possibly or probably might be true. These sidelights on the inner processes of his mind are valuable and instructive, all the more so because they are somewhat rare. Connected with these memoirs may be mentioned the paper entitled "A Sketch of Recent Researches upon Quartic and Quintic Surfaces," likely to be useful to future explorers of the subject. Here is exemplified the way in which Cayley frequently made his own preliminary reading of service to others engaged in the same field. Another kindred paper, "On the Surfaces each the Locus of the Vertex of a Cone, which passes through m given Points, and touches $6-m$ given Lines," is described by the author as a construction and development of the researches by Dr. Hierholzer. In this memoir Cayley obtains the equations of the seven surfaces and determines their singularities. The orders of the surfaces are respectively 4, 8, 16, 24, 24, 14, 8. Evidently the treatment of equations of such high orders needed very skilful handling. I may mention also the interesting memoirs "On Geodesic Lines, in particular those of a Quadric Surface."

At an early period, Cayley directed the attention of our Society to the important and then recent theories of correspondence and trans-

formation. Thus in the first volume of the *Proceedings* we find papers on the "Transformation of Plane Curves," and "On the Correspondence of Two Points on a Curve." In the third volume appear a paper "On the Rational Transformation between two Spaces," and the first communication "On the Rational Transformation between Two Planes, and on Special Systems of Points." These investigations are not only geometrically valuable, but present features of considerable analytical interest, marked by singular arithmetical symmetries in the tabular schemata. The subject has since been further studied, and would probably reward still more attention.

In the domain of analysis Cayley gave us some interesting notes "On Elliptic Functions," and "On the Binomial Equation $x^n - 1 = 0$," and specially "On Trisection, Quartisection, and Quinquisection." These papers may be described without impropriety as "chips from his workshop." Those on Elliptic Functions were subsequent to the first edition of the *Elementary Treatise on Elliptic Functions.* I do not know how far the substance of them has been incorporated in the second and posthumous edition of that work.

Notwithstanding his decided predilection for the speculations of pure mathematics, Cayley did not fail to leave his mark on those applications, which are but slightly affected by empirical data. His "Report on the Recent Progress of Theoretical Dynamics," published by the British Association for the Advancement of Science in 1858, remains a monument of his industry and systematizing power. On this subject there are in our *Proceedings* several communications by him which the Society would not willingly have lost. They relate to the theory of attraction, and the potentials of polygons and polyhedra, the ellipse and circle, and of an ellipsoidal shell, and are contained in the sixth volume. The subject-matter can scarcely be distinguished from pure algebra and geometry.

Professor Cayley therefore enriched the *Proceedings* with papers on most of the subjects which occupied his attention elsewhere. Exception must be made of astronomy, in the mathematical theory of which we know he took great interest and did valuable work.

I have still to refer to one department of Cayley's versatility that possesses general and even popular interest. He liked to study models of surfaces, and sometimes contrived them for himself. He highly appreciated beauty of form and motion. His diagrams were elaborated with great care, on a sufficient scale, and with a certain artistic skill. The valuable series of Plucker's models now in the

possession of our Society were carefully examined and described by him, and many of them identified. The mechanical description of curves also peculiarly interested him, and on several occasions he described curve-tracing arrangements, in some instances constructed by his own hands. He was evidently pleased to see a curve revealing itself beneath the traversing pencil in smooth and graceful motion, reminding an observer of the dictum of Newton—"At æquatio non est sed descriptio quæ curvam geometricam efficit."

Professor Forsyth remarks that, so late as 1893, Cayley exhibited at a meeting of the Cambridge Philosophical Society a curve-tracing apparatus connected with three-bar motion, a subject on which he prepared an interesting paper for our Society, specially treating of the triple generator of three-bar curves.

But the influence of Professor Cayley can only be imperfectly measured by the length and number or even the importance of his printed contributions. The older members recall with pleasure how much the presence of Professor Cayley enhanced the interest of ordinary meetings. His attendance at them was for a long time frequent, in spite of non-residence in London; and when his visits became rare, from accidental causes and failing health, a sense of privation was distinctly felt by his colleagues, long accustomed to his genial participation in their work. He was always ready to offer suggestive remarks on papers brought before the Society by other members, and the interest shown by so eminent an expert in even slight communications was very encouraging. The occasions were rare indeed that he did not add something of value to the programme of the evening, but never obtrusively; on the contrary, with the modest self-effacement belonging to his character, he seemed to think only of the beauty or utility of a theorem or process. His enthusiastic devotion to his science, scarcely veiled under a quiet and balanced manner, communicated itself more or less to those who could associate themselves with his labours.

In other ways Cayley placed the Society under obligation. He was often on the Council, and in that capacity his advice and assistance were always at its service. Particularly his legal knowledge and business ability were laid under contribution. Mathematicians know too well how trying it is to be called upon to read and give an opinion on papers relating to subjects possibly long abandoned, or, at all events, foreign to the lines of thought at the moment occupying them. Cayley appeared exempt from any weakness of that kind. He responded without remonstrance to the

frequent calls upon him to act as referee, and he undertook the duty in no perfunctory manner, often making suggestions readily adopted by the authors and very advantageous to their papers. To individual members who sought information or advice from him he was always accessible, and his courtesy and interest in the work of others will long remain a tradition of our Society.

From the time of his adhesion in 1865 until his attendance at the meetings ceased, Cayley was, with the exception of a few sessions, on the Council. He was President in 1868 and 1869, and for five or six sessions, not consecutive, acted as Vice-President. The Society could hardly confer adequate honours on its illustrious member, but it did what it could, and awarded to him the first De Morgan Medal.

Arthur Cayley was born at Richmond, Surrey, on the 16th August, 1821, and died at Cambridge on the 26th January, 1895.

On behalf of the Society a message of condolence was conveyed to Mrs. Cayley, and the Society was represented at the funeral by the President and a distinguished delegation.

James Cockle, second son of James Cockle, was born at Great Oakley, near Harwich, in Essex, on January 14th, 1819. From 1825 to 1829 he was educated at a private school in Kensington, whence he proceeded to the Charterhouse. Here he remained during the years 1829–31, and during his stay showed considerable power in writing Latin verses. Subsequently he was placed with the Rev. Dr. Lenny, of St. John's College, Cambridge, and under his care young Cockle first manifested his mathematical talent. He left England in November, 1835, for a year's travel in the West Indies and the United States. On his return he entered Trinity College, Cambridge (October 18th, 1837), and graduated B.A. in 1842, having come out thirty-third Wrangler in the Tripos List of the previous year (Stokes' year). He proceeded to M.A. in 1845. Mr. Cockle had previously been entered as a student at the Middle Temple in 1838. He practised as a special pleader from 1845 to 1849, was called to the Bar at the Middle Temple in 1846, and joined the Midland Circuit in 1848. In 1862 he drafted the "Jurisdiction in Homicides Act," and in 1863, on the recommendation of Chief Justice Erle, he was appointed by the English Government first Chief Justice of Queensland. He resigned his post as Chief Justice in 1879, having been previously knighted by patent in 1869, and returned to England with Lady Cockle and his family of eight children.

Sir James Cockle, as we must now call him, though he attained considerable eminence as a lawyer and a Judge, yet found time in the intervals of his official labours for his favourite science. He was the author of upwards of eighty papers, most of which, with the exception of four papers on the motion of fluids (*Q. J. of Math.*, x., xi.) deal entirely with pure mathematics. We have noted many papers by him in the *Lady's and Gentleman's Diaries* during the years 1848 to 1863, in the various forms of the *Messenger of Mathematics*, in the *Quarterly Journal*, and elsewhere, but we are happily spared the labour of going through these critically, as the Rev. R. Harley has most kindly placed at our service his full and interesting notice of his friend of forty-eight years, which has been published in the *Proceedings of the Manchester Literary and Philosophical Society* (Vol. ix., 1894–5). "His papers may be grouped for the most part under two heads, viz., Common Algebra, and the Theory of Differential Equations. In algebra he worked mainly among the higher equations; and for many years his labours in this department were inspired and directed by the hope of 'solving the quintic,' or, to be more exact, expressing a root of the general equation of the fifth degree by a finite combination of radicals and rational functions. . . . He found not what he sought for, but other things which amply repaid the toil of effort, and he opened up new methods of working and new lines of research which are of acknowledged value in themselves. . . . By an indirect but ingenious process he succeeded in determining the explicit form of a certain sextic equation, on the solution of which that of the general quintic may be shown to depend." The attention of Mr. Harley and of Prof. Cayley was subsequently turned to the matter, and so the methods devised by Sir James Cockle and the results he obtained largely directed the course of subsequent speculation on the subject.

Again, "his mode of dealing with the theory of differential equations was equally marked by originality and independence of mind. . . . He found, for instance, that from any rational and entire algebraic equation of the degree n, whereof the coefficients are functions of a single parameter, we can derive a linear differential equation of the order $n-1$, which is satisfied by any one of the roots of the algebraic equation. Out of this germ has grown the theory of differential resolvents. . . . To Cockle also belongs the honour of being the first to discover and develop the properties of those functions of the coefficients of linear differential equations called Criticoids or Differential Invariants. . . . Criticoids seem destined to play an important part in the theory of linear differential equations."

His work, Mr. Harley sums up, was eminently initiatory. He started theories, but left others to elaborate and perfect them.*

Sir James Cockle was elected a member of the Society June 9th, 1870, and was admitted February 13th, 1879. He attended the meetings occasionally during the sessions 1879–81, and from April 14th, 1881, to November, 1891, he was never absent. He attended once only in 1892, and then his seat was vacant. He served on the Council from November 10th, 1881, to November 13th, 1890; was Vice-President in the sessions 1882, 1883, 1888, 1889, and President in 1886, 1887. Lady Cockle writes: "His ambition was to become a President of the Mathematical Society," and she recalls to our recollection what Sir James told us at the time. "So determined was he to be present at his installation as President, that, although suffering from a bad attack of congestion of the lungs, he had himself wrapped in blankets, which were only removed on his arrival."†

The following are the titles of the communications made to the Society by Sir James Cockle, with the dates and their places in the *Proceedings*:—

February 13th, 1879, "Construction of Magic Squares," Vol. x., 75.

"On Differential Equations, Total and Partial," and "On a New Soluble Class of the First, and an Exceptional Case of the Second," Vol. x., pp. 105–120.

"On a Binomial Biordinal and the Constants of its Complete Solution," Vol. xi., pp. 123–131.

"Supplement on Binomial Biordinals," Vol. xii., pp. 63–72.

"On the Explicit Integration of certain Differential Resolvents," Vol. xiv., pp. 18–22.

"On the Equation of Riccati," Vol. xviii., pp. 180–202.

"On the General Linear Differential Equation of the Second Order" (complementary of the last paper), Vol. xix., pp. 257–278.

"On the Confluences and Bifurcations of certain Theories" (Presidential Address), November 8th, 1888, Vol. xx., pp. 4–14.

Esse quam videri. This was Sir James' motto, and as far as we, who were associated with him so long on the Council, can judge, his whole life was a beautiful illustration of the motto. Sir James passed

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* Prof. Forsyth, whose numerous engagements prevented him from drawing up an analysis of Sir James Cockle's work ("my inability is solely owing to want of time"), refers us to a note to his memoir "On Invariants . . .," *Phil. Trans.*, 1888 (A), p. 383.

† An obituary notice in the *Annual Report of the Royal Astronomical Society* (February, 1895) truly remarks: "On committees or councils he was singularly reticent, rarely venturing a suggestion unless appealed to, but the regularity of his attendance testified to the keen interest he took in the management of business."

away on Sunday, January 27th, 1895, the day after Prof. Cayley's lamented death.*

Arthur Cowper Ranyard was born on June 21st, 1845, at Swanscombe, in Kent, whence his family subsequently removed to 13 Hunter Street, Brunswick Square. He was educated at University College School (1857–60), at which school George Campbell De Morgan, son of Prof. De Morgan, was a pupil (1856–57), and subsequently mathematical master, in succession to Dr. Hirst, in the year 1865. The two young men attended Prof. De Morgan's classes at the College, and became intimate friends. In the year 1864, whilst they were discussing problems during a walk in the streets, it struck one of them that "it would be very nice to have a society to which all discoveries in mathematics could be brought, and where things could be discussed, as at the Astronomical." † It was agreed between them that this should be proposed, and that "George should ask his father to take the chair at the first meeting." ‡

In October, 1864, Ranyard appears to have been at Caius College, whence he migrated to Pembroke College and graduated Senior Optime in 1868.§

In due course of time a lithographed circular,‖ signed by both as "Hon. Secs. *pro tem.*" (*Life*, p. 282), was sent out, "to request the honour of your attendance at the first meeting of the 'University College Mathematical Society.'" From a *letter* (September 6th,

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* For such particulars as we have not given above, we must refer to Mr. Harley's memoir, which has been our authority throughout.

† *Life of Prof. De Morgan*, by Mrs. De Morgan, p. 281.

‡ *Life*, *l.c.* In a collection of letters, from G. De Morgan to Ranyard, bearing upon the foundation of the Society, and which Mr. Ranyard left by will in a bound volume to the Society, De Morgan, under date of October 30th, 1864, writes: "You will remember that my father asked that he might have a notice sent him of the meeting of the Mathematical Society. As it was you who asked him to preside, would you send a note reminding him of the date?" De Morgan had a bad memory for names and such things (*Letter*, November 24th, 1865), and he has certainly misdated by a year a letter relating to the initial meeting of the Society.

§ We are indebted to the Master of Pembroke for the following extract from the admission register:—"Ranyard Arturus Cowper filius natu secundus Benjamini Ranyard de Londino generosi natus apud Swanscombe in Com. Cantiæ atque annum agens vicesimum literis testimonialibus de Coll. Caiensi ornatus se ad nos contulit atque admissus est ad mensam secundam sub tutore Mro. Power. February 18th, 1866."

‖ *Letters*, September 5th, October 8th, October 19th, 1864, are concerned with this matter of lithographing the circular and the form in which the circular should be drawn up.

1864) it appears that Prof. De Morgan had objected to the original title of "London University Mathematical Society," and so the *first* change in title was made and gave rise to the report that "the Society was only an upper higher senior class of De Morgan's." *

Unfortunately, De Morgan's health broke down (*Letters*, October 30th, November 4th, 1864), and so he was unable to be present at the meeting on November 7th, 1864. "You must make my excuses to every one for not being present." He further expresses the hope that his absence will not harm the Society, and that Dr. Hirst will be able to attend, "as I think he will be an important member, and may take an interest in the affair." Under date November 9th he writes to Ranyard: "I have not heard much about the meeting and should be much obliged to you for an account of it, if you have time for it." From this we gather that Ranyard was present, but all attempts on our part have failed to elicit a report of what took place. From the "minutes" of the first meeting of the London Mathematical Society, on January 16th, 1865, we learn that Mr. B. Kisch† was the *acting* Secretary. From the active part Dr. Hirst took in the early days of the Society we surmise that he was present at the November meeting and proposed that the scope of the Society's operations should be enlarged, and the *second* change of name be made. When application was made to Mr. Ranyard for a subscription to the De Morgan Medal fund, he wrote (November 1st, 1880), enclosing his subscription and also an extract from a letter from Prof. De Morgan.‡

In consequence of this communication, we were directed by the Council to write to Mr. Ranyard, and under date of November 30th, 1880, we received the following reply from that gentleman:—

"Thank you for your very courteous letter. I should have much regretted it if the Council had withdrawn the circular—for there can be no doubt that the real founders of the Society were Prof. De Morgan

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* Letter from Sir Philip Magnus.

† Mr. Kisch writes (16th January, 1896): "I believe Ranyard was at the meeting of November, 1864. So far as I can recollect, I was never appointed Secretary, even *pro tem.*, but it is quite possible that I may have so acted on one occasion, in the absence of both Ranyard and De Morgan."

‡ This runs "I make out distinctly, from written evidence [*i.e.*, the *Letters*] that you [Ranyard] and he [De Morgan] were the projectors." It was by the Professor's wish that the Letters have been preserved. The copier of the extract has made one or two clerical errors, viz., 1864 for 1867. A "minute" of the Society records, November 14th, 1867, "the great loss sustained by the Society through the death of its late Secretary," and dwells upon the fact that "it was probably known to few that he and Mr. Ranyard took the first steps towards establishing the Society by sending out circulars inviting gentlemen to attend the first meeting of the Society."

and the other mathematicians of standing who early gave the Society their countenance. Without them it would have remained a students' society, and would probably soon have died a natural death."

The names of the twenty-seven members at the formation of the Society on January 16th, 1865, are given in the "minutes" of the meeting; of these, at the present date (January, 1896), the sole remaining names are those of Prof. Clifton, Sir Philip Magnus, and Dr. Routh.

Most of the obituary notices say that Mr. Ranyard was one of the first Secretaries. This was not the case. We have indicated what part he took in the inception of the *students'* society. At the January meeting Mr. Kisch, as *acting* Secretary, read the proposed rules, presumably drawn up at the previous November meeting, and at the same meeting Mr. H. M. Bompas and Mr. Cozens-Hardy were elected Secretaries, the latter gentleman also undertaking the duties of Treasurer. The President and Vice-President were respectively Prof. De Morgan and Dr. Hirst. Mr. Bompas remained in office until November 20th, 1865, but Mr. Cozens-Hardy retired from both his offices at the second meeting, held on February 20th, 1865. Mr. W. Jardine was elected in his room, but he too retired at the same time as Mr. Bompas, on being appointed to a professorship in the Government College at Lahore. At this November meeting Mr. M. Jenkins was appointed to act in Mr. Jardine's place until the annual meeting on January 15th, 1866, when he and Mr. George De Morgan were elected Secretaries. The latter died in October, 1867, and Mr. Tucker was elected to fill the vacancy, November 14th, 1867, the date to which the annual meeting was transferred. This arrangement has continued in force until the recent retirement of Mr. Jenkins.

Mr. Ranyard read only one paper before the Society, viz., that on "Determinants," at its first meeting.*

He was naturally elected on the Council, or committee as it was called in those early days, and he was again elected January 5th, 1866, and served until November 8th, 1866, the new date of the annual meeting. He never served on the Council after this retirement.

We have entered into so much detail because the facts relating to the beginning of the Society are getting more and more difficult to trace year by year. The thought of these young men was a very happy one,

* Under date 29th January, 1864 (it should be 1865), De Morgan writes: "The only way to get the papers printed is to get more members, and the only way to get members is to get the papers printed." He then makes suggestions for having the papers copied by the College Beadle.

and has been productive of most beneficial results in the influence of the Society on the advancement of mathematical research (*cf.* Mr. S. Roberts' notice of Prof. Cayley, *supra*). The thought has not been a barren one, as is amply proved by the now long list of similar societies which have been established in other countries.*

For other details of Mr. Ranyard's active and useful career we need only refer to an article by Mr. W. H. Wesley on "Arthur Cowper Ranyard and his Work," in *Knowledge* (February 1st, 1895), of which journal Mr. Ranyard was the able editor on the death of its projector and first editor, Mr. R. A. Proctor. There is also a full account of him in the *Monthly Notices of the Royal Astronomical Society*, of which Society Mr. Ranyard became a Fellow at the early age of eighteen.†

Mr. Ranyard died on December 14th, 1894, at 13 Hunter Street.

We are indebted to Prof. E. B. Elliott for the following notice of his friend Prof. A. M. Nash, who was elected a member of the Society, November 10th, 1887.

Alfred Moses Nash was a native of Twyford, Berks, and was educated at Bristol Grammar School. He was elected scholar of Queen's College, Oxford, in 1870, and graduated with First Class Mathematical Honours in 1873. Very shortly afterwards he gained an appointment in the Indian Education Department (January, 1875). His premature death now deprives the Indian service of one of its most indefatigable and conspicuous educational workers. His principal appointments have been, at different times, those of Professor in the Presidency College, Calcutta (November, 1890); Acting Registrar of Calcutta University (May, 1891), and Inspector of Schools. His mathematical interests have always continued keen, but in his busy life they have been followed mainly in intervals of recreation, so that his actual publications, though suggestive and indicative of keen insight, have been of a minor character, appearing for the most part in the *Educational Times*. More was confidently expected from him when early in the present year (1895) he was

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* We are not unmindful of the old "Mathematical Society," founded in 1717 (see *Budget of Paradoxes*, pp. 230, &c., and *Life*, pp. 123, 150), the Manchester Society, in 1718, and the Oldham Society, in 1794, and possibly some others, but these did not "come to stay."

† In the *Annual Report of the Council of the Royal Astronomical Society* (Vol. LV., pp. 198–201) the statement is made "While at University College he collaborated with Mr. George De Morgan in founding a 'Students' Mathematical Society,' which had a most successful career [of two months!], developing eventually into the present 'London Mathematical Society.'"

granted a prolonged rest from official cares. Two letters written by him to the compiler of the present notice in January and April last were full of hopeful plans for the future, and in particular of a work in which he was engaged, and to which he was intending to devote much of his leisure, upon *Irregular Determinants in the Theory of Binary Quadratic Forms.* Alas! within a few days of the second letter came the news of the sad mischance at sea (June 3rd, 1895) which put an end at once to this application of his mathematical powers and to his career of public usefulness. He was a true friend, and a man of indomitable courage and self-reliance. He leaves a widow to mourn his loss.

Edward Hawksley Rhodes, B.A., was born on July 9th, 1836. He was educated at Cheltenham and at Clare College, Cambridge, where he held a classical scholarship. He graduated thirty-fourth Wrangler in 1859. He was for many years Deputy Keeper of the Land Revenue Records and Enrolments of the Woods and Forests Department, Whitehall. Some eight years ago his mind was affected through over-work and over-study, and he had at that time a year's holiday. Latterly he was again worried on account of certain extra responsibilities which had devolved upon him, and the end came on November 2nd, 1895. He was elected a member June 10th, 1875, but, although he always studied the *Proceedings* of the Society with deep interest, he only attended some six of the meetings. Whilst he continued an ardent student of mathematics, he was also a student of languages and philosophy, and was much interested in the work of the Aristotelian Society, of which he was a member. He was the writer of several philosophical papers.

ERRATA.

VOL. XXVI.

P. 1, *for* June 4th *read* June 14th.

P. 3, *for* Brickmore *read* Bickmore.

P. 116, *for Math. Ann.*, LXV. *read* XLV.

P. 488, line 11, *for* $(n+1)\,P_{n-1}$ *read* $(n+1)\,P_{n+1}$.

Dele from p. 196, line 18, to p. 197, line 21; from p. 197, line 35, to p. 198, line 16. [I owe these corrections to the kindness of Herr O. Hölder. See his paper "Die Gruppen mit quadratfreier Ordnungszahl," *Göttingen Nachrichten*, May, 1895. The true number of distinct groups of order 210 is 12.—W. BURNSIDE.]

P. 210, footnote. The proof that Herr Frobenius gives of the non-existence of simple groups of order $p_1^4 p_2^m$ is correct. I regret that, owing to my having misunderstood a point in his argument, I should have stated it was not.—W. BURNSIDE.

P. 237, *for*

$$\wp' a = \wp' b \quad \ldots\ldots (106),$$

$$\wp' a = \wp' b \quad \ldots\ldots (107),$$

read

$$\wp' v_1 = \wp' v_2 \quad \ldots\ldots (106),$$

$$\wp' v_1 = \wp' v_2 \quad \ldots\ldots (107).$$

P. 290, line 10, *for* unity *read* 2.

P. 291, lines 22 and 27, *for* n *read* $n\,(a+1)$.

P. 291, line 28, *for* $\dfrac{n}{a+1}$ *read* n.

P. 303, line 6, *for* 150223 *read* 150233.

P. 306, third paragraph, *for* $n = 6$ *read* $n = 8$; *for* ·06136, ·18816, ·43660, ·37524, 17 *read respectively* ·04592, ·14109, ·45241, ·40649, 11.

P. 311, lines 7 and 9 from bottom, *put* > before the bracketed expressions.

P. 313, line 17, *for* unity *read* 2.

P. 313, line 20, *for* improper *read* intermediate.

P. 313, line 7 from bottom, *for* $\dfrac{D}{n}$ *read* $\dfrac{D}{n^{\frac{3}{2}}}$.

VOL. XXIV.

P. 257. The fourth entry in the reciprocal factor of 2161 *should be* 24, *not* 14.

P. 258. The prime 3371 has been omitted. The entries should be

$$3371 \mid 127,\ \dot{2}\dot{3} \mid \dot{2} \mid 3,\ 1,\ 1,\ 2,\ \dot{6} \mid 0\dot{5}\dot{4}28 \mid 6,\ 8,\ \dot{4}\dot{6},\ 15,\ 16 \mid 55$$

The author has to thank Mr. Bickmore for pointing out these errata.

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